1441

WAVE MOTION AND SOUND

Extract from *The Purple Island* by Phineas Fletcher (1633)
(Canto V, 47 and 48)

"As when a stone, troubling the quiet waters,
Points in the angry stream, a wrinkle round
Which soon another and another scatter
Till all the lake with circles now is crown'd;
All so the aire struck with some violence nigh,
Begets a world of circles in the skie;
All which infected move with sounding qualitie.
These at Auditus' palace soon arriving,
Enter the gate and strike the warning drum;
To these three instruments fit motion giving,
Which every voice discern; then that third room
Sharpens each sound and quick conveys it thence;
Till by the flying poast 'tis hurried hence,
And in an instant brought into the judging sense."

WAVE MOTION AND SOUND

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PREFACE

Numerous books on sound have appeared in recent years, and the addition of yet another volume may require some explanation. In submitting this book the authors feel that they are filling a gap which exists in the literature between the elementary textbook, with its presentation of the subject as a matter of " strings and pipes," and the advanced treatise, with its copious references to original researches. Furthermore, the authors have felt a strong urge to dispel the notion still prevalent in many teaching circles that sound is merely a Cinderella subject; the theory of vibrations is in fact fundamental to most other branches of physics, and this is emphasised throughout the book by constant reference to analogous problems in electricity, light, etc.

The main text of the present volume, with the omission of certain paragraphs and chapters, *e.g.* Chapter 15, should cover the needs of the ordinary degree student, while the book as a whole, including the mathematical appendix, should meet the requirements of the honours degree physicist. In the earlier chapters only a modest knowledge of mechanics and acquaintance with the calculus is assumed. Moreover, the bulk of the more mathematical theory has been placed in the appendix, and it is hoped that this will help the average reader to more readily assimilate the essential physical concepts of acoustics as set forth in the text.

To assist the student who wishes to extend his knowledge in particular parts of the subject, a list of authoritative books, in preference to original research references, is included at the end of certain chapters. This procedure is adopted as it provides the sources of information most readily accessible to the majority of students. Finally, the set of examples at the end of the book should provide the earnest student with a useful testing ground for estimating the degree of success, or otherwise, with which he has absorbed the fundamentals of his subject.

Although primarily intended as a textbook, it is the authors' hope that within these pages the general reader will find much of interest to hold his attention and to impress upon him the importance of acoustics in everyday life.

In a work of this type much of the subject matter and ideas cannot be claimed as original; furthermore, when these become absorbed into a lecture course their source of origin, after many years, becomes forgotten. Our indebtedness is therefore due to those unnamed authors whose ideas have been incorporated in the text. We are also grateful to the various persons and institutions who have given permission for the reproduction of figures from their publications, and in particular we thank Professor E. N. da C. Andrade, F.R.S., Dr. Mary Waller, and Professor J. Read. Our sincere thanks are also due to our friends Mr. E. Nightingale, M.Sc., and Mr. C. W. Celia, M.Sc., for their invaluable help in reading and correcting the galley and page

proofs of the book, to Dr. A. Kjerbye Nielsen for many helpful suggestions in the preparation of the manuscript, and to Mr. T. Clare of Messrs. Arnold who has always been most courteous and responsive to our demands during the passage of the book through the press.

Jan. 1950.

R.W.B.S.
A.E.B.

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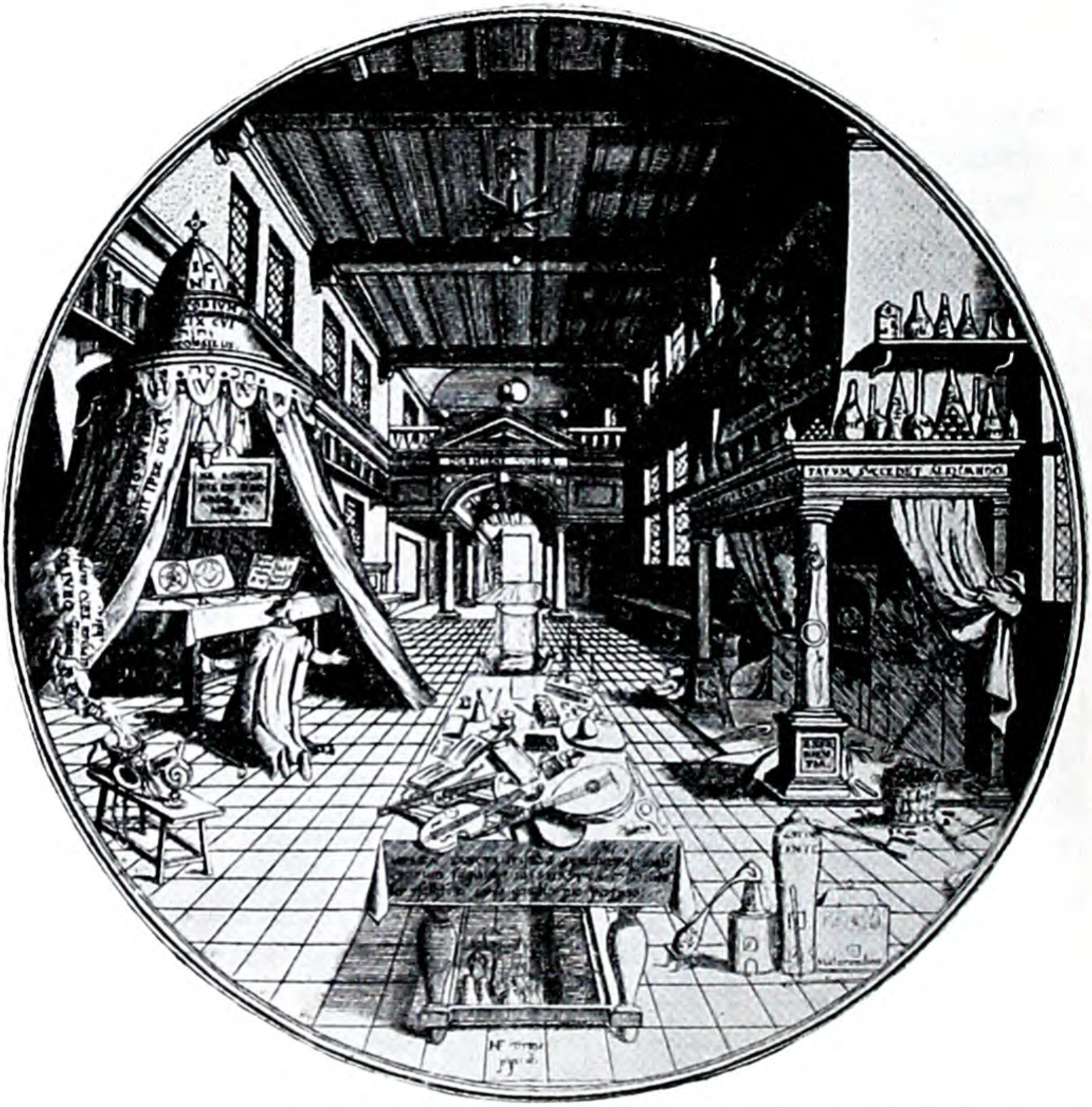

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KHUNRATH'S ORATORY-LABORATORY

(16th century)

CHAPTER 1

HISTORICAL INTRODUCTION

The study of sound can be said only to have developed into a science of measurement in the period between the world wars, although considerable theoretical advances had been made in the eighteenth and nineteenth centuries. Consequently its historical aspect is lacking in background when compared with that of other branches of physics and only a scanty reference to acoustics, as the science of sound is usually termed, is to be found in books devoted to the history of physical science. In order to trace out the earlier growth of the subject it is necessary to refer to musical works, and the superficial attitude towards sound as a science, at any rate up to the middle of the seventeenth century, is aptly expressed by the words of the English philosopher Bacon, "The nature of sound hath in some sort been enquired as far as concerneth music."

The early history of sound is therefore linked closely with that of music, an art which was practised by Hindus, Egyptians, Chinese and Japanese as long ago as 4000 B.C., although our existing system of music has been derived from the Greek era of civilisation. It was the Greek mathematician Pythagoras, about 2500 years ago, who was mainly responsible for originating the studies of musical intervals and ratios. He carried out experiments with a simple apparatus, usually known as a monochord, which is essentially a string maintained in uniform tension, and passing over two wooden "bridges" which define the wave-length of the transverse vibrations of the string when it is plucked or bowed. As a result of his experiments Pythagoras found that, under the *same* tension, the ratio of the length of string sounding the fundamental and the octave respectively was exactly two to one. Unfortunately it was not until towards the middle of the seventeenth century that Mersenne carried the experiments of Pythagoras a stage further, and examined the effect on the frequency of varying both the mass and the tension of the strings. In consequence of this delay the evolution of the modern pianoforte with its range of $7\frac{1}{4}$ octaves was considerably retarded, for such an instrument using similar strings under *equal* tensions would require a ratio between the longest and shortest strings of the order of 150 to 1 [*i.e.* $2^{7.2}$ to 1], which is impracticable within reasonable limitations of space.

Pythagoras' discovery, instead of acting as the stimulus for further experiments, became the basis of fantastic philosophical and mathematical speculations, such as the famous "harmony of spheres," and so for another 2000 years sound was mainly involved in a semi-mystical arithmetic of music. The extent to which mystical ideas prevailed at the time is shown by the writings of the Greek mathematician, astronomer and geographer Ptolemy, who devised, about the year A.D. 130, a geometrical diagram called the "Helicon." This figure

exhibited certain ratios which were supposed to represent harmonic relationships between colours and musical tones, and Ptolemy went even further to suggest that these harmonic relationships permeated the entire universe. It is interesting to note that Newton himself was obsessed with this linking of metaphysical speculation with physical facts, and he was a staunch believer in the idea that the seven colours of the rainbow did not show continuous graduation, but that each occupied a proportional space in the spectrum, due to some inherent property of the colour. In his "Opticks," when referring to the colours of thin films, Newton states that the limits of the seven colours—red, orange, yellow, green, blue, indigo and violet, in order, are to one another as the cube roots of the squares of the eight lengths of a chord which sound the notes of the musical scale.

Some 200 years after Pythagoras, the great philosopher Aristotle was achieving a reputation by his writings, many of which contained allusions to sound and music. In particular, reference should be made to a series of short dissertations entitled "Sound and Hearing," in which Aristotle proposes an interesting explanation of the mode of propagation of a sound wave in air. He says, "Sound takes place when bodies strike the air, not by the air having a form impressed upon it, as some think, but by it being moved in a corresponding manner; the air being contracted and expanded and overtaken, and again struck by the impulses of the breath and the strings. For when the air falls upon and strikes the air which is next to it, the air is carried forward with an impetus, and that which is contiguous to the first is carried onward; so that the same voice spreads every way as far as the motion of the air takes place." During the next fifteen or sixteen hundred years, that is to say until the time of Galileo, the influence of Aristotle spread and his writings in physics became authoritative. Two other writers of the Aristotlean era should be mentioned; Aristoxenus, a pupil of Aristotle, who wrote amongst others, three books styled "Elements of Harmony," and Euclid who wrote "Introduction to Harmonics." Neither of these Greek philosophers was at all concerned with the physical aspect of sound and the same may be said of the contemporary Romans with the possible exception of the famous architect Vitruvius who wrote, about 50 B.C., a treatise on the acoustical characteristics of theatres. This work is noteworthy as Vitruvius clearly pointed out how reverberation, interference and echoes occurred in an auditorium.

It may be said that the period of experiment in scientific investigation did not commence until the second half of the sixteenth century and it was an Englishman, Sir Francis Bacon (1561-1626), who was mainly responsible for inaugurating the philosophy of inductive science, although Bacon himself did not make any notable experimental contribution. It was his contemporary, the great Italian scientist Galileo Galilei (1564-1642), who really laid the foundations of experimental science, certainly of experimental acoustics. He deduced the complete laws of vibrating strings, although his results were not published until after Marin Mersenne (1588-1648) had independently discovered the laws, and hence their discovery has come to be wrongly attributed to the French Franciscan friar. In "Two New Sciences," published in

1638, Galileo gave a concise explanation of consonance, dissonance and resonance, and he described various experiments including that of the consonant pendulums (Appendix 5). A few extracts from the writings of Galileo will serve to indicate their breadth and clarity:—"One must observe each pendulum has its own time of vibration so definite and determinate that it is not possible to make it move with any other period than that which nature has given it." . . . "Agreeable consonances are pairs of tones which strike the ear with a certain regularity; this regularity consists in the fact that the pulses delivered by the two tones in the same interval of time shall be commensurable in number so as not to keep the ear-drum in perpetual torment, bending in two different directions in order to yield to the ever-discordant impulses." . . . "Waves are produced by the vibrations of a sonorous body, which spread through the air, bringing to the tympanum of the ear a stimulus which the mind interprets as sound."

The seventeenth century is often spoken of as "the century of genius," since during this period the foundations of modern philosophy and science were laid. The universities, however, were almost solely occupied with classical studies and this led to an important innovation with regard to the organisation of scientific investigation. In 1657 the Accademia del Cimento was founded in Florence, followed by the formation of the Royal Society of London in 1662, and the Paris Académie des Sciences in 1666. The Accademia was inspired by the work of Galileo and, among others, of his famous pupil Torricelli (1608-1647), and the creation of this institution was a direct challenge to the deductive science of the *quadrivium* (geometry, astronomy, arithmetic and music which were respectively classified as stationary, moving, pure and applied numbers). During the ten years of its existence (the Accademia was disbanded in 1667) numerous experiments in physics were carried out, and included among them was a new determination of the velocity of sound in air. The experimenters were Borelli, an anatomist, and Viviani, a disciple of Galileo, and they obtained a value for the velocity of 1148 feet per second. Similar experiments were undertaken for the French Academy by Cassini, Römer, Picard and Huygens, and they found the velocity to be 1142 feet per second.

The first direct experimental determination of the velocity of sound in air is usually attributed, however, to Mersenne, who took readings of the time between seeing and hearing the report of a gun as observed at a distant point. Mersenne was also the author of a comprehensive treatise on music and sound which was published in 1636 under the title "Harmonic Universelle." This book included sections on musical scales and intervals, dissonance and consonance, musical instruments and what was at that time an important advance in acoustics, namely the description of an experiment to determine the actual number of vibrations per second in a musical tone. Pierre Gassendi (1592-1655) repeated and extended the velocity experiments of Mersenne by finding the effect of the size of the gun (he used a musket and a cannon) upon the value obtained. In this way he disproved Aristotle's theory that a sharp sound was transmitted more quickly than a low one, *i.e.* a crack than a boom. Gassendi furthermore deduced from his experiments that the velocity

of sound was unaffected whether the direction of propagation was with or against the prevailing wind. The true influence was not discovered until some years later by William Derham (1657-1735) who was also the first to suggest that a knowledge of the velocity of sound could be used to estimate the "seat" of a storm, by noting the time difference between the lightning flash and the thunder-clap. It was not until some years later, about 1740, that measurements were made on the change of velocity with temperature. Two experimenters, Bianconi and La Condamine, independently formed a general conclusion that the velocity increased with temperature. Bianconi made his observations at the same place both in summer and in winter, while La Condamine carried out his experiments at the same period of the year but at two places which differed widely in temperature.

The early measurements of the velocity of sound in solids and in liquids were also of a large scale, *i.e.* non-laboratory, type, and the first recorded determination of the velocity of sound in a solid was not made until 1808, by Biot. This Frenchman, who is probably better known as the discoverer of the familiar law in optics bearing his name, used an iron pipe over half a mile long. By striking a bell mounted at one end of the pipe and noting the difference of the times of arrival of the waves which travelled through the material of the pipe and those which were propagated in the air within the pipe, an estimate of the velocity of sound in the material was obtained. Some twenty years later Biot applied his acquired optical technique to devise an elegant method of rendering the modes of vibration visible by placing the specimen rod of a transparent material between the analyser and polariser of a polariscope. About this time, 1826, Colladon and Sturm made their celebrated experiments on the determination of the velocity of sound under water in Lake Geneva, the source of sound being a submerged bell, which when struck, simultaneously ignited a small charge of gunpowder. The interval between seeing the flash and hearing the sound was determined by means of a stop-watch.

The theoretical aspect of sound had its beginning with Newton's derivation of the expression, velocity $= \sqrt{\frac{\text{elasticity}}{\text{density}}}$, for the velocity of propagation of a pulse in an elastic fluid, the elastic modulus concerned being appropriate to the type of deformation considered. The proof of the formula was published in Book II of the "Principia" in 1687, and by suitable substitution Newton calculated that the velocity of sound in air was 968 feet per second, which was much lower than the experimentally observed value. Some 70 years later Lagrange (1736-1813) pointed out that the mathematical analysis failed to take into consideration the changes in the elasticity of air due to temperature changes brought about by the propagation of the sound waves. In effect, therefore it is the adiabatic and not the isothermal elasticity which is involved, for the condensations and rarefactions in the wave take place too rapidly for isothermal conditions to prevail. Actually it was not until nearly a hundred years later, in *Ann. de Chimie* 1816, that the French mathematician Pierre Laplace modified Newton's equation on the basis of Lagrange's argument and the velocity calculated

from the corrected formula was in agreement with the standard experimental value.

The theoretical study of sound and wave motion received a considerable stimulus as a result of the development of the mathematical calculus by Leibniz (1646-1716) and by Newton, and there followed an era in which many famous mathematicians such as Euler (1707-1783), Lagrange, Poisson (1781-1840), D'Alembert (1717-1783), D. Bernouilli (1700-1782) applied calculus methods to many physical, including acoustical, problems.

In view of the present-day importance of wave propagation in such periodic structures as crystal lattices or transmission lines, it is interesting to note that Newton was the first to attempt the derivation of an expression for the velocity of sound in a one-dimensional lattice. He assumed a very simple structure consisting of equal masses at equal distances apart along the direction of wave propagation, these masses attracting each other with a constant elastic force. By considering the wave-length to be large compared with the distance between neighbouring masses he was able to treat the lattice as a continuous structure, and so was able to calculate the velocity of sound in air. It was left to John Bernouilli and his son David to show that a one-dimensional system consisting of *n point-masses* possessed *n* independent modes of vibration or *n proper* frequencies, as they are termed. Furthermore, the younger Bernouilli propounded the important principle of superposition, namely, that the general motion of any vibrating system is given by the superposition of its proper vibrations. By means of partial differential equations, just being introduced at that period, Euler completed the work on the solution of the case of a *continuous* string which had been started by Taylor (1713). It was Lagrange (1759), however, who treated this case as the limiting example of Newton's one-dimensional lattice of point-masses, a result which was later applied by the engineer Pupin to the electrical problem of a "loaded" cable. Incidentally, both Euler and Lagrange refused to accept the principle of superposition, which in effect was but a special case of a Fourier's series, destined to be developed a few years later.

Newton's simple periodic structure received further application, by Cauchy (1830), in his endeavour to explain the dispersion of optical waves by associating the latter with elastic waves of very small wave-length. This length now became comparable with the distance between the discrete masses of the structure, and the velocity of propagation was no longer independent of wave-length. The quantitative agreement with experiment was not good and this led Baden-Powell to consider a cubic lattice structure. The discovery of the modern dispersion formula, although usually attributed to Lorenz, was really due to Lord Kelvin, but his work attracted little notice at the time for it was contained in a paper dealing with the size of atoms. In his theory Kelvin assumes a structure in which a small mass is associated with every large mass so that the latter is restrained by two elastic forces, due respectively to each size of mass. The first mechanical filter constructed on these lines was due to Vincent (1898) and the electrical counterpart was developed by Heaviside, Vaschy,

Pupin, and Campbell (1906). It is interesting to note that the modern engineer or physicist tends to solve his mechanical, acoustical or even thermal problem, by translation first into the equivalent electrical system.

The two men who were mainly responsible for the establishment of sound as an exact science were Hermann V. Helmholtz (1821-1894) and Lord Rayleigh (1842-1919). Both Helmholtz and Rayleigh were very versatile, the former occupying in turn the professorial chair of physiology (Königsberg, 1849), anatomy (Bonn, 1855), physiology (Heidelberg, 1858) and of physics (Berlin, 1871). Helmholtz's contribution both to optics and to sound was mainly physiological and his classical work in the latter subject was contained in his "Sensations of Tone" (1862). Helmholtz developed the theory of summation and difference tones, and also the theory of resonators, including in particular the spherical form of resonator bearing his name. Until the advent of this resonator the ear alone was used for the analysis of complex tones. It is interesting to note that musical instruments up to this time had been mainly evolved on empirical lines, the guiding principle being to fashion the instrument to give the maximum amount of pleasure to the hearer. A change of attitude, however, of the manufacturer towards scientific method was heralded by Helmholtz's collaboration with a pianoforte maker, both in the suggestion and in the testing of improvements in the instrument.

Lord Rayleigh was primarily interested in physics and chemical physics. Although his work in sound was essentially mathematical it must not be forgotten that he possessed considerable skill in carrying out experiments with simple apparatus and, moreover, could be exceedingly accurate when occasion demanded, as shown by his very careful determination of the density of nitrogen, which led to the discovery of argon. In his monumental work "The Theory of Sound," published in 1877, and often spoken of as "The Principia of Acoustics," Rayleigh gave a comprehensive survey of the subject up to that time but, furthermore, developed considerable additional theory which opened up a vast field of research for experimenters of the following generation. By means of a delicately suspended disc, which tends to set so that its plane is perpendicular to the direction of propagation of a wave, Rayleigh was able to make measurements of the absolute intensity of sound. Before this time the detection of sound waves had been chiefly qualitative, such detectors as singing flames, singly and combined with manometric capsules, being employed to locate the nodes and anti-nodes in the path of a wave. Sand or other fine powder was also often used as an indicator, as for example in the well-known experiment of Kundt's tube, first described by the German, A. Kundt in 1866. Incidentally by suspending the Rayleigh disc in the mouth of a Helmholtz resonator the sensitivity of the latter as an analyser was increased, and also the uncertain characteristics of the human ear as the detector were avoided. The phonic wheel used in accurate frequency measurements is usually attributed to Lord Rayleigh, although it was also devised independently by La Cour (1878).

Two other workers in acoustics of the last half of the nineteenth century deserve, in their different ways, to be mentioned with Rayleigh and Helmholtz. John Tyndall (1820-1893), an Irishman who succeeded Michael Faraday as Director of the Royal Institution, is probably best known for his investigation into the colours of the sky, but he also carried out original experiments on singing and sensitive flames. Tyndall is chiefly notable, however, for his outstanding gift as a lecturer, and his efforts to popularise sound and physical science in general. K. R. Koenig (1832-1901), a German, was a student at Königsberg under Helmholtz but became an instrument maker, and devoted all his spare energy and capital to furthering his research work in sound. Amongst his most notable achievements was the construction of a tonometric apparatus involving about 600 tuning-forks. The well-known clock-fork was another of Koenig's efforts and his "phonautograph" was the predecessor of Edison's reproducing phonograph which was invented some 20 years later. Koenig tested and calibrated every tuning-fork which he made and his "standards" are to be found in widely scattered parts of the world.

In these days when research tends to become highly organised, at least outside of universities, the individual becomes a member of a team and so his sphere of work is restricted, possibly, to a small sub-division of a section of physics. By contrast, therefore, it is interesting to note a few prominent workers in other fields of physics during the nineteenth century, additional to those already mentioned, who have made useful contributions to the development of acoustics. Sir Charles Wheatstone (1802-1875), of Wheatstone-bridge fame, carried out a number of experiments on audition and was the originator of the term *microphone*, which he applied to an instrument invented by himself and which resembled a stethoscope. Wheatstone was also the first to employ a rotating mirror to examine the motion of rapidly vibrating systems. Lord Kelvin was interested in mathematical machines and invented the first harmonic *synthesiser* in 1872 which he used for the prediction of tides, and he followed this by producing the first harmonic *analyser*. These and more elaborate machines have proved of great use for the solution of various acoustical problems when used in conjunction with Fourier's theorem* and Ohm's law for sound. Fourier (1768-1830) himself was not interested in sound and he applied his theorem solely to the solution of problems in heat. It was left to the celebrated Georg S. Ohm (1789-1854), after whom the unit of electrical resistance is named, to indicate the theoretical application of Fourier's theorem to acoustic problems by the formulation of an Ohm's law for sound. This law states that all types of tone quality are due to certain combinations of a larger or smaller number of simple tones whose frequencies are commensurable; and further, that a complex musical assembly of tones may be analysed into a sum of simple tones, which may each be separately heard by the ear. The author of Henry's law in electromagnetism, Joseph Henry (1799-1878), a professor of physics at Princeton, U.S.A., carried out a

* Fourier's theorem states that any finite and continuous periodic motion may be represented by a series of simple harmonic motions of suitable phases and amplitudes.

number of important investigations in connection with sound signalling through fogs and with acoustics of buildings. The notable contribution of the great Michael Faraday (1791-1867) of electrical fame, was his correct explanation of the "mechanism" of singing flames and the difference in behaviour of a light and of a heavy powder when used to indicate the modes of a vibrating plate. These powder figures were first discovered by Chladni (1756-1827) and are described in his book "Die Akustik."

The rapid advancement made in experimental acoustics during the last decade was fostered by the intense interest in the subject created as a necessary consequence of the submarine campaign of the first world war. Another factor responsible for the amazing progress has been the linking up of the close analogy existing between the theory of mechanical vibrations and that of alternating electric currents, coupled with the increasing use of these currents and the advent of the thermionic valve. The development of the valve and its application to broadcasting and sound-motion pictures has led to an increased demand for sound recorders and reproducers of continually improved standards. Furthermore, the increasing mechanisation of the life of to-day has accentuated the problem of noise, and so a great deal of attention has been focused upon methods of sound insulation, a factor also of prime importance in the design of sound-recording studios. Photography has also played a useful part in the solution of a number of acoustical problems, chiefly hydrodynamical, and the basic principle utilised here is dependent upon the change in refractive index of a fluid medium due to an alteration of density created, for example, by the passage of a sound wave. The idea was first employed by A. Toepler in 1866. The cumulative effects of this widening field of acoustical applications have led to an increasing army of investigators and of people who are sound-conscious. As evidence of the growth of acoustical research may be cited the Bell Telephone Laboratories, which are the research organisation of the Bell Telephone Manufacturing Company of America. Many of the publications of these laboratories are of considerable academic as well as technical interest. In Britain the Post Office, the National Physical Laboratory, the Building Research Station, etc., are some government-sponsored groups, besides various industrial organisations, who are all actively engaged in different aspects of acoustical research. Also in some countries, *e.g.* Italy, Denmark, etc., there are colleges devoted solely to acoustical research and instruction. Journals dealing entirely with acoustics have appeared within comparatively recent times in America, France and Germany respectively under the titles the *Journal of the Acoustical Society* (1929), the *Revue d'Acoustique* (1932), and the *Akustiche Zeitschrift* (1936).

This short historical survey could not be justifiably closed without a brief mention, at least, of some of the experimenters of the last two or three decades. An outstanding worker of this era was W. C. Sabine (1868-1919) who may be rightly regarded as the "father of architectural acoustics," and the term a "sabinised" room is synonymous with one of good acoustical properties. F. R. Watson, P. E. Sabine, F. R. Bolt, V. O. Knudsen, A. H. Davis (and others of the National Physical Laboratory) are also prominent names in this sphere of acoustics.

The more complex the acoustic problem has become the greater has been the emphasis upon the importance of a previous theoretical analysis of the problem; in this theoretical field of sound notable names stand out like Stewart, associated with the theory of acoustical filters, Crandall, Kennelly, Olson (microphones), McLachlan (loudspeakers), Morse and Wente (microphones). In physiological acoustics H. Fletcher, of the Bell Telephone Laboratories, has been a prominent worker, while the lawyer-scientist Sir Richard Paget has carried out valuable investigations on the formation of vowel sounds. Dayton C. Miller's name will always be associated with his exhaustive analysis of musical sounds. Erwin Meyer has been largely responsible for the development of electro-acoustic measurements which have attained an accuracy comparable with corresponding measurements in other branches of physics. These measurements are not necessarily restricted to audible sound, and in consequence of the improvement of electrical technique the generation of mechanical vibrations above the audible limit (approximately 20,000 cycles per second) has become an important branch of acoustics and is known as ultrasonics. The wave-lengths of these inaudible sounds become comparable with ordinary-sized objects and so the analogous optical effects of diffraction, etc., are readily observable. Ultrasonics first came into prominence during the first world war as a means of detecting submarines, but in peacetime it has been utilised for "depth-sounding," for iceberg detection and for many other purposes. The great possibilities of research work in this field of acoustics has attracted the interest of scientific workers all over the world, as shown by the diversity of the countries represented in the list of investigators mentioned below. G. W. Pierce was responsible for much of the earlier quantitative work on the subject including that on the acoustic interferometer, and this was further developed by a fellow-American, J. C. Hubbard. W. G. Cady, also an American, and W. D. Dye of the National Physical Laboratory, were responsible for valuable work on the properties of quartz crystals in electrical circuits, which has led to their use as frequency standards. Other prominent investigators have been Bergmann and his co-workers of Breslau, Parthasarathy (India), Biquard (France), Giacomini (Italy) and R. W. Boyle and colleagues in Canada.

It is hoped that this brief summary has enabled the reader to obtain some idea of the development in the science of acoustics through the ages, and how it is intimately concerned with many phases of everyday life.

For further reference

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CHAPTER 2

PERIODIC MOTION

Waves

When a gust of wind passes over a field of ripe corn the stalks are observed to bend in the direction of the wind, and the observer receives the impression of a disturbance or wave passing over the field. After the disturbance has died away the field becomes quiescent and the stalks regain their upright positions, until displaced by another gust of wind. A closer examination reveals that the stalks do not return directly to their positions of equilibrium, but that they execute gradually decreasing displacements backwards and forwards until they come to rest. In the case of a small disturbance in a liquid surface, however, the individual particles do not move in the direction of the resulting waves, but perform an up-and-down movement perpendicular to the direction of propagation. This is shown by the bobbing up and down of a floating cork. In both of these examples there is no bodily movement of the medium as a whole, despite the progress of the disturbance, and so the conception of a wave motion is obtained, viz. the propagation of a certain condition or disturbance in a medium from one point to another, without any bodily motion of the medium itself.

The two types of wave motion cited above are termed (*a*) *longitudinal*, which corresponds to the to-and-fro movement of the cornstalks, and takes place in the direction of the propagation of the wave, and (*b*) *transverse*, which resembles the movement of the cork on the liquid surface. Sound waves in fluid media are of the longitudinal type.

Simple Harmonic Motion. When waves succeed one another at regular intervals or periods, the particles, whether moving to-and-fro, or up-and-down, execute a periodic motion. The characteristics of this motion are readily investigated by experiments with a *simple* pendulum, which consists of a small weight or bob attached to the lower end of a light inextensible string fixed at its upper end to a rigid support. The word *simple* is used to distinguish this pendulum from the *compound* pendulum, which may be any irregular rigid body supported at a point other than at its centre of gravity. Both types of pendulum swing in *simple harmonic motion* (S.H.M.) if the angular displacement from the vertical is small, *i.e.* less than about 5°. The maximum angular displacement is termed the *angular amplitude* of the motion.

If a simple pendulum is made to swing so that the bob traces out a circle in a horizontal plane, the speed is uniform and the string sweeps out the slant surface of a cone. This arrangement is known as a conical pendulum. If the bob is illuminated by a horizontal parallel beam of light (Fig. 2.1), its shadow on a vertical screen will be seen to move in S.H.M. This can be verified by arranging a simple pendulum of

suitable length to vibrate in a plane parallel and near to that of the vertical screen. Subject to the condition that the angle of swing is small, the bob of the simple pendulum can be adjusted to move in the shadow of the bob of the conical pendulum. In contrast to the conical pendulum, the speed of the bob of the simple pendulum varies continually, in a manner to be described later. This comparison suggests the following definition of S.H.M.:—

When a particle moves with uniform motion in a circle, its projection on to a plane perpendicular to the plane of the circle moves in S.H.M. (Fig. 2.1).

The motion of the shadow of the bob of the conical pendulum is independent of the actual location of the screen, *i.e.* of the plane on to which the moving particle is projected, and, for convenience of graphical

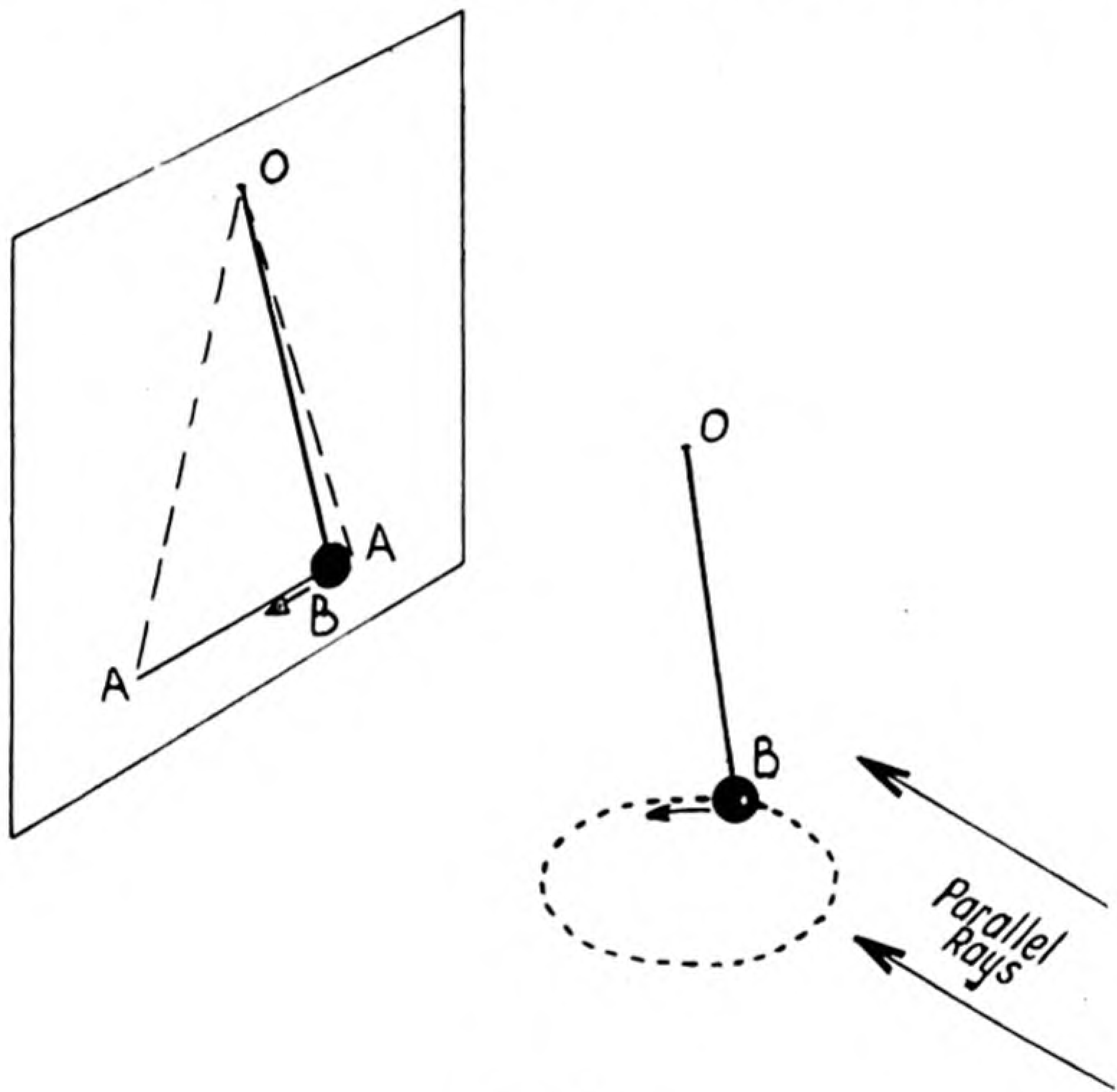


Fig. 2.1.

construction the plane through the diameter of the circle is taken, and the definition becomes:—

When a particle moves with uniform motion in a circle, its projection on to a diameter of that circle moves in simple harmonic motion.

Graphical relationship between displacement and time for a particle moving in S.H.M. If the amplitude or maximum displacement is denoted by a, a circle, termed the circle of reference, of radius a is drawn with X and Y axes through its centre O, and the circumference is divided into a number of equal parts, say 16, by the points A, B, C, etc. (Fig. 2.2). These points are projected on to the X axis as shown, b being the projection of B, etc. Consider a particle to be moving anti-clockwise round the circle with a uniform speed V; it follows

from the definition that the projection of the particle will move in S.H.M. along the diameter AJ.

Now as the speed of the moving particle is constant, the times it takes to traverse the arcs AB, BC, CD, etc., are equal, hence the projection moves through the corresponding distances Ab, bc, cd, etc., in equal intervals of time, in this case $\frac{T}{16}$, where T is the time taken for the particle to travel round the circumference of the circle, *i.e.* the time taken to perform one cycle. This time interval T is a constant which is characteristic of the circular motion and is known as the periodic time or *period*, and clearly it represents also the periodic time of the motion of the projection of the particle. Referring to Fig. 2.2, it is evident that the actual displacements of the particle executing S.H.M. along AJ after equal intervals of $\frac{T}{16}$ are successively given by $+OA$, $+Ob$, $+Oc$, $+Od$, zero, $-Of$, $-Og$, etc. The relationship between displacement and

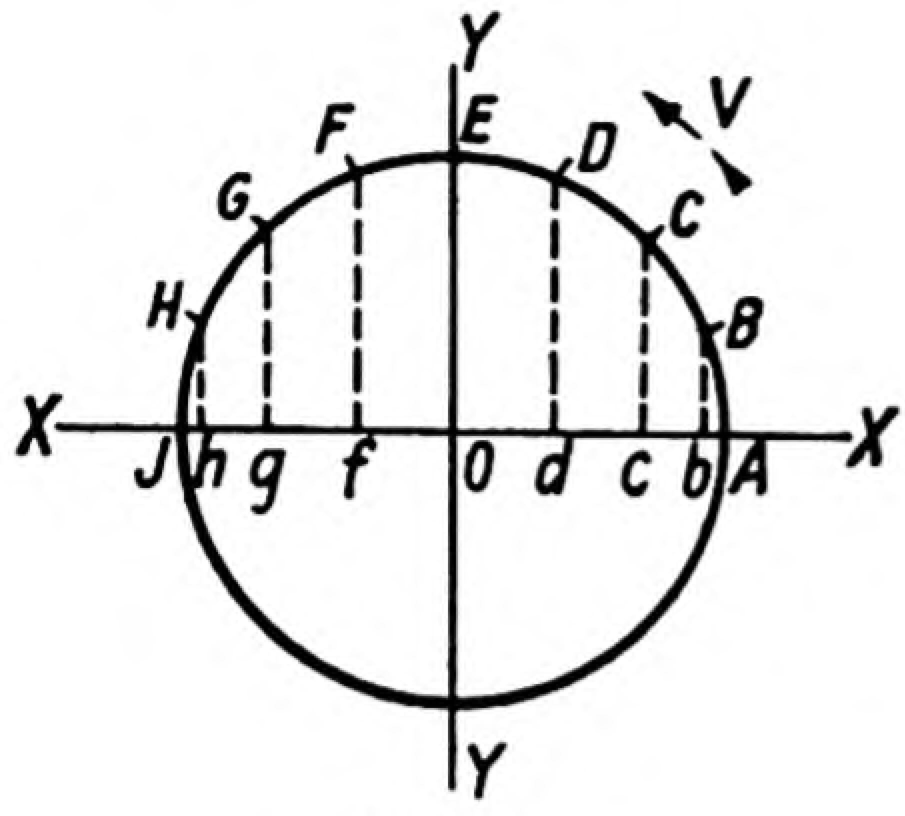


Fig. 2.2.

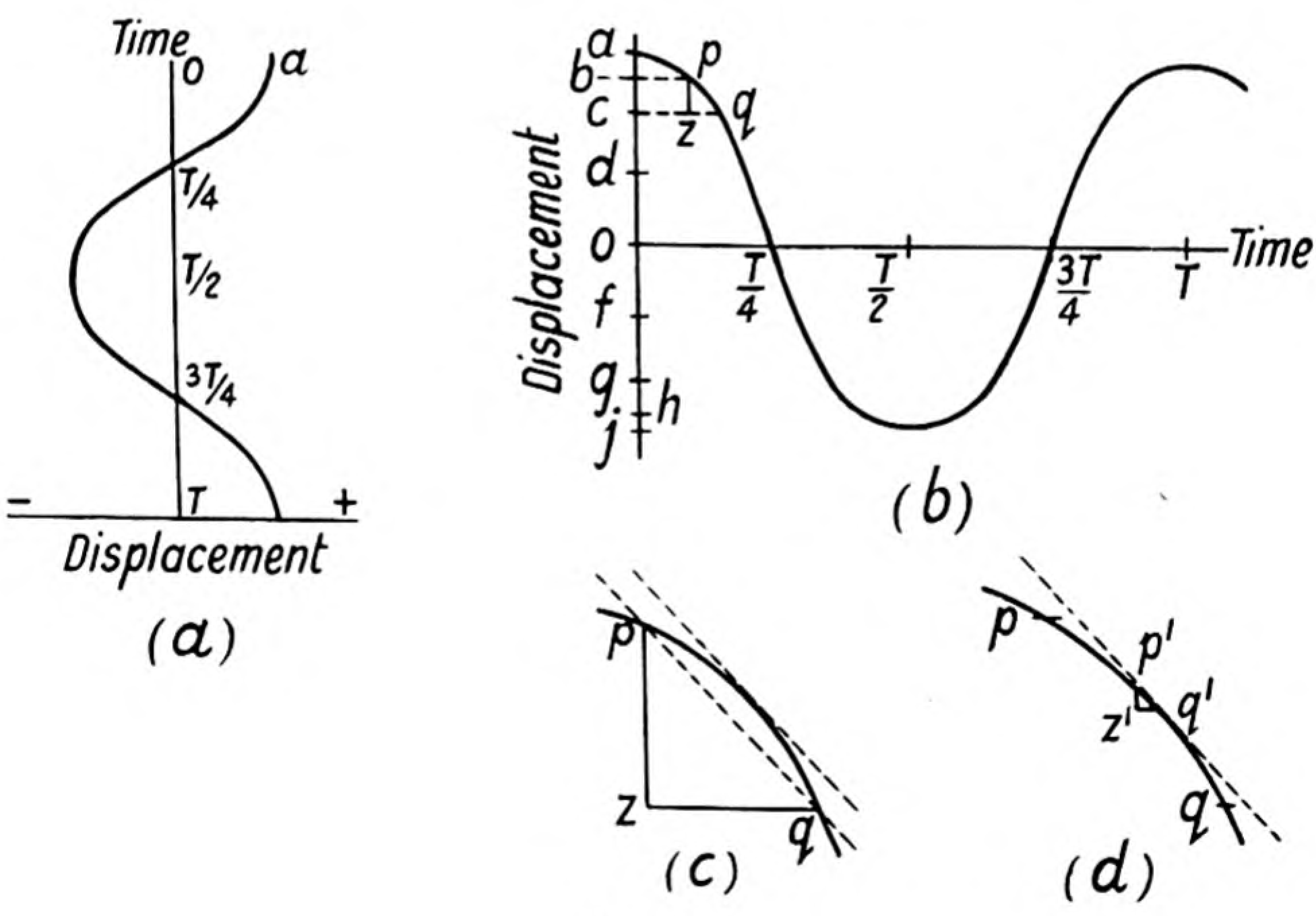


Fig. 2.3.

time can conveniently be shown graphically as in Fig. 2.3*a* in which the horizontal axis is the displacement along the diameter of the circle of Fig. 2.2, and *oa* is the amplitude (OA in Fig. 2.2). It is usual to represent time horizontally as in Fig. 2.3*b*.

Deductions from the graph. Velocity is defined as the rate of change of displacement of a body, hence the *average* velocity of the particle projection in any time interval is $\dfrac{\text{(distance moved)}}{\text{time}}$.

Thus $\dfrac{ab}{bp}$ is the average velocity in the first interval

„ $\dfrac{bc}{zq}$ „ „ „ „ second „ , and so on (Fig. 2.3*b*). It is clear that the smaller the time interval, the more nearly does the portion of the curve intercepted approximate to the hypotenuse of a right-angled triangle as shown in Fig. 2.3*c*, which represents the portion *pq* of the graph, and that for infinitesimal time intervals this hypotenuse coincides with the tangent to the curve at that particular time. Thus, the slope of the *tangent* to the curve at a point, say $p'q'$ at z' in Fig. 2.3*d*, is a measure of the instantaneous velocity at the time and displacement represented by that point; hence:

(1) the velocity is zero when the displacement is maximum, for the tangent is horizontal;

(2) the velocity is maximum when the displacement is zero;

(3) the slope of the curve—and therefore the velocity—changes continuously, implying that the particle projection is moving with an acceleration;

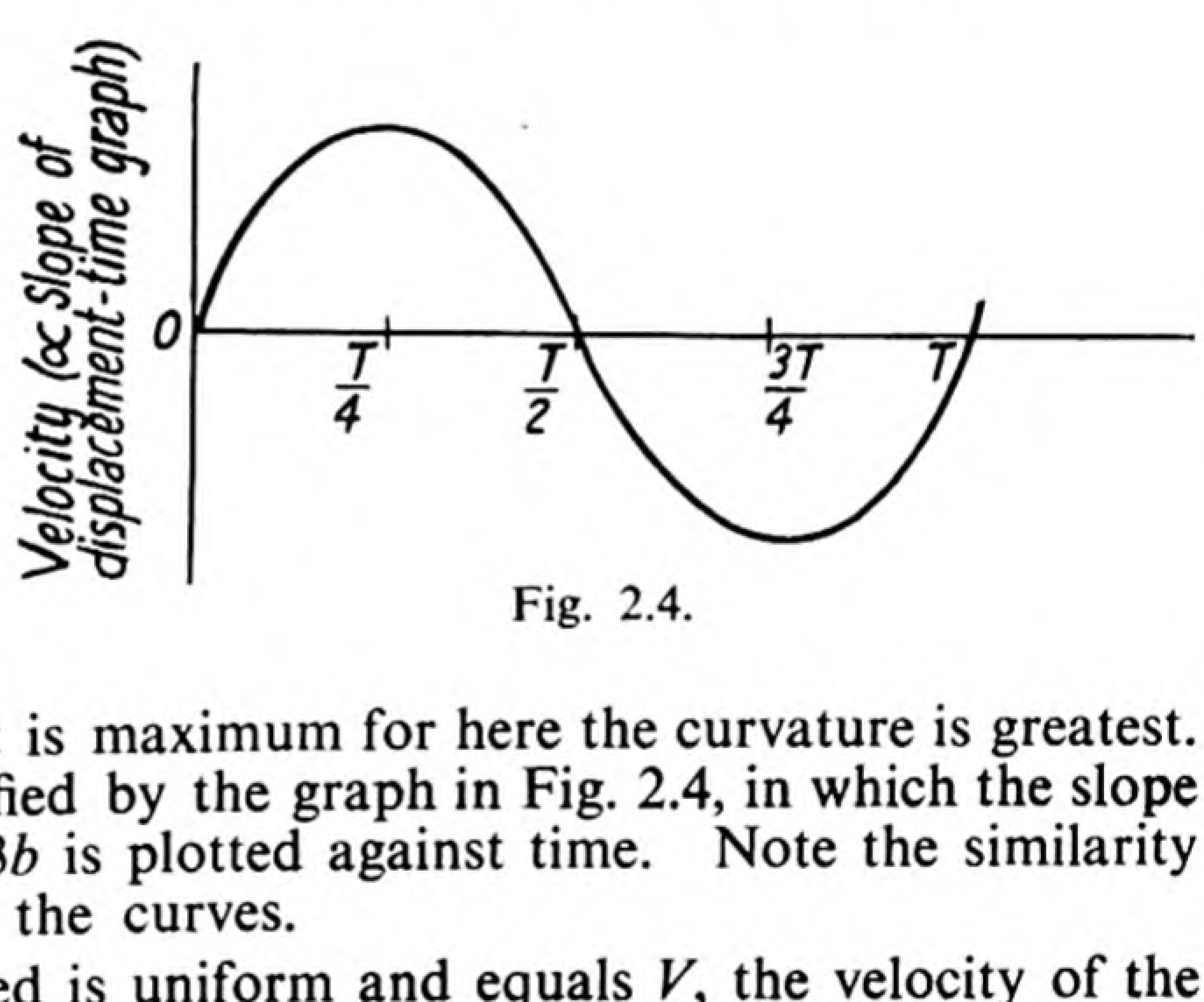


Fig. 2.4.

(4) the greatest *change* in velocity is indicated by the greatest rate of change of slope; this occurs when the displacement is maximum for here the curvature is greatest. This statement is justified by the graph in Fig. 2.4, in which the slope of the curve in Fig. 2.3*b* is plotted against time. Note the similarity between the forms of the curves.

(5) The particle speed is uniform and equals V, the velocity of the projection, when the displacement is zero; this occurs at the "uniform speed instants," $\frac{T}{4}$, $\frac{3T}{4}$, etc. In S.H.M. the acceleration is always directed towards the position of rest, O in Fig. 2.2, and is proportional to the displacement.

The equation to the displacement-time curve. Consider a particle P (Fig. 2.5) moving in a circle with a uniform speed V. The angular velocity of the radius OP $(=a)$ is obtained by dividing the length of the arc described in unit time by the radius. Thus if the time taken by the particle starting from A to go to B is 1 sec., then $\dfrac{AB}{OA}$ is the angle swept out in unit time, i.e. $\dfrac{AB}{OA}=\dfrac{V}{a}$ is the angular

velocity (ω). Thus angular velocity is expressed in radians per second and is constant since V is constant.

If P is the position of the particle after t sec., the angle AOP, θ say, swept out by the radius is ωt radians. The projection of P on OA is M and the displacement is given by

$$OM=x=OP\cos\theta=OP\cos\omega t=a\cos\omega t \quad . \quad . \quad . \quad (1)$$

which indicates the significance of a as the amplitude or maximum displacement of the S.H.M. This is the equation to the curve of Fig. 2.3b which is a cosine curve relating x and t.

Epoch (or Phase) Angle. If the angular displacement of OP be measured, not from the axis OX (Fig. 2.5), but from some position OB where angle $AOB=\delta$, then the displacement of the projection of P at any time t will be given by $x=a\cos(\omega t+\delta)$. This angle δ is known as the epoch or *initial* phase angle and is measured in radians. Since, however, $\omega=\frac{2\pi}{T}$ (see (2)), then the displacement may also be expressed as $x=a\cos\frac{2\pi}{T}(t+\delta')$, where δ' expresses the epoch in terms of time.

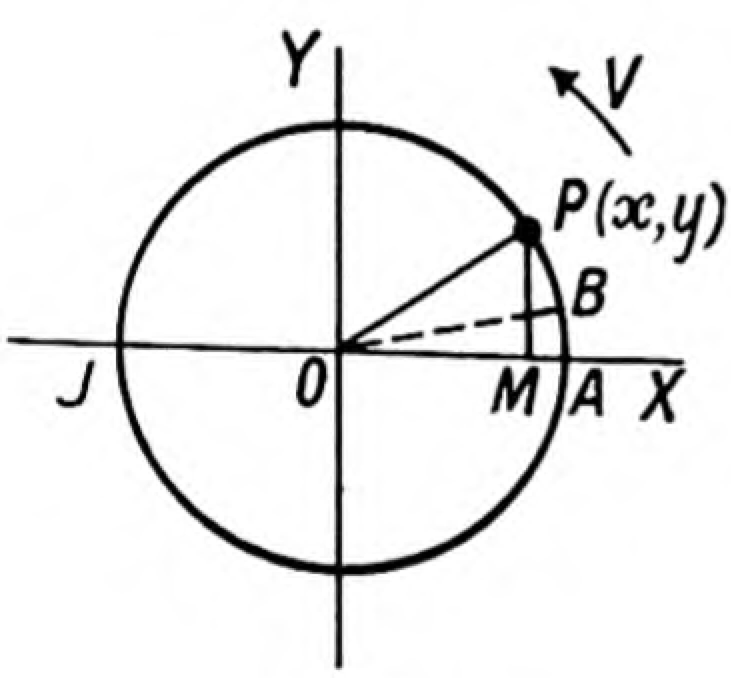


Fig. 2.5.

In T sec. the radius OP makes one complete revolution, *i.e.* it traces out an angle of 2π radians

$$\therefore\ \omega=\frac{2\pi}{T} \quad . \quad . \quad . \quad (2)$$

It is sometimes necessary to express the angular velocity ω in terms of the frequency (n) and as *frequency of vibration is the number of cycles performed in one second*, then the time T occupied in performing one cycle is $\frac{1}{n}$ sec.

$$\therefore\ nT=1 \quad . \quad . \quad . \quad (3)$$

Substituting in (2) the expression for ω becomes

$$\omega=2\pi n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The angular velocity ω is a property of the circle of reference (p. 11) and in order to identify it with the S.H.M. it is given the name *pulsatance.*

The equation of S.H.M., then, can be written in the alternative forms

$$x=a\cos\omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$x=a\cos\frac{2\pi t}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$x=a\cos 2\pi nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

in which x is the displacement of the particle.

It should be evident that the S.H.M. can be expressed alternatively in terms of the projection along the OY axis (Fig. 2.5), and in this case the displacement of the particle at any instant would be

$$y=a\sin\omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1c)$$

assuming P to be in position A at $t=0$.

To find the velocity and acceleration of a particle moving with S.H.M. As the velocity is continually changing, it is necessary to consider the change of displacement Δx in a very small interval of time Δt. The equation

$$x=a\cos\omega t \quad \text{at time } t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

becomes $$x+\Delta x=a\cos\omega(t+\Delta t) \text{ at time } t+\Delta t \quad . \quad . \quad (5)$$

Subtracting (1) from (5)

$$\Delta x=a[\cos\omega(t+\Delta t)-\cos\omega t] \quad . \quad . \quad . \quad . \quad . \quad (6)$$

whence* $$\frac{\Delta x}{\Delta t}=\frac{-2a\sin\omega\left(t+\frac{\Delta t}{2}\right)\sin\left(\frac{\omega\Delta t}{2}\right)}{\Delta t}$$

Now in limit as $\Delta t\to 0$, $\left[\sin\left(\frac{\omega\Delta t}{2}\right)\right]\Big/\Delta t\to\frac{\omega}{2}$,

and thus the instantaneous velocity $v=\dfrac{\Delta x}{\Delta t}=-a\omega\sin\omega t \quad . \quad . \quad (7)$

When ωt lies between 0° and 180° the value of $\sin\omega t$ is positive, hence v is negative, *i.e.* from right to left in Fig. 2.5. When ωt lies between 180° and 360°, $\sin\omega t$ is negative and v is positive, *i.e.* from left to right.

If Δv is the increment of velocity of the particle in time Δt it follows from equation (7) that

$$v+\Delta v=-a\omega\sin\omega(t+\Delta t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Subtracting (7) from (8),

$$\Delta v=-a\omega\,[\sin\omega(t+\Delta t)-\sin\omega t]$$

Hence $$\frac{\Delta v}{\Delta t}=\frac{-2a\omega\,[\cos\omega\,(t+\Delta t/2)\,.\sin\omega\,.\,\Delta t/2]}{\Delta t}$$

or the acceleration† $$\alpha=Lt.\,\frac{\Delta v}{\Delta t}=-a\omega^2\cos\omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\alpha=-\omega^2(a\cos\omega t)$$
$$=-\omega^2 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

since $$x=a\cos\omega t, \text{ from (1).}$$

The minus sign in equation (10) implies that the acceleration acts in a direction opposite to the displacement. This is made evident by considering the position of the pendulum bob at various stages in its motion (Fig. 2.6).

This result is very important; it provides an alternative definition of S.H.M.:—

S.H.M. is the motion of a particle about a point in a straight line such that its acceleration is proportional to its displacement from the point, and is always directed towards it. Thus, when x is negative, α is positive, since $\alpha=-\omega^2.(-x)=\omega^2 x$.

* $\cos B-\cos A=2\sin\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)$.

† $\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$.

The expressions derived above are readily deduced by application of the differential calculus:—

As the displacement $x = a \cos \omega t$, differentiation with respect to t gives

$$\text{velocity } v = -a\omega \sin \omega t.$$

A second differentiation with respect to t gives the acceleration

$$a = -a\omega^2 \cos \omega t = -\omega^2 x.$$

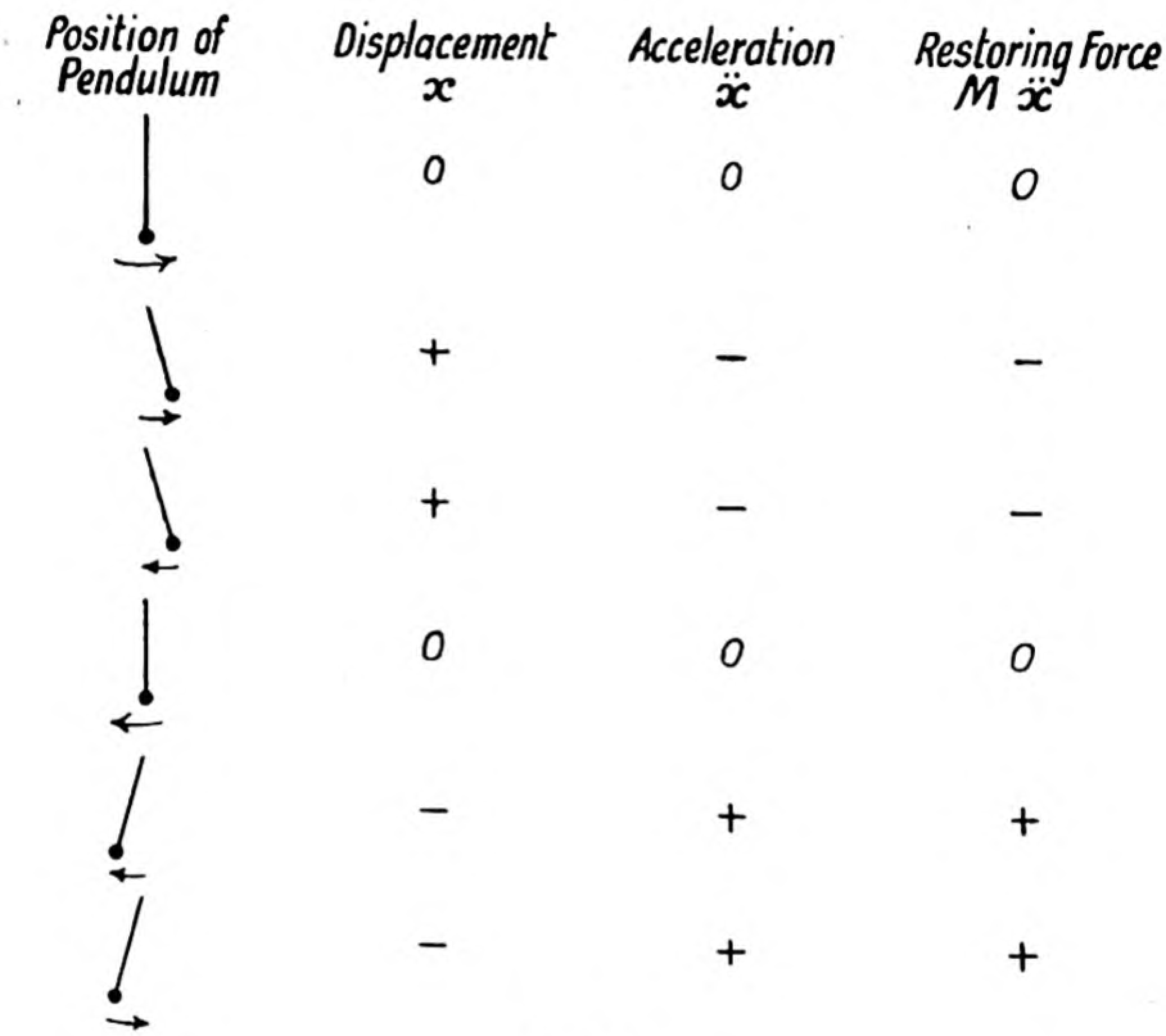


Fig. 2.6.

In the notation of the calculus, which will be used in succeeding chapters, the velocity and acceleration are respectively denoted by $\dot{x}$ and $\ddot{x}$; and equations (9) and (10) may be written as

$$\dot{x} = -a\omega \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\text{and} \quad \ddot{x} = -a\omega^2 \cos \omega t = -\omega^2 x \quad . \quad . \quad . \quad . \quad (12)$$

The conclusions previously drawn from the graphs may now be directly confirmed by means of these formulae.

Graph	Abscissa (time)	Ordinate (displacement x)	Slope $\dot{x}$ (Velocity v)	Rate of change of slope (acceleration a)
Fig. 2.3b	$0 \quad \frac{T}{2} \quad T$	maximum	0	maximum
	$\frac{T}{4} \quad \frac{3T}{4}$	0	maximum	0

It should be noted that the maximum *numerical* value of the acceleration along the diameter of reference AJ (Fig. 2.5) is $a\omega^2$ and it occurs at the extremities; and further, that this is the acceleration of the particle P towards the centre O of the circle. This is to be expected as $\omega=\frac{V}{a}$, whence $a\omega^2=a\left(\frac{V}{a}\right)^2=\frac{V^2}{a}$; V being the uniform speed of the point P round the circle of reference.

The alternative definition of S.H.M. can be modified, as below, by using the expression, from Newton's laws of motion, that force= mass × acceleration.

Let the particle performing S.H.M. have mass M, then for a displacement $\pm x$, the acceleration $\mp \alpha$ *towards* the centre is due to a force $\mp M\alpha=\pm M\omega^2 x$, hence the definition:

S.H.M. is the motion in a straight line of a particle about a point in that line such that the *restoring* force is proportional to its displacement from that point.

The *magnitude* of the ratio $\frac{\ddot{x}}{x}$ is a measure of the acceleration per unit displacement of the vibrating system and is given in this instance, from (12), by the value of $\omega^2=\frac{4\pi^2}{T^2}$.

The more general form of the equation representing a simple harmonic motion is

$$\frac{d^2x}{dt^2}=-\beta x \quad . \quad . \quad . \quad . \quad . \quad . \quad (13a)$$

where β is a *constant* and x represents the displacement of the body or system from its rest position. This displacement may be linear or angular, and the period of the motion is given by

$$T=\frac{2\pi}{\sqrt{\beta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13b)$$

In this preliminary study of S.H.M. the resistive forces, which are always present to bring the motion finally to rest, are for the present neglected, but will receive consideration later in the book.

Energy of a strained body. When a helical spring is extended in length the effort required increases from zero to a maximum value which is governed by the material, dimensions, and extension of the spring. Denoting the extension by x and the force necessary to produce unit extension by β, the final effort will be βx, with a mean effort of $\frac{\beta x}{2}$. Thus the work done on the spring will be $\left(\frac{\beta x}{2}\right)x=\frac{\beta x^2}{2}$, and this is the energy stored. This is equally true for wires, rods, etc., providing Hooke's Law holds, whether in tension or compression.

Energy of a small mass attached to a light spring vibrating in S.H.M. The energy (U) of this system will be partly kinetic and partly potential, except at the positions of maximum and zero displacement of the attached body,

i.e. $$U=\tfrac{1}{2}Mv^2+\tfrac{1}{2}\beta x^2=\text{constant} \quad . \quad . \quad . \quad (14)$$

where M is the mass of the suspended body and v is its velocity appropriate to a displacement x. Such a system as this is said to be *conservative* and the above equation is expressive of the principle of the conservation of mechanical energy.

Rewriting equation (13*a*) as $M\ddot{x}=-\beta x$, it follows by multiplying each side by $\dot{x}dt$ and integrating (see also Appendix I) that

$$\tfrac{1}{2}M\dot{x}^2=-\frac{\beta x^2}{2}+C \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where C is a constant of integration,

i.e.
$$\tfrac{1}{2}M\dot{x}^2+\frac{\beta x^2}{2}=C$$

which is the form of the energy equation previously derived. Hence it may be said that the first integral of the equation of motion of a conservative system is an equation of energy.

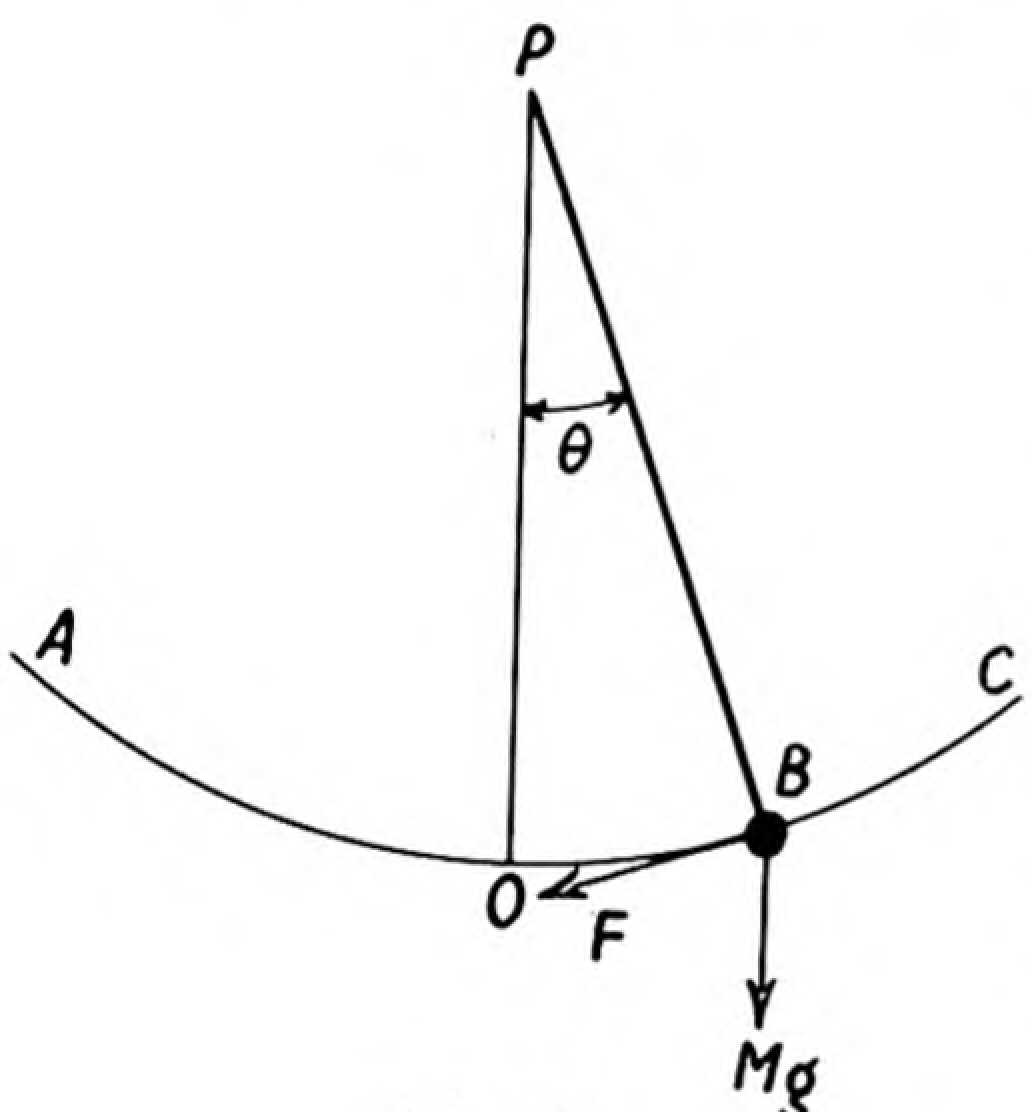


Fig. 2.7.

Examples of S.H.M.

This section is concluded with the derivation of formulae for the periodic times in some particular cases of S.H.M.

1. The simple pendulum. This consists (Fig. 2.7) of a small heavy body or bob B, suspended by a light inextensible string from a point P in a rigid support. When displaced from its position of rest O the bob moves along the circular arc $AOBC$. Consider the pendulum to be performing *small* oscillations about its rest position and that the angular displacement at a particular instant is θ. If M is the mass of the bob, l the effective length of the string and g the acceleration due to gravity, then the component of the gravitational force which is responsible for the motion is $F=Mg \sin \theta$ and is directed towards O along the tangent to the circular path at B.

The differential equation of the motion is therefore

$$M\frac{d^2s}{dt^2}=-Mg \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where s is the displacement OB measured *along* the arc from the rest position O. But $\frac{s}{l}=\theta$ and hence $\frac{d^2s}{dt^2}=l\frac{d^2\theta}{dt^2}$, so equation (16) now becomes $\frac{d^2\theta}{dt^2}=-\frac{g}{l}\sin\theta$,

i.e.
$$\ddot{\theta}=-\frac{g}{l}\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now for small angles $\sin\theta$ may be replaced by θ to a sufficient order of accuracy, thus for an angle of 5°, $\sin 5° = \cdot 0872$, whereas its angular measure is equal to ·0873.

It follows, therefore, that equation (17) may now be written as

$$\ddot{\theta} = -\frac{g}{l}\cdot\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In words, this equation implies that the angular *acceleration* is proportional to the angular *displacement* which is the characteristic of a S.H.M., and so from the relation (13*b*) the period

$$T = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

A motion in which the periodic time, as above, is independent of the amplitude is said to be *isochronous*. It is only because Hooke's law, viz. stress is proportional to strain (see p. 55), is obeyed for small strains that elastically controlled vibrations of small amplitude are isochronous. For larger amplitudes the period of the pendulum is no longer constant and is given by the expression (for proof see Appendix)

$$T = 2\pi\sqrt{\frac{l}{g}}\cdot\left(1+\frac{\alpha^2}{16}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where α is the amplitude of the angular displacement.

2. Rubber cord or helical spring supporting a load. Assume the mass of the spring or cord to be negligibly small in comparison with that of the load, M. If μg be the force necessary to produce unit extension, an additional extension of x calls into play a restoring force of $\mu g x$, and the forces acting on the mass M are:—

downward : Mg,

upward : $Mg + \mu g(-x) = Mg - \mu g x$.

The resultant upward force $= -\mu g x$, which produces an acceleration of $\ddot{x}$ upward in M, giving an upward force of $M\ddot{x}$;

i.e. $$M\ddot{x} = -\mu g x;$$

$$\therefore \quad -\frac{\ddot{x}}{x} = \frac{\mu g}{M} = \frac{4\pi^2}{T^2}$$

and $$T = 2\pi\sqrt{\frac{M}{\mu g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This formula is applicable to the vertical oscillations of a helical spring. If such a spring has an appreciable mass, say m, it has the same effect on the periodic time as increasing the load by $\frac{m}{3}$, and the expression becomes

$$T = 2\pi\sqrt{\frac{M+\frac{m}{3}}{\mu g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The direct determination of m can be avoided by observing the periods T_1 and T_2 for two known loads, M_1 and M_2 respectively, and subtracting; thus

$$T_1{}^2-T_2{}^2=\frac{4\pi^2}{\mu g}(M_1-M_2) \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Derivation of formula. If l be the length of the spring (or cord), the linear density is $\frac{m}{l}$, and the mass of an element situated between points s and $s+\delta s$, measured along the wire of the spring from the point of support is $\frac{m\,.\,\delta s}{l}$. At any instant the velocity of the mass M is given by $\dot{x}$ and therefore the velocity of the element at the same instant will be $\frac{\dot{x}s}{l}$ and its Kinetic Energy (K.E.)

$$\frac{1}{2}\left(\frac{m\,\delta s}{l}\right)\left(\frac{s\dot{x}}{l}\right)^2=\frac{m\dot{x}^2s^2\,\delta s}{2l^3}$$

The K.E. of the whole spring is

$$\frac{m\dot{x}^2}{2l^3}\int_0^l s^2ds=\frac{1}{2}\frac{m}{3}\dot{x}^2.$$

Now at the instant considered, the K.E. of the mass M is $\frac{1}{2}M\dot{x}^2$ and the total K.E. is $\frac{1}{2}\left(M+\frac{m}{3}\right)\dot{x}^2$, *i.e.* the spring increases the effective mass of the system by $\frac{m}{3}$.

The Potential Energy (P.E.) of the system in its displaced position x is given by $\frac{\mu g x^2}{2}$, hence the total energy of the system is

$$\frac{1}{2}\left(M+\frac{m}{3}\right)\dot{x}^2+\tfrac{1}{2}\mu g x^2=\text{constant};$$

Differentiating with respect to t.

$$\frac{1}{2}\left(M+\frac{m}{3}\right)2\dot{x}\ddot{x}+\tfrac{1}{2}\mu g\,.\,2x\,.\,\dot{x}=0;$$

$$\therefore\ \left(M+\frac{m}{3}\right)\ddot{x}=-\mu g x,$$

which is the characteristic equation of S.H.M. of period

$$T=2\pi\sqrt{\frac{M+\frac{m}{3}}{\mu g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

3. Oscillation of a liquid column in a vertical uniform U-tube. Let LL be the normal level (Fig. 2.8), a the uniform cross-sectional area, and ρ the liquid density. When the liquid is oscillating the elevation on one side equals the depression on the other. Noting that the

effects of surface tension on each liquid column tend to cancel, the only force to be considered is that due to gravity. (The tube is assumed to be sufficiently wide for the effect of viscosity to be negligible.)

When the displacement is x, the excess mass $2xa\rho$ above the depressed level in the right-hand limb causes a downward force of $2xa\rho g$, which acts on the whole liquid column of mass $la\rho$, l being the mean length of the column, and an acceleration in a direction opposite to the displacement is produced. It is left to the student as an exercise to verify

$$T=2\pi\sqrt{\frac{l}{2g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Note: The effect of viscosity will be to cause damping and to increase T.

4. Hydrometer of the Nicholson type. Let a be the cross-sectional area of the neck of the instrument (Fig. 2.9) at the water line, its

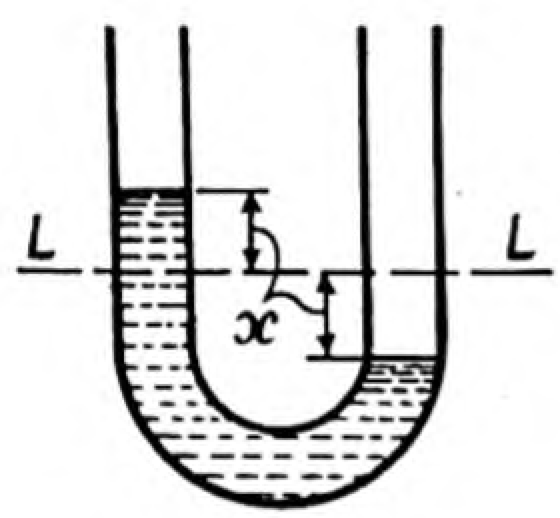


Fig. 2.8.

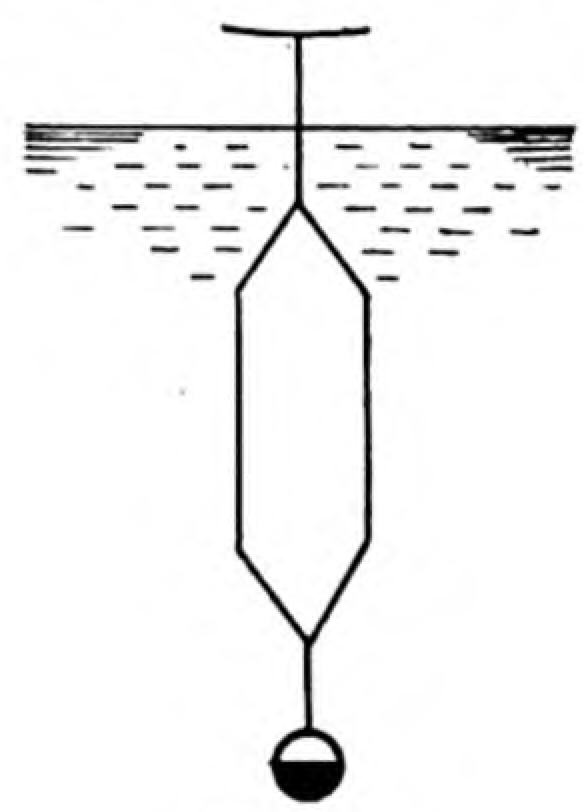


Fig. 2.9.

displacement in a vertical direction x, its mass M, and the liquid density be ρ. Show that the time of oscillation after a slight vertical displacement in a wide vessel is given by

$$T=2\pi\sqrt{\frac{M}{g\rho a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

5. The resonator. The derivation of the formula for the natural frequency of a resonator follows along similar lines, but is deferred until Chapter 4.

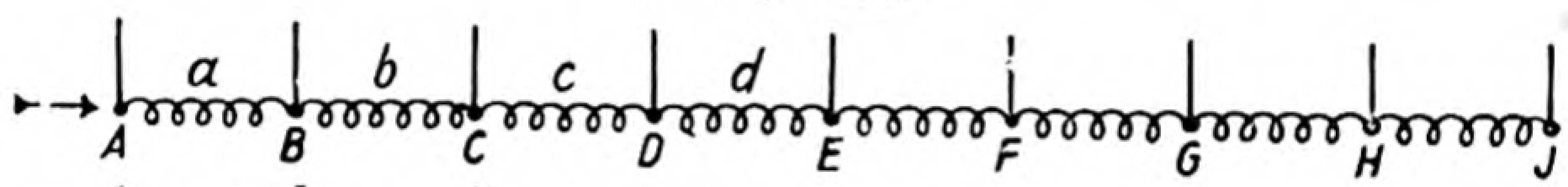
Wave motion

So far the motions of discrete objects have been considered. Now suppose such objects to be connected, the motion of any one being communicated to its neighbours. For example, if several identical pendulums are arranged in line and are connected by light springs as in Fig. 2.10, on *momentarily* displacing the first bob A inwards, the spring a will be compressed, so that bob B will move away and compress the spring b. In this manner a compression will travel along the

springs from left to right until all the bobs are set in motion, the energy of which will be derived from the work done in compressing the system of springs.

Consider 9 particles to be represented by $A, B, C, \ldots H, J$ (Fig. 2.11). When at rest they will be as in the first row. Assume the particle A to move to the right in S.H.M. The movement of A will cause B to commence moving a little later, and for the purpose of this discussion, and for convenience, the lapse of time will be taken as $\frac{T}{8}$ sec., T being the periodic time of each pendulum, assumed uninfluenced by the rest of the system. After $\frac{T}{8}$ sec. the disposition of the particles will be as in the second row, the displacement of A being determined by the construction given on p. 12, and indicated by the circle at the top of the diagram. Successive intervals find the particles in the

Fig. 2.10.



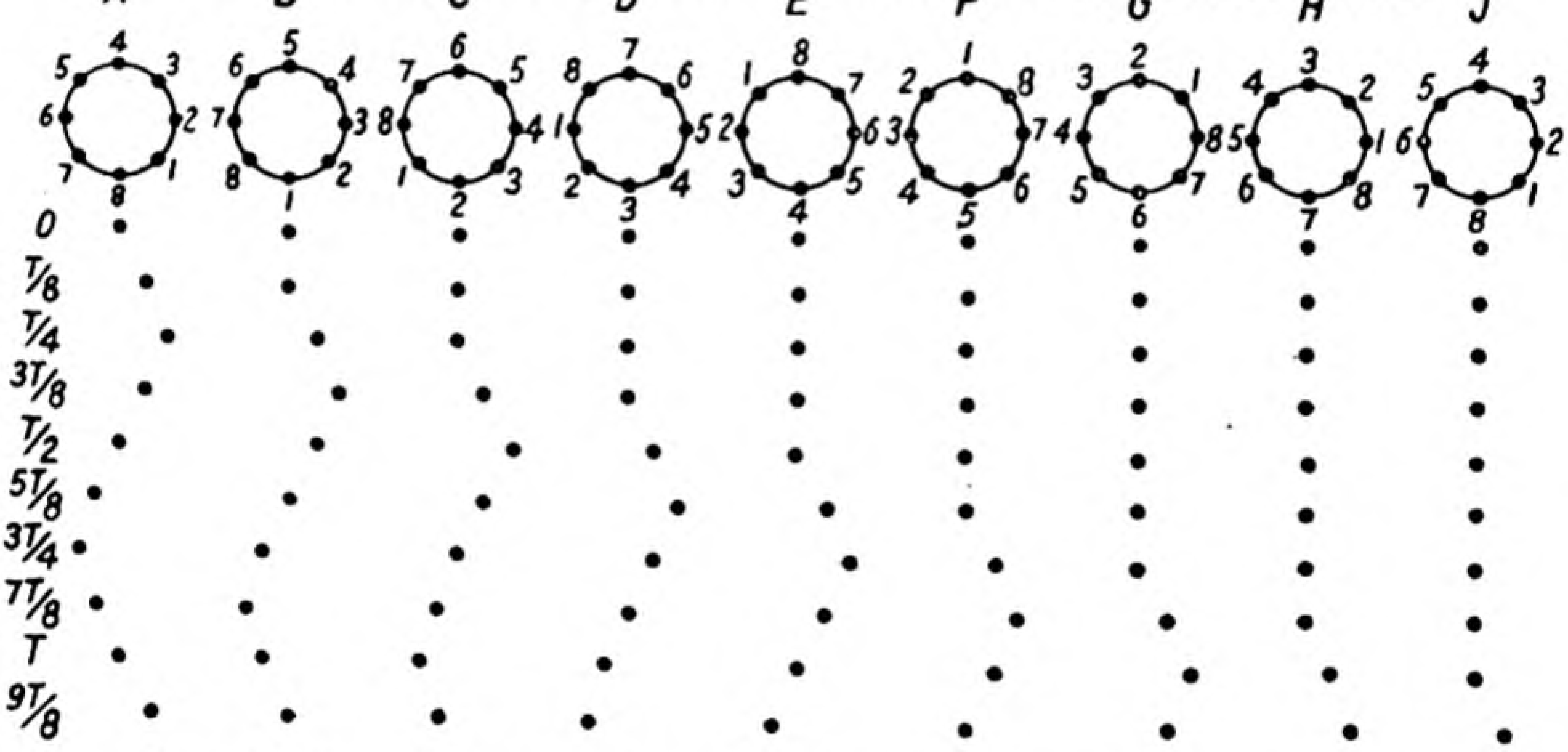


Fig. 2.11. (Numbers in circles denote equal time intervals.)

positions shown; in the last row but one, A, E and J are seen to be in their original positions, A and J moving to the right and E to the left. A and J are in phase, E is 180° out of phase with A and J, the meaning of the 180° being evident from the row of reference circles. It will be seen that B is 45° out of phase with A. Further, this phase difference between any pair of particles is always the same, thus A and B differ by 45° in each row. Similarly A and C differ in phase by 90° in all rows but the first, in which C is at rest. Usually this is expressed by saying that C suffers a *phase lag* of 90°. The wave-length is the distance between two particles in phase in a particular row. Reference to the last row shows that the particles are closer at the left than at the right: this indicates a compression on the left and a rarefaction on the right. The wave is said to be longitudinal as the particle displacements are in the direction of the motion of the wave.

Graphical representation of a longitudinal wave. As the axes of a graph are at right angles, the displacement of each of the particles is plotted against its position of rest. Thus the graph showing the wave at time T of Fig. 2.11 is obtained by drawing the displacements vertically, as in Fig. 2.12*c*.

Displacements to the left, being negative, are shown below the X axis, and those to the right, above. This graph gives the displacements of *all the particles at a particular instant.* The reader should draw two similar curves, on the same axes, for two successive instants as shown in Fig. 3.7. The direction of motion of the wave is thus rendered evident. The arrow-heads in line (*b*) of Fig. 2.12 indicate the direction of motion of particular particles at time $t=T$, and it is to be noted that at a position of rarefaction (in centre of system shown) the particles move in a direction opposite to that of the wave, but move in same direction at the location of a compression.

Frequently it is necessary to represent graphically the movements of *one particular particle in successive instants,* and in such a case the resulting curve for the particle A (Fig. 2.10) can be obtained by connecting the dots in the first column, which represent the successive

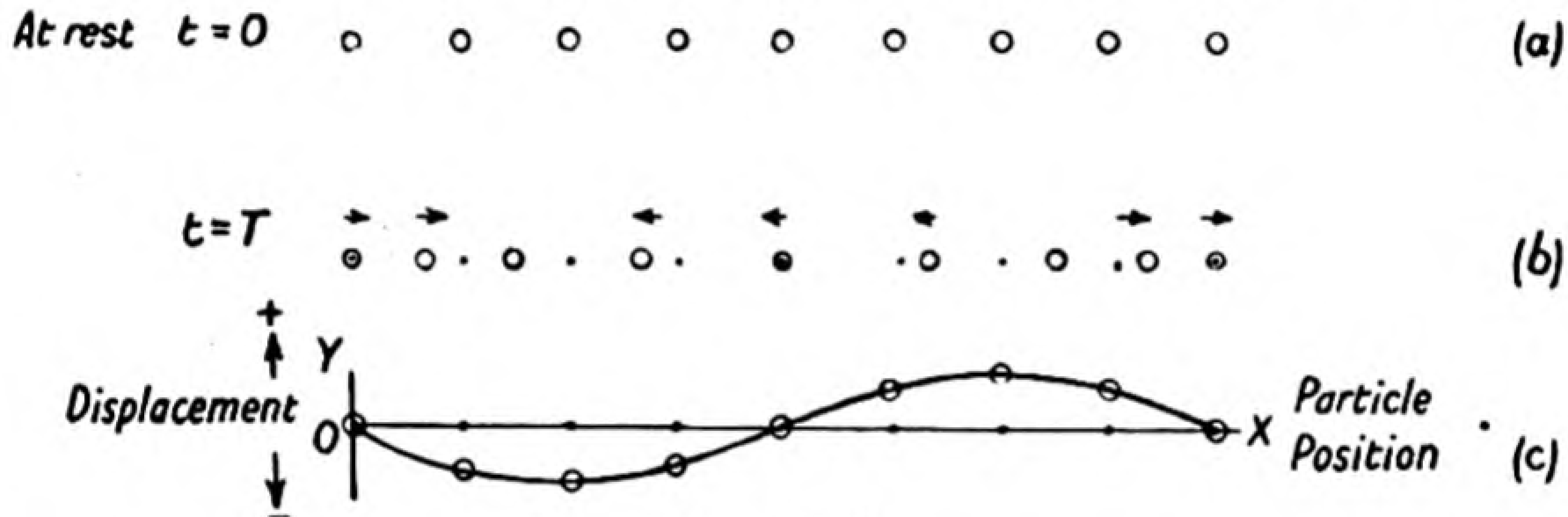


Fig. 2.12.

positions of A. In practice, time is plotted horizontally and displacement vertically. This displacement-time curve and the displacement-position curve (Fig. 2.12) are both sine curves, but great care must be taken to distinguish between them. Both are included in the expression

$$y=a \sin \frac{2\pi}{\lambda}(x-vt)$$

in which the wave is assumed to travel in the positive direction of x. The symbols a, v, t, x, y and λ respectively stand for amplitude, wave velocity, time, distance from the origin to the undisturbed position of the particle, particle displacement and wave-length. This expression is the general equation of a plane progressive wave, and is discussed more fully in Chapter 3.

Composition of two rotating vectors. If OP_1 and OP_2 (Fig. 2.13) are the rotating vectors corresponding to two particles P_1 and P_2 moving in circles with constant angular velocity, then the resultant displacement on the X axis at any instant is given by

$$\begin{aligned} x &= OP_1 \cos \delta_1 + OP_2 \cos \delta_2 \\ &= OP_1 \cos \delta_1 + P_1R \cos \delta_2 \\ &= OR \cos \theta \end{aligned}$$

since OP_1RP_2 is a parallelogram.

The magnitude of OR is given by

$$OR^2 = OP_1^{\,2} + OP_2^{\,2} + 2OP_1 . OP_2 \cos ROP_1$$
$$= OP_1^{\,2} + OP_2^{\,2} + 2OP_1 OP_2 \cos(\delta_2 - \delta_1).$$

Hence, since $(\delta_2 - \delta_1)$ is constant, the resultant displacement at any instant is given by the projection of the vector OR which forms the diagonal of the parallelogram of sides OP_1, OP_2.

If $OP_1 = OP_2 = a$, say, then $x = a(\cos \delta_1 + \cos \delta_2)$ and for n terms

$$x = a\sum_1^n \cos \delta_n \text{ and similarly } y = a\sum_1^n \sin \delta_n \quad . \quad . \quad . \quad (26)$$

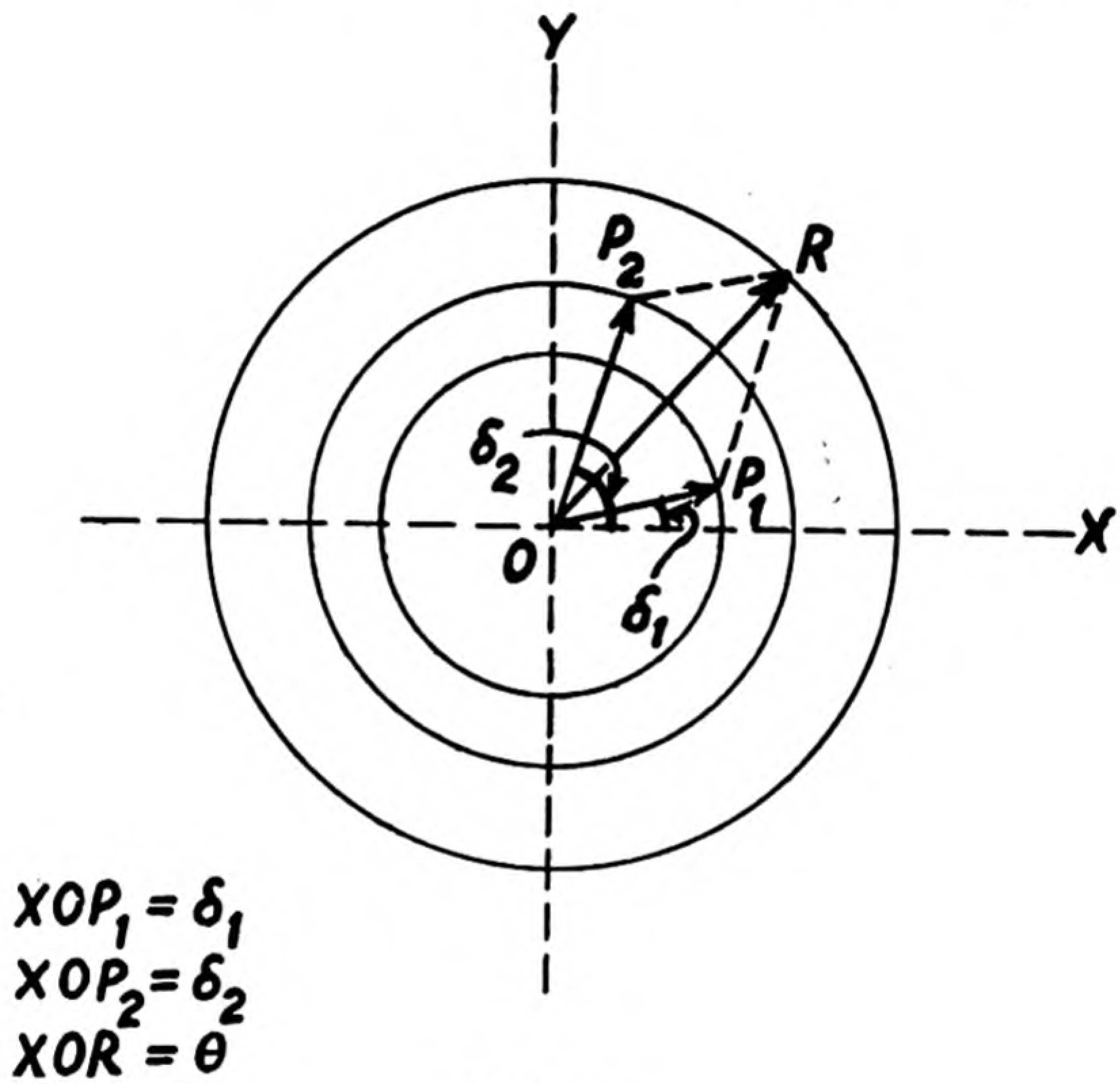


Fig. 2.13.

Superposition of simple harmonic motions. The graphical method is useful in the compounding of a number of S.H.Ms., and the particular case will be considered where they have equal amplitudes a and progressively differ in phase by a constant angle δ. The geometrical construction is self-evident from Fig. 2.14, the successive vectors OP_1, P_1P_2, P_2P_3, etc., representing the component vibrations each being set at an angle δ with its neighbours. Following from Fig. 2.13 the resultant of OP_1 and P_1P_2 will be given by OP_2 and this is compounded, in turn, with P_2P_3 to yield the resultant OP_3. The final resultant is obviously given in magnitude (A) and phase angle (θ) by the closing side (OP_6) of the vector polygon. From the geometry of the figure the reader should verify that the phase angle (θ) with respect to the OX axis is equal to $\frac{5\delta}{2}$, which is one-half of the phase difference

between the first and last component vibrations. Hence it follows that for n component vibrations the resultant vibration will be expressed by $x=OP_n \cos\left[\omega t+\left(\frac{n-1}{2}\right)\delta\right]$.

The case considered above has important application in diffraction phenomena, for example, in the determination of the intensity pattern of a diffraction grating, the function of the latter being to produce an equal phase difference in the beams transmitted through each successive pair of apertures.

The analytical treatment of the case of n vibrations of equal amplitude and constant phase difference follows as an extension of equation (26).

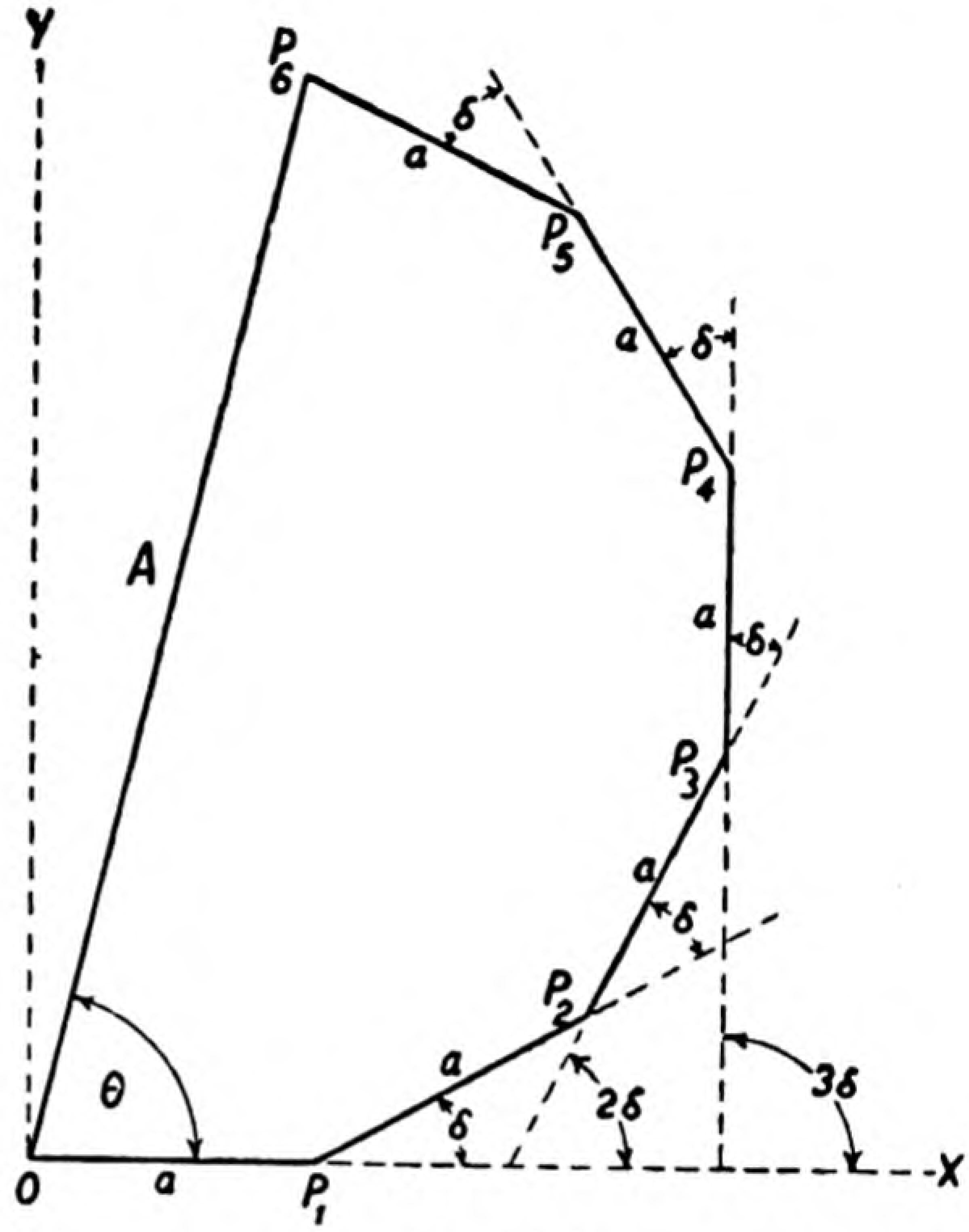


Fig. 2.14.

For $\sum_1^n a.\sin n\delta = a\{\sin\delta+\sin 2\delta+\ \ldots\ +\sin n\delta\}$*

* Applying formula $2\sin mx \sin nx=\cos(m-n)x-\cos(m+n)x$.

$$2\sin\delta\sin\delta/2=\cos\delta/2-\cos 3\delta/2$$
$$2\sin 2\delta\sin\delta/2=\cos 3\delta/2-\cos 5\delta/2$$
$$\cdots\cdots\cdots$$
$$2\sin(n-1)\delta\sin\delta/2=\cos\left(\frac{2n-3}{2}\right)\delta-\cos\left(\frac{2n-1}{2}\right)\delta$$
$$2\sin n\delta\sin\delta/2=\cos\left(\frac{2n-1}{2}\right)\delta-\cos\left(\frac{2n+1}{2}\right)\delta$$
$$2\sin\frac{\delta}{2}\sum_1^n \sin n\delta=\cos\delta/2-\cos\left(\frac{2n+1}{2}\right)\delta$$
$$=2\sin\left(\frac{n+1}{2}\right)\delta.\sin\left(\frac{n\delta}{2}\right)$$

Hence
$$\sum_1^n \sin n\delta=\frac{\sin\left(\frac{n\delta}{2}\right)}{\sin\delta/2}.\sin\left(\frac{n+1}{2}\right)\delta$$

Similarly, si $2\sin mx\cos nx=\sin(m+n)x+\sin(m-n)x$ it is easily shown that

$$2\sin\frac{\delta}{2}\sum_1^n \cos n\delta=\sin\left(\frac{2n+1}{2}\right)\delta-\sin\frac{\delta}{2}$$
$$=2\sin\left(\frac{n\delta}{2}\right).\cos\left(\frac{n+1}{2}\right)\delta$$

$$=a\frac{\sin\left(\frac{n\delta}{2}\right)}{\sin\delta/2}\,.\,\sin\left(\frac{n+1}{2}\right)\delta$$

and $$\sum_1^n a\,.\,\cos n\delta=a\frac{\sin\left(\frac{n\delta}{2}\right)}{\sin\delta/2}\,.\,\cos\left(\frac{n+1}{2}\right)\delta$$

Hence $$A^2=a^2\left[\frac{\sin\left(\frac{n\delta}{2}\right)}{\sin\delta/2}\right]^2.\left[\sin^2\left(\frac{n+1}{2}\right)\delta+\cos^2\left(\frac{n+1}{2}\right)\delta\right]$$

or $$A^2=a^2\left[\frac{\sin\left(\frac{n\delta}{2}\right)}{\sin\delta/2}\right]^2 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The phase difference between the first and last component vibration is $(n-1)\delta=2\theta$, but if n is very large this relation becomes $\delta=\frac{2\theta}{n}$. Therefore from (27) it follows that

$$A=a\frac{\sin\theta}{\sin\theta/n}\rightarrow na\frac{\sin\theta}{\theta} \quad . \quad (28)$$

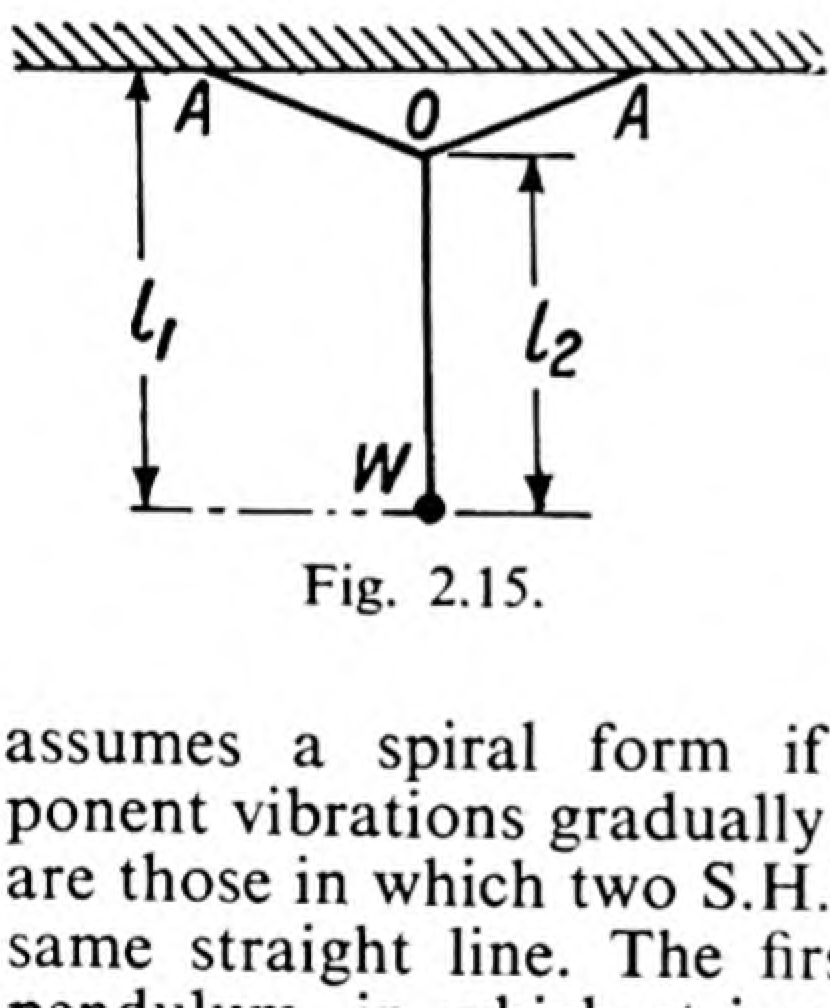


Fig. 2.15.

It is to be noted that when $\theta=m\pi$, where m is an integer, the value of A is zero, *i.e.* the polygon is completely closed. In the limiting case when the amplitude a and the phase difference δ are infinitesimal, n being very large, the vector polygon will become a continuous curve, in fact part of the circumscribing circle, closed by a chord. This curve assumes a spiral form if the successive amplitudes of the component vibrations gradually diminish in magnitude. The simplest cases are those in which two S.H.Ms. are acting (*a*) at right angles, (*b*) in the same straight line. The first type is demonstrated by the Blackburn pendulum, in which string in the form of a **Y** is attached to a beam (Fig. 2.15), and supports a weight W. The arrangement acts as a pendulum supported at AA and of length l_1 when swinging at right angles to the plane of the paper, and as a pendulum of length l_2 when swinging in the plane of the paper.

If the bob is moved out, say, at 30° to the plane of the paper, and to one side of the line WO and then released, the two motions occur simultaneously, and the bob traces out a figure which is more complex. The actual shape of these Lissajous figures, as they are termed, depends on the ratio $\frac{l_1}{l_2}$, and on the respective amplitudes of the two S.H.Ms., and also on the position of the bob at the particular instant of release. The latter governs the phase difference between the two S.H.Ms., and the effect is indicated in Fig. 2.16, which refers to two S.H.Ms. of

the same period and amplitude. Such figures are important in the comparison of frequencies by means of a cathode ray oscillograph (see Chapter 13). A simple demonstration of the figures formed with the apparatus of Fig. 2.15 is obtained by replacing the bob with a funnel containing sand; as the sand falls it is caught on a stationary tray, and reveals the figure described.

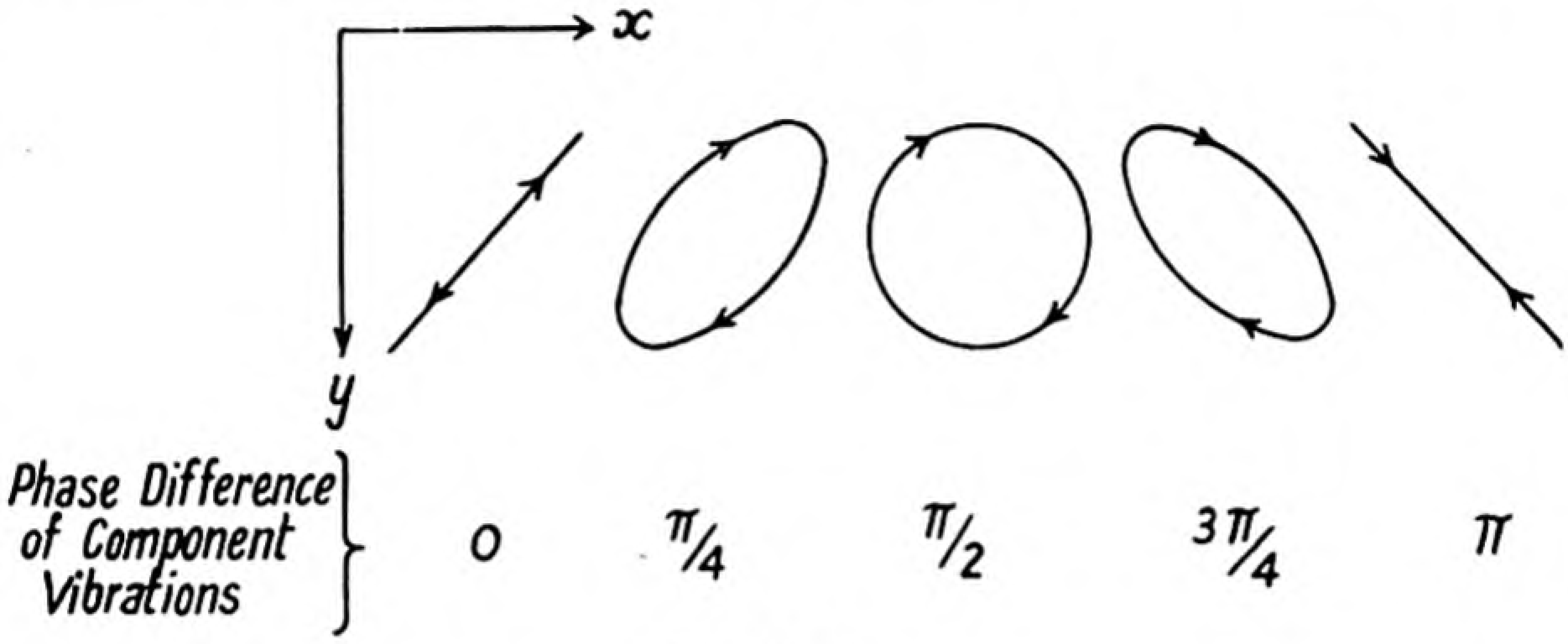


Fig. 2.16.

To trace a Lissajous figure in which the period of one S.H.M. is twice that of the other S.H.M. Draw a rectangle in which the sides are in proportion to the respective amplitudes OX, OY (Fig. 2.17). The period in the X direction is double that in the Y direction. At rest, the particle occupies the centre O of the rectangle; when moving in the S.H.M. of greater amplitude only, its path is along XX', and when in the other S.H.M. alone, the path is along YY'. Consider the particle to be released from the position A under the influence of the forces producing both S.H.M. As the period in the XX' direction is double that in YY' direction, the particle cuts the YY' axis when it reaches CF, *i.e.* it is at Y'. Similar reasoning shows that it continues until it reaches the point B on the path. In order to fix intermediate points, draw a semicircle on each side of the rectangle as diameter as shown in the diagram. Divide the larger arc into eight, and the smaller into four, equal parts, and project each part on to the corresponding diameter, and complete the diagram as shown. The times taken for the particle to traverse relevant portions of its respective diameters are equal, from the definition of S.H.M., hence, by joining the appropriate points of intersection of the horizontals and verticals by a smooth curve, the required path is obtained. If the particle is projected from G in the direction of GA, the Lissajous figure resembles a figure 8. This should be checked by the reader, and other figures drawn commencing at K and M. Fig. 2.18 shows figures in which the ratio of the periods is 3 : 1.

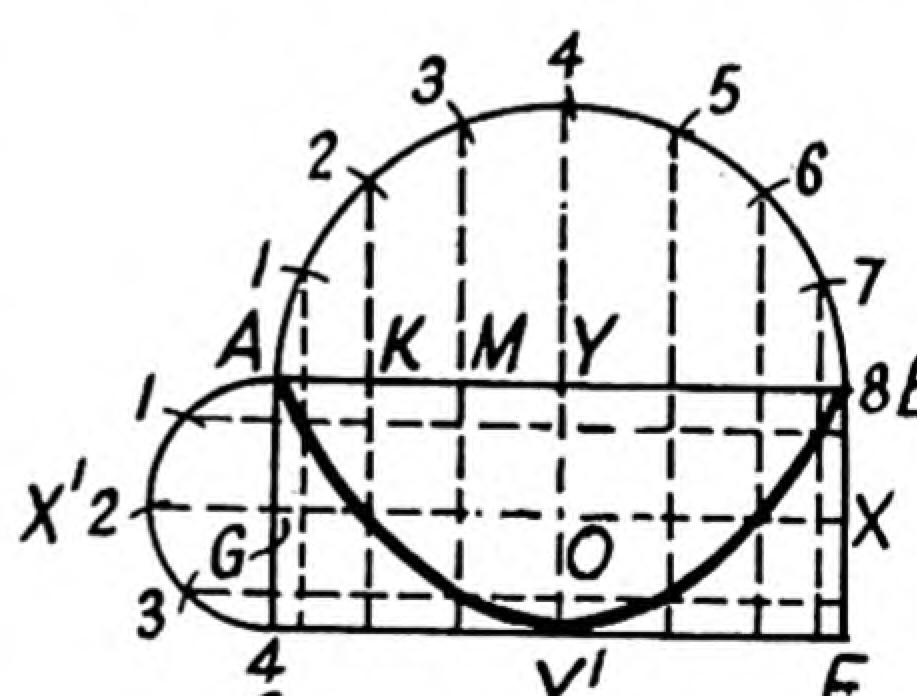
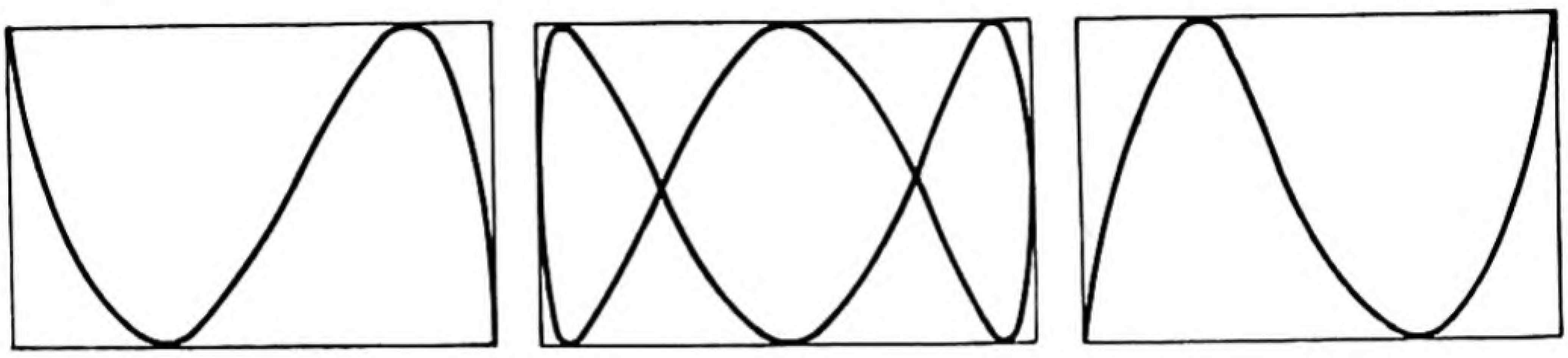


Fig. 2.17.

Two S.H.Ms. acting in the same direction. The second type of superposed S.H.M. may be demonstrated by attaching a short pendulum with a light wooden bob m to the bottom of a long one with a massive lead bob M (Fig. 2.19). When the two are swinging in the same plane the lower bob moves with two S.H.Ms. Each S.H.M. is drawn as a displacement-time curve to the same scale (Fig. 2.20 *a*



(*a*) (*b*) (*c*)

Fig. 2.18.

and *b*), and the displacements are added algebraically (Fig. 2.20*c*). The resulting curve is complicated, but repeats itself at regular intervals, *i.e.* it has a definite periodicity or frequency of repetition.

Composition of two S.H.Ms. at right angles to each other—analytical treatment. The rotating vector OP of Fig. 2.5 is equivalent to two component perpendicular projections on the OX and OY axes, and as P moves round in a circle these projections move "to" and "fro" in S.H.M. along their respective axes. These component vibrations have equal amplitudes but differ in phase by $\pi/2$ (cf. Fig. 2.16), for 1(*c*) may be written as $y=a \sin \omega t = a \cos(\pi/2-\omega t)=a \cos\left(\omega t-\frac{\pi}{2}\right)$, indicating that the phase of y lags behind that of x by $\pi/2$. The converse of the above is also true, viz. that two mutually perpendicular S.H.Ms. of equal amplitude having a phase difference of $\pi/2$, are equivalent to a uniform circular motion, the radius of the circle being equal to the amplitude of the S.H.Ms. It should be noted that the motion of P is essentially a planar one, as distinct from the one-dimensional motion of the S.H.Ms. previously considered.

Fig. 2.19.

The more general case where the component vibrations have different amplitudes will now be considered. Let these be $x=a \cos \omega t$ and $y=b \cos(\omega t+\delta)$, where δ is the phase *advance* of y with respect to x, time being counted from the position of the maximum value of the x displacement.

Now
$$\frac{y}{b}=\cos(\omega t+\delta)$$
$$=\cos \omega t \cos \delta-(1-\cos^2 \omega t)^{\frac{1}{2}} \sin \delta$$
$$=\frac{x}{a}\cos \delta-\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}} \sin \delta$$

$$\therefore \quad \left(\frac{y}{b}-\frac{x}{a}\cos\delta\right)^2=\left(1-\frac{x^2}{a^2}\right)\sin^2\delta \quad . \quad . \quad . \quad (29)$$

which on expanding and simplifying gives

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\delta+\frac{y^2}{b^2}=\sin^2\delta \quad . \quad . \quad . \quad . \quad (30)$$

This is an equation of the second degree, and is the equation of an ellipse whose form varies with the amplitudes a and b and the phase difference δ, as tabulated below. When

(1) $\delta=0$, the curve becomes two coincident straight lines, $y=\frac{b}{a}x$, passing through the origin and inclined to the x axis at an angle $\tan^{-1}b/a$;

(2) $\delta=\pi/2$ or $3\pi/2$, the curve is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is an equation of an ellipse of semi-axes, a and b, coincident with the x and y axes respectively. The orbital path degenerates into a circle when $a=b$, the origin being at the centre, and

(3) $\delta=\pi$, the curve is again two coincident straight lines, $y=-\frac{b}{a}x$, passing through the origin but now inclined to the axis at an angle $\tan^{-1}-b/a$.

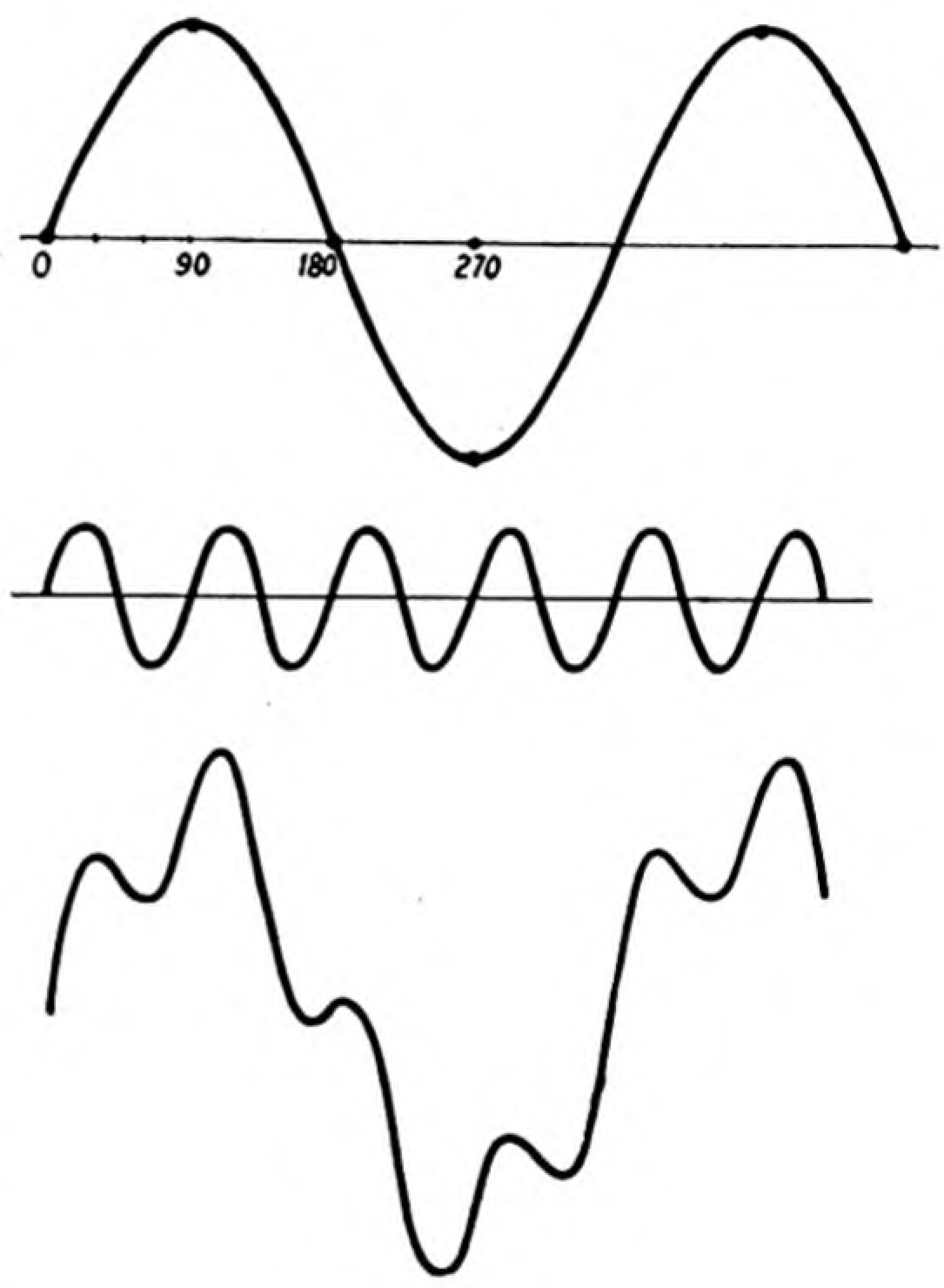


Fig. 2.20.

The above paths will remain quite stationary if the two frequencies are exactly equal and constant; if, however, there is a small constant difference of frequency the form of the path will slowly change. Since this frequency difference will have the effect of gradually changing the phase angle δ, it follows that the path will assume sequentially the various forms under (1), (2), and (3) above, and the cycle will be carried out at a frequency equal to the difference of the frequencies of the component vibrations. This effect provides the means for the measurement of small frequency differences, by timing the rate at which the cycle is performed when comparing frequencies by a method involving Lissajous curves, *e.g.* cathode-ray oscillograph (p. 257).

If the ratio of the frequencies is 1 : 2, *i.e.* $x=a\cos\omega t$ and $y=b\cos(2\omega t+\delta)$ represent the component vibrations, then the resultant path of the particle, on elimination of t between the above relations, may be shown in general to be a curve having two loops. Two

particular values of δ will be taken, as the solution of the general case is very laborious.

(1) $\delta=0$.

Here
$$x=a\cos\omega t \quad \ldots\ldots \quad (31)$$
$$y=b\cos 2\omega t \quad \ldots\ldots \quad (32)$$

From (32) it follows that $\cos^2\omega t=\frac{1}{2}(1+y/b)$. Hence combining this relation with (31) the result obtained, on eliminating t, is

$$x^2=\frac{a^2}{2b}(b+y) \quad \ldots\ldots \quad (33)$$

Similarly for $\delta=\pi$ it is easily shown that

$$x^2=\frac{a^2}{2b}(b-y) \quad \ldots\ldots \quad (34)$$

Equations (33) and (34) represent parabolas, the one being the inversion of the other; one such curve is shown in Fig. 2.17, only in this case the frequency of x is double that of y, so the equation would be $y^2=\frac{b^2}{2a}(a+x)$.

(2) $\delta=\pi/2$.

Here
$$x=a\cos\omega t \quad \ldots\ldots \quad (35)$$
$$y=b\cos(2\omega t+\pi/2)=-b\sin 2\omega t \quad . \quad (36)$$

whence it is easily shown that

$$\frac{y^2}{b^2}=\sin^2 2\omega t=4\sin^2\omega t\cos^2\omega t=4\frac{x^2}{a^2}\left(1-\frac{x^2}{a^2}\right)$$

or rearranging
$$4\frac{x^2}{a^2}\left(\frac{x^2}{a^2}-1\right)+\frac{y^2}{b^2}=0 \quad \ldots\ldots \quad (37)$$

The path represented by this equation is a symmetrical figure of 8.

In employing Lissajous figures to determine the ratio of two frequencies it is important to be able to recognise readily the frequency ratio from the pattern observed. If this ratio r is a whole number then the number of loops in the figure will also be r. This fact is not necessarily observable, as when the pattern assumes a stationary position giving a line as in Fig. 2.17, but if either of the frequencies can be altered very slightly the figure will commence to turn and the loops will be rendered quite evident.

When r is not a simple ratio let $x=a\cos\omega_1 t$ and $y=b\cos(\omega_2 t+\delta)$ represent the two S.H.Ms. where $r=\omega_1/\omega_2$. Then the resultant Lissajous figure will be inscribed in a rectangle of sides $2a$ and $2b$, and therefore in a given time τ each of the extreme values of x will be attained $\tau\div 2\pi/\omega_1=\omega_1\tau/2\pi$ times and those of y, $\omega_2\tau/2\pi$ times. Hence the ratio of the number of traverses in the x and y directions is given by $\omega_1/\omega_2=r$, the ratio of the two frequencies. This result provides an obvious means of finding r by counting the ratio of the number of intersections of the pattern on two lines parallel to the x and y axes respectively. In this way the reader should verify that Fig. 2.18 shows Lissajous figures for a frequency ratio 3 : 1.

Pure tones and overtones. It will be shown later (p. 85) that the sine curves shown in Fig. 2.21 resemble the vibrations produced in air by tuning-forks, which have practically *pure tones*, whereas those of a violin are more complicated (thick line in Fig. 2.21). It is the superposition of *overtones* (tones of higher frequency) on the fundamental note that gives an instrument its characteristic tones and quality; the fundamental determines the pitch of the instrument, and has the largest

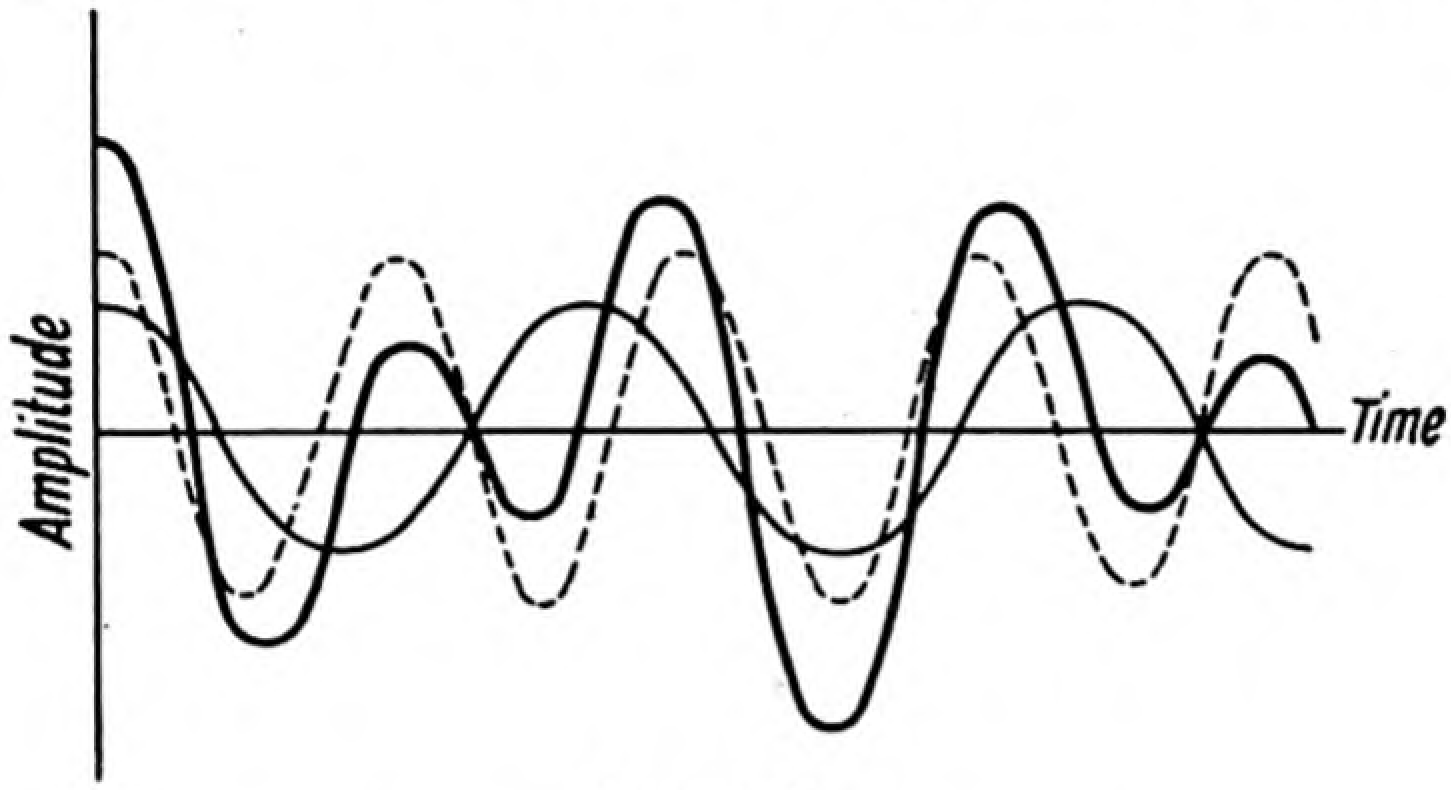


Fig. 2.21.

amplitude usually, whereas the overtones are of smaller amplitude. The same remarks apply to the human voice. Overtones also affect articulation, *i.e.* the ease with which spoken words are heard. This is discussed in Chapter 14.

It is possible to simulate various musical sounds by taking tuning-forks of the appropriate frequency and striking them simultaneously, the lowest with greater force than the others to accentuate the fundamental.

For further reading

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CHAPTER 3

WAVE PROPAGATION

The wave-equation

If a particle is part of an infinite medium which is subjected to a periodic force applied at a point near the particle, the latter will be displaced from its position of rest. Its displacement at any instant will depend on its position relative to the direction of propagation of the disturbance in the medium. As a result of the impulses transmitted by neighbouring particles situated nearer to the origin of the force, each particle will execute a simple harmonic motion in the direction in which the disturbance travels. Consequently the velocity of propagation, v, will determine the time taken, $\frac{x}{v}$, for the disturbance to reach a point P situated a distance x from the origin, *i.e.* it will be the factor which determines the actual displacement y of the particle at P *at any given instant*. Since the disturbance is a simple harmonic motion and is continuous with time, then the motion of every individual particle will be repeated after an interval of time, T sec. say, and in this manner a wave is propagated through the medium. The interval T is the period of the motion and is equal to $\frac{1}{n}$, where n is the frequency of the source which may also be expressed as $\frac{\omega}{2\pi}$, ω being known as the pulsatance (p. 14). The shortest distance between two particles whose displacements, at any given instant, are similar is known as the *wave-length* λ of the periodic motion; *i.e.* it equals the distance travelled by the wave during the time T taken for the disturbing force to make one complete vibration or cycle;

$$\therefore\ \lambda = vT = \frac{v}{n}.$$

A simple harmonic wave is represented by an equation of the type $y=a \sin \theta = a \sin \omega t$ (equation 1(c)). Such a wave is reproduced in Fig. 3.1. Now

$$\begin{aligned}\omega t &= \frac{2\pi}{T} t\\ &= \frac{2\pi v t}{\lambda}\\ &= \frac{2\pi x}{\lambda}\end{aligned}$$

$$\therefore\ y = a \sin \frac{2\pi}{\lambda}(x) \qquad \qquad (1)$$

When x is a multiple of $\frac{\lambda}{2}$, $y=0$, and the curve cuts the axis; these multiple values are set out above the X axis in Fig. 3.1. This

equation gives the actual *contour*, at a particular instant, of a transverse wave, for example, travelling along a string in which the actual motion of the particles is at right angles to the length of the string, *i.e.* perpendicular to the direction in which the wave is propagated. Alternatively, as already noted (p. 23), it may represent the instantaneous longitudinal displacements of the particles of a medium through which

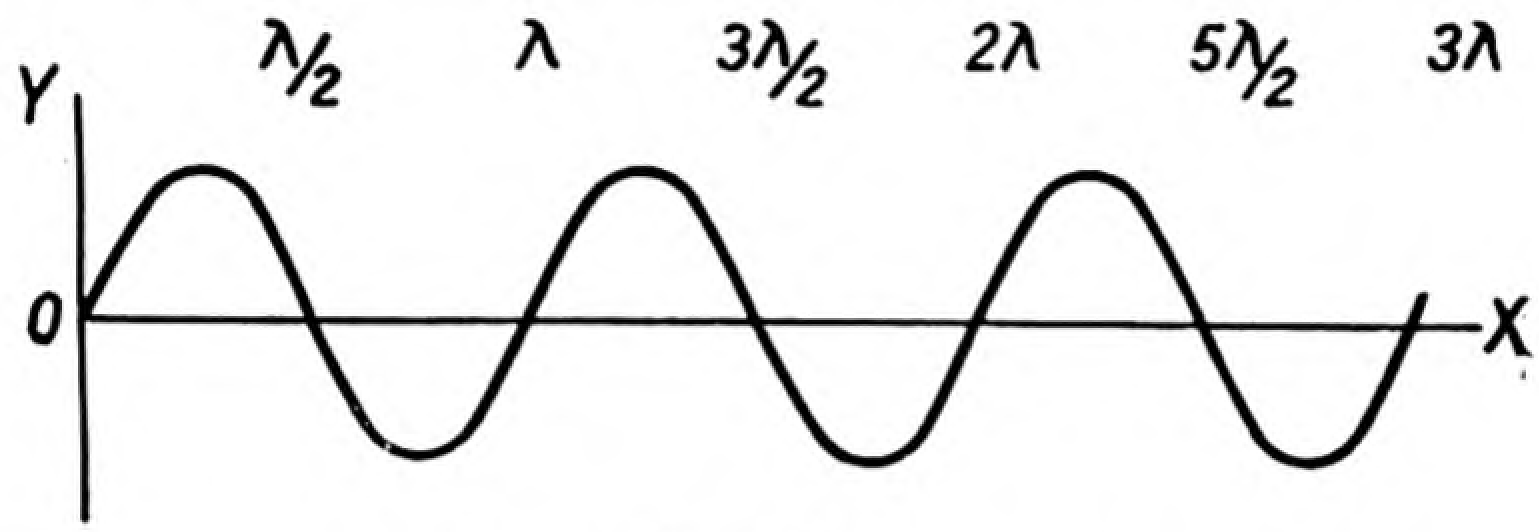


Fig. 3.1.

a longitudinal progressive wave is travelling. Either type of displacement may be represented, therefore, by the equation $y=f(x)$, denoting that the displacement is a *function* of the position x of the particle, but it does not specify the motion completely as the effect of time is not included.

If the outline of the wave remains constant in form as it progresses through the medium it is said to be of "constant type." Sound waves and light waves propagated through a vacuum conform to this designation. Hence the contour of the wave may still be represented by the same equation, $y=f(x)$, at any other time t provided that the old origin, from which x was initially measured, moves forward with the velocity v of the waves. In order to incorporate the time factor, consider the displacement of a particle situated at x in the path of a *train* of waves, *i.e.* a *group* of successive waves proceeding from the

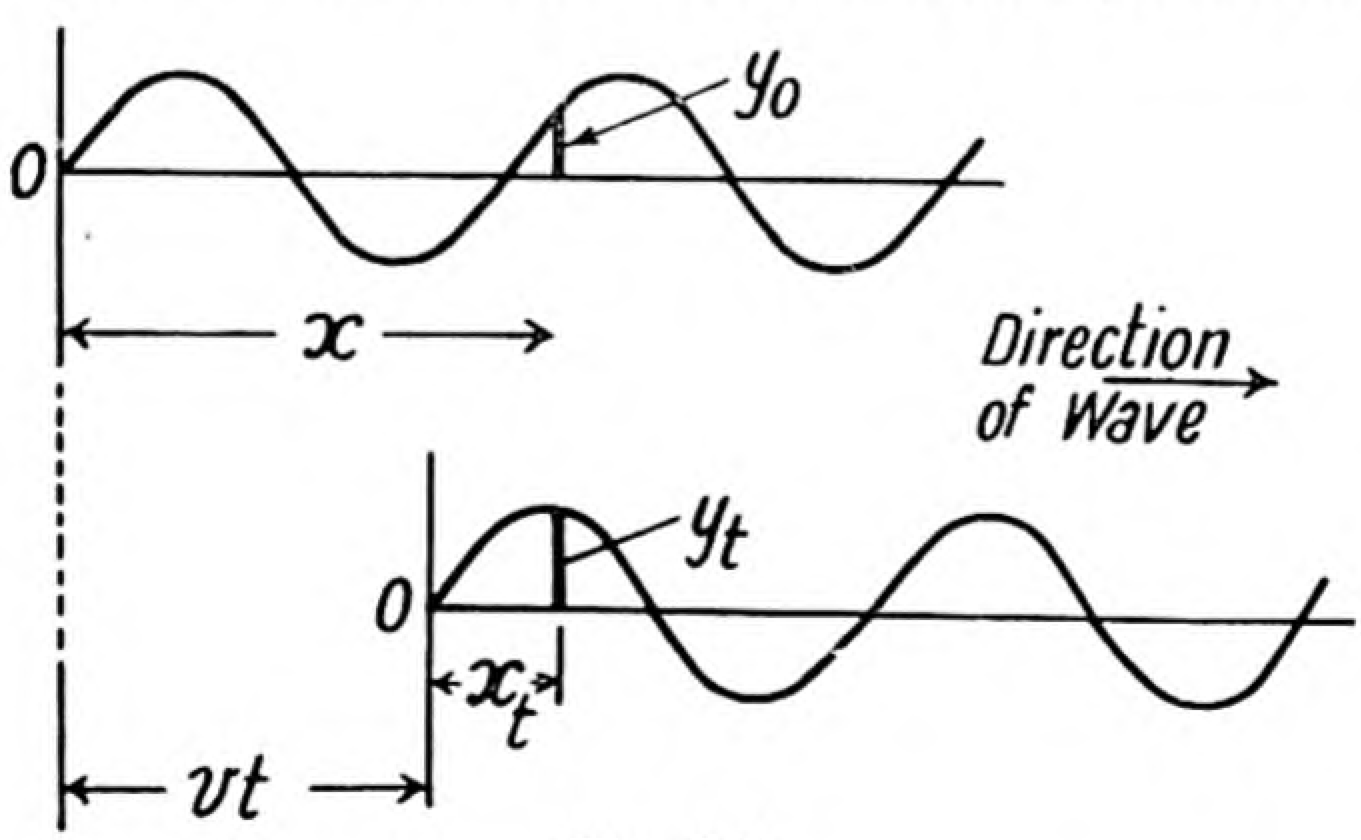


Fig. 3.2.

same source. Two positions of this train are shown in Fig. 3.2, and they differ in time by an interval denoted by t. The displacement of the particle under consideration at the respective instants will be given by

$$y_0=f(x) \quad \text{and} \quad y_t=f(x_t) \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

where the distance (x) is measured from the same point O *in the wave*

system. Referred to the old origin the equation $y_t = f(x_t)$ will be given by $y_t = f(x - vt)$, since O has moved through a distance vt.

Hence $$y = f(x - vt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1b)$$

is the general equation of a wave of constant type propagated in the positive direction of x with a velocity v, and it gives the displacement y at any time t in terms of the velocity of the wave and of the distance x of the particular particle from the *original* position of the origin. For a wave propagated in the negative direction of x, v is negative and the corresponding equation is therefore

$$y = f(x + vt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1c)$$

A particular and important case of equation (1*b*) is

$$y = a \sin \frac{2\pi}{\lambda}(x - vt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Alternative forms of the equation are:

(*a*) Substituting $n\lambda$ for v,

$$y = a \sin 2\pi\left(\frac{x}{\lambda} - nt\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2a)$$

(*b*) Substituting n for $\frac{1}{T}$, T being the period of vibration,

$$y = a \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2b)$$

(*c*) Substituting κ for $\frac{1}{\lambda}$,

$$y = a \sin 2\pi(\kappa x - nt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2c)$$

κ is known as the wave number, *i.e.* it is the number of waves per unit distance. It is often used in optics to express the frequency of spectrum lines.

(*d*) Substituting $\frac{2\pi}{T} = \omega$,

$$y = a \sin \frac{2\pi}{T}\left(\frac{xT}{\lambda} - t\right)$$

$$= a \sin \omega(cx - t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2d)$$

Observations:

(1) There are three variables in the equation, the dependent variable, the displacement y, and the independent variables time t and distance x of the particle from the origin.

(2) If t is fixed at a particular instant t_0, the equation becomes

$$y = a \sin 2\pi\left(\frac{x}{\lambda} - \frac{t_0}{T}\right)$$

$$= a \sin\left(\frac{2\pi x}{\lambda} - \alpha\right)$$

in which α is constant. A graph of this expression corresponds to

that of Fig. 3.2, the position of the ordinate being governed by the magnitude of α. It gives the disposition of all the particles at the n stant t_0.

(3) If x is fixed, say, at x_0, then

$$y=a \sin 2\pi\left(\frac{x_0}{\lambda}-\frac{t}{T}\right)$$

$$=a \sin\left(\beta-2\pi\frac{t}{T}\right)$$

gives the displacement of the particular particle situated at a point distan x_0 from the origin. This expression gives the history of the motion of this particle, and its graph is similar to Fig. 2.3b. In cases (2) and (3) α and β are epoch angles.

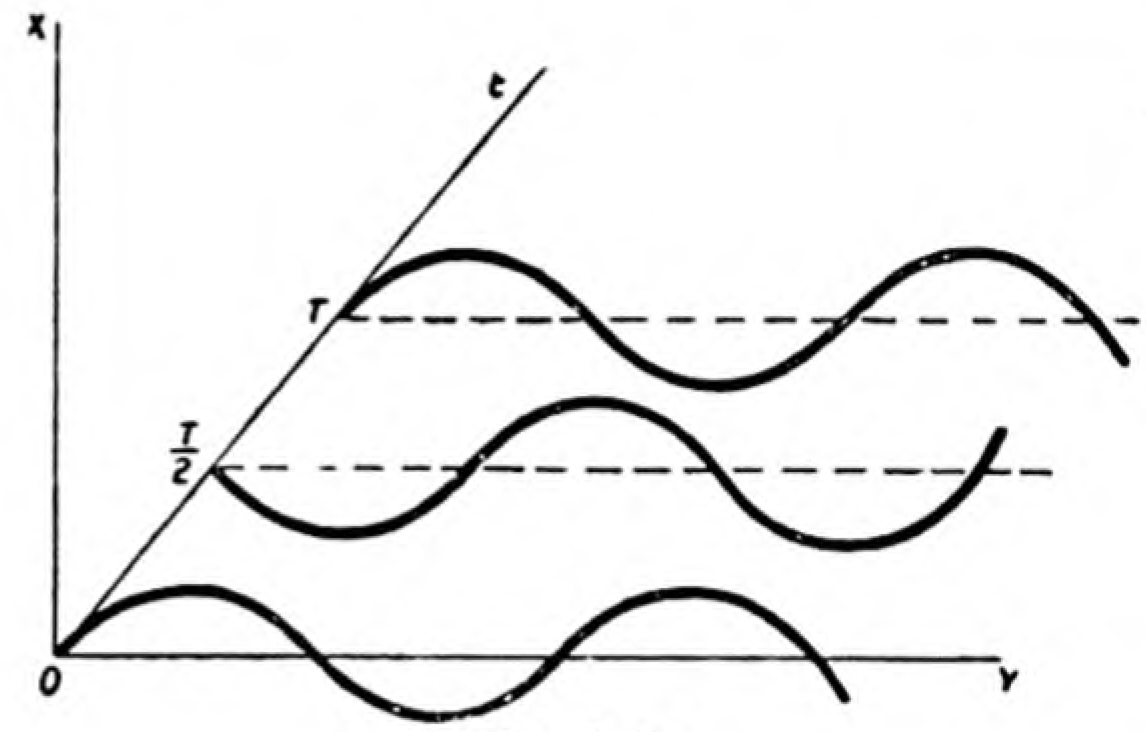


Fig. 3.3a.

(4) The two graphs are termed displacement curves and may be plotted in three dimensions, in which the three axes are at right angles, as the three edges at the corner of a box. To draw these in two dimensions the third axis is inclined (Fig. 3.3a). Suitable scales are chosen and the graph for time $t=0$ is drawn on the XY axes.* Repeating this for $t=\frac{T}{2}$, *i.e.* half a period later, the graph, which is an inversion of the first, is drawn as shown. Further graphs for

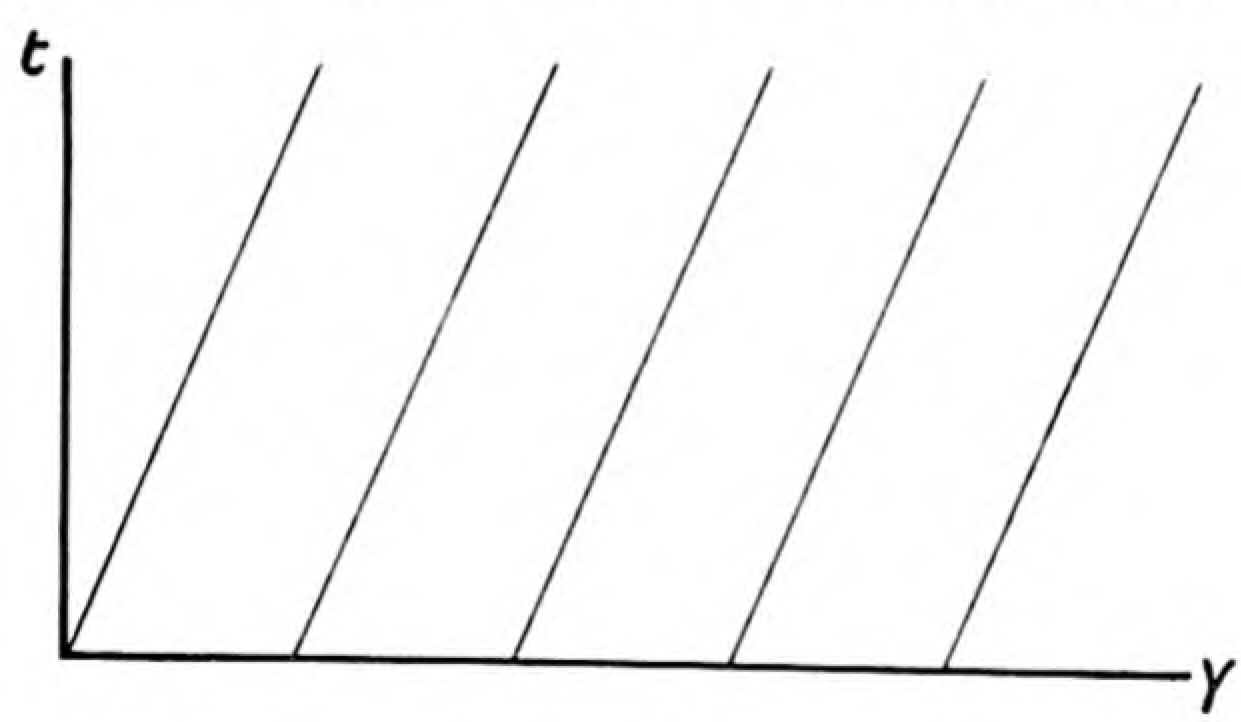


Fig. 3.3b.

$t=T$, etc., are drawn, and the result is that, whereas a two-dimensional graph is a line, this three-dimensional graph is a surface which resembles a sheet of corrugated iron lying on a plane parallel to the $Y-t$ plane, the directions of the corrugations being inclined to the t axis as shown in the plan (Fig. 3.3b).

The *general* expression for wave motion is from equation (1b)

$$y=f(x-vt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

y, being the displacement of the *particle*,

* In Figs. 3.3 (a) and (b), x denotes the displacement and y the particle position.

x, the distance of the mean position of the particle from some arbitrary origin,

t, the time which has elapsed, and

v, the velocity of the *waves*.

To find the velocity of the *particle*, differentiate the above expression with respect to time, then

$$\dot{y}=\frac{\partial y}{\partial t}=-v.f'(x-vt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As the particle velocity is variable, the value obtained is the instantaneous velocity. The *particle acceleration* is

$$\ddot{y}=\frac{\partial^2 y}{\partial t^2}=v^2.f''(x-vt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Differentiating with respect to x,

$$\frac{\partial y}{\partial x}=f'(x-vt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{\partial^2 y}{\partial x^2}=f''(x-vt) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting (6) in (4) and (7) in (5) give

$$\frac{\partial y}{\partial t}=-v\frac{\partial y}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$\frac{\partial^2 y}{\partial t^2}=v^2.\frac{\partial^2 y}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

respectively, two important results, particularly the latter.

These results should be checked by using the particular equation

$$y=a \sin \frac{2\pi}{\lambda}(x-vt).$$

The significance of (9) will be seen in the following section.

Formulae for the velocity of sound

The velocity of sound in a uniform homogeneous rod. Fig. 3.4 represents a portion of a rod, the section at A being distant x from some arbitrary origin in the rod.

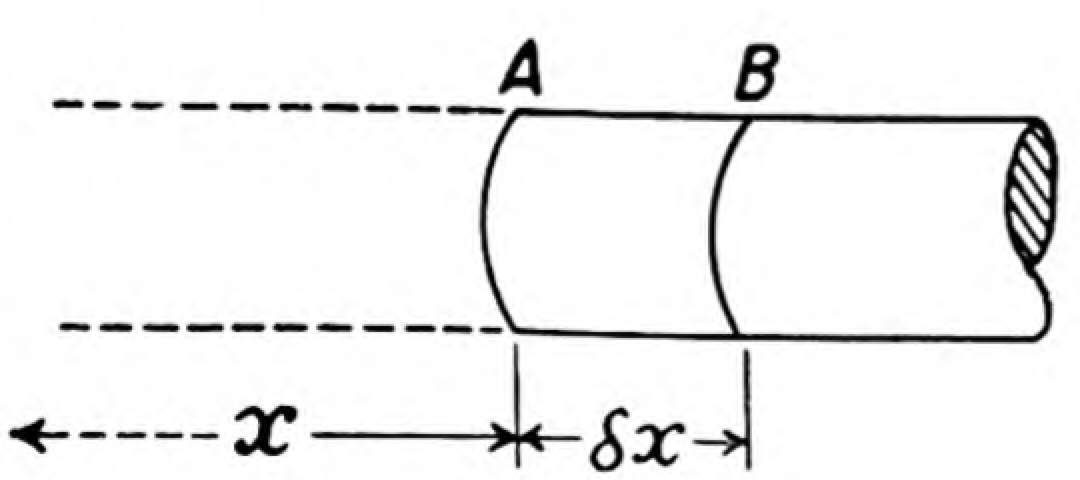


Fig. 3.4.

Assume the section at A to suffer a displacement η, in the x direction at a particular instant when the rod is in a state of *longitudinal* vibration, and that the section at B, which is very near to A, is displaced by an amount $\eta+\delta\eta$ at the same instant. The whole rod is strained and it is necessary to express this in terms of small lengths such as AB, because if A and B were taken to be an integral number of wave-lengths apart, the displacement of the section A would be equal to that at section B, and no strain would be apparent. By taking small lengths this is

eliminated. The need for this is further emphasised by reference to the graphs in Fig. 3.5. Both graphs show the displacement of each point along the rod due to an applied stress, but whereas (*a*) relates to a steady stress, (*b*) deals with a periodic stress which produces compressional waves in the rod. Clearly, the strain in the first case is constant at all points, but in the second case the strain varies with the alternating stress, and at any point will be measured by the tangent of curve (*b*) at that point, which is expressed by $\frac{\partial\eta}{\partial x}$.

The strain at A, then, at any instant is $\frac{\partial\eta}{\partial x}$, and the *rate* of change of strain with distance is $\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial x}\right)=\frac{\partial^2\eta}{\partial x^2}$. In the distance AB, *i.e.* δx, the *actual* change of strain is $\frac{\partial^2\eta}{\partial x^2}.\,\delta x$, so that the actual strain at the section B is $\frac{\partial\eta}{\partial x}+\frac{\partial^2\eta}{\partial x^2}.\,\delta x$. The elastic modulus called into play by a simple longitudinal pull is Young's modulus,* E, so the stress at the section A, assuming Hooke's Law is obeyed, is equal to $E\times\text{strain}=E.\frac{\partial\eta}{\partial x}$, and that at $B=E\left(\frac{\partial\eta}{\partial x}+\frac{\partial^2\eta}{\partial x^2}.\,\delta x\right)$. The difference between these stresses is $E.\frac{\partial^2\eta}{\partial x^2}.\,\delta x$, and thus the resultant *force* acting over the whole sectional area S of the rod AB is $E.\frac{\partial^2\eta}{\partial x^2}.\,\delta x.S$. This force causes

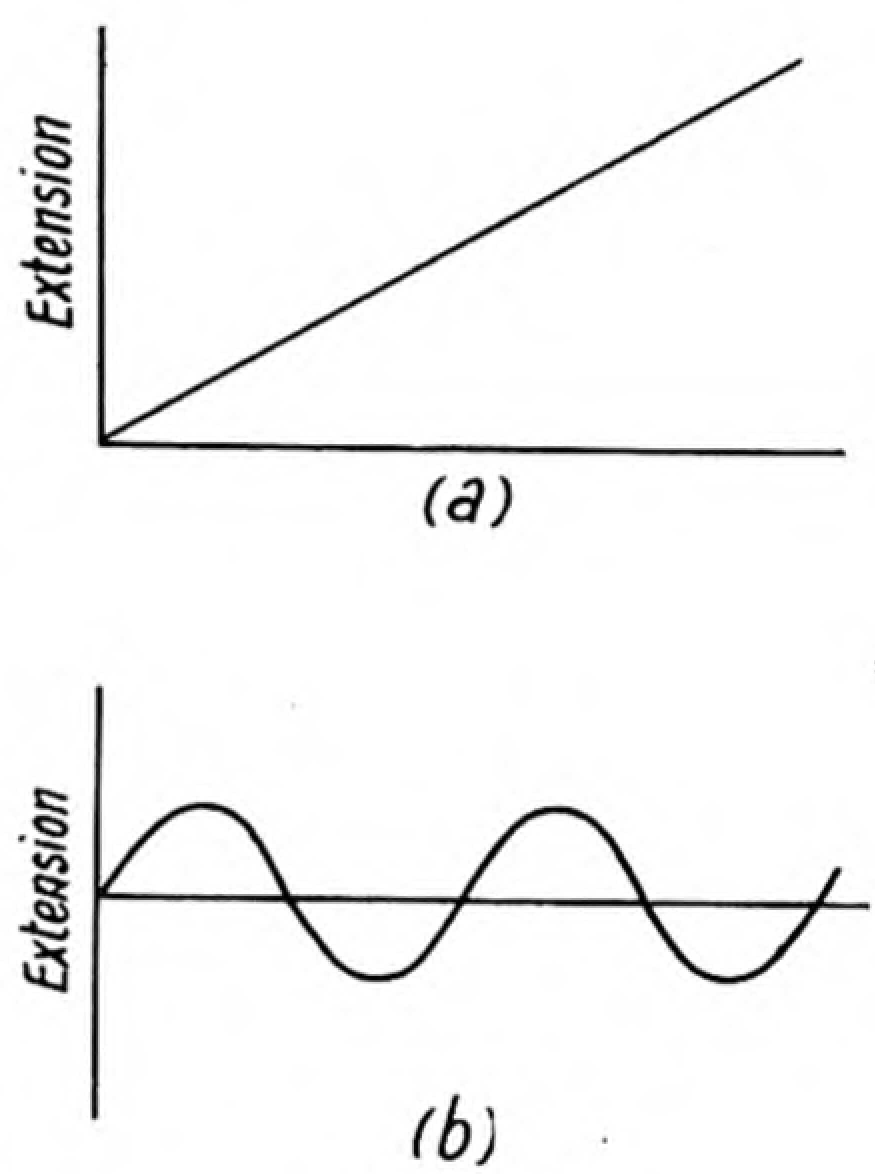


Fig. 3.5.

an acceleration of the mass AB, which is represented by $\frac{\partial^2\eta}{\partial t^2}$, so that $E.\frac{\partial^2\eta}{\partial x^2}.\,\delta x.S=(S\delta x\rho)\frac{\partial^2\eta}{\partial t^2}$, in which ρ stands for the density of the rod, whence

$$\frac{\partial^2\eta}{\partial t^2}=\frac{E}{\rho}.\frac{\partial^2\eta}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Since this equation must be dimensionally correct it follows that $\frac{E}{\rho}$ has the dimensions of (velocity)^2, which is verified by reference to equation (9) which states that

$$\frac{\partial^2\eta}{\partial t^2}=v^2.\frac{\partial^2\eta}{\partial x^2},$$

$$\therefore\ v^2=\frac{E}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

* See Chap. 5.

where v now stands for the velocity of longitudinal elastic waves in the material, hence: The velocity of sound in a solid rod is proportional to the square root of Young's modulus for the material of the rod, and is inversely proportional to the square root of its density.

A method of finding Young's modulus for substances which are difficult or impossible to procure in the form of a wire, *e.g.* wood or glass, is at once suggested, for these substances are readily obtained as rods for which the value of v is obtained by the dust tube method devised by Kundt (p. 137). It should be mentioned, however, that the modulus obtained under these essentially adiabatic conditions may differ slightly from that obtained under the isothermal conditions of normal static experiments.

The velocity of sound in a gas. The same argument holds as for a solid, the gas being considered to be contained in a cylindrical pipe. Young's modulus, however, does not apply, for the lateral dimensions of the gas remain constant. The adiabatic bulk modulus* replaces Young's modulus, and is equal to γP, γ being the ratio (C_P/C_V) of the principal specific heats C_P and C_V of a gas, and P the absolute pressure of the gas.

Originally Newton assumed that the compressions and rarefactions to which the gas was subjected while transmitting sound waves were isothermal changes, and hence Boyle's law held. The isothermal elasticity is calculated by assuming that an increase of pressure p produces in a given mass of gas of volume V a *contraction* equal to δV. If P stands for the initial pressure, then

$$\begin{aligned} PV &= (P+p)(V-\delta V) \\ &= PV+pV-P\delta V-p\delta V, \end{aligned}$$

so that $P\delta V=p(V-\delta V)$. Now δV is of the order $10^{-5}V$, and so $p\delta V$ may be neglected, hence

$$P=p\,.\frac{V}{\delta V}=\frac{p}{\delta V/V}$$

where p is the stress which produces the strain $\dfrac{\delta V}{V}$,

i.e. $$\text{Isothermal bulk modulus}=\frac{\text{stress}}{\text{volume strain}}=P \quad . \quad . \quad (12)$$

The values obtained for the velocity of sound from the formula $v=\sqrt{\dfrac{P}{\rho}}$ were considerably lower than the actual values, and it was later pointed out by Laplace that the elasticity involved was the adiabatic bulk modulus, for, owing to the rapidity with which rarefactions succeed compressions in a gas transmitting sound, the changes could not be isothermal. Substituting γP for q in equation (11)

$$v=\sqrt{\frac{\gamma P}{\rho}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

* See Chap. 5.

Results calculated from this expression are in agreement with those obtained experimentally. To show that the elasticity$=\gamma P$, write $PV^{\gamma}=(P+p)(V-\delta V)^{\gamma}$, expand the last bracket to three terms, and proceed as before.

By using a modified Kundt's apparatus to measure the velocity of sound in a gas available only in small quantities, it is possible to utilise the above formula (13) to calculate γ. This has been done for the rare gases.

Effect of temperature on the velocity of sound in a gas. The gas equation may be written

$$\frac{P_1V_1}{T_1}=\frac{P_0V_0}{T_0}$$

for m gm. of a gas, T_1 signifying absolute temperature and V_0 the volume occupied by the gas at standard temperature (T_0) and pressure (P_0). Dividing by m, the equation becomes finally $\frac{P_1}{(T_1\rho_1)}=\frac{P_0}{(T_0\rho_0)}$; and substituting for $\frac{P_0}{\rho_0}$ and $\frac{P_1}{\rho_1}$ from equation (13);

$$\frac{v_1{}^2}{T_1\gamma}=\frac{v_0{}^2}{T_0\gamma}$$

or

$$\frac{v_1}{\sqrt{T_1}}=\frac{v_0}{\sqrt{T_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Thus, the velocity of sound in a gas which obeys the gas equation is proportional to the square root of its absolute temperature.

In air at 0° C., $v_0=331{\cdot}5$ m. per sec., and by calculation $v_{100}=387{\cdot}4$ m. per sec. at 100° C. This agrees closely with the experimental values, but it should be pointed out that γ decreases slightly with increase of temperature, and that the relationship should be used for moderate differences of temperature only. The value of γ for all gases lies between $\frac{5}{3}$ and unity, $\frac{5}{3}$ being the theoretical value for monatomic gases, and $\frac{7}{5}$ for diatomic gases.

Writing equation (14) in terms of the centigrade scale,

$$\frac{v_1}{v_0}=\sqrt{\frac{(273+t)}{273}}=\left(1+\frac{t}{273}\right)^{\frac{1}{2}}=1+\frac{t}{546} \text{ very nearly, if } t \text{ is small;}$$

$\therefore\ v_1=v_0+\frac{v_0 \,.\, t}{546}$, or, for an increase of $t°$ C. the velocity at 0° C. increases by $\frac{331{\cdot}5t}{546}$ m. per sec.$=\cdot 61t$ m. per sec., a result which has been chiefly of academic interest until the recent interest in flight at supersonic speeds, *i.e.* at speeds greater than that of sound. At sea-level in our climate the speed of sound is about 760 m.p.h., but the air becomes colder and less dense with height and at 30,000 ft. is calculated to be 660 m.p.h.

Effect of pressure. The density of a gas is proportional to its pressure, hence $v=\sqrt{\frac{\gamma P}{\rho}}$ shows that velocity is independent of pressure.

Properties of displacement curves

As previously stated, the graphs which show the relationship between y and t for a particular particle and that between y and x at a particular time as expressed by the equation $y=a \sin \frac{2\pi}{\lambda}(x-vt)$ are known as displacement curves.

Let the angle $\frac{2\pi}{\lambda}(x-vt)$ be denoted by θ; increasing x by $\pm\lambda$ increases θ by $\pm 2\pi$, but $\sin(\theta \pm 2\pi)$ is equal to $\sin\theta$, hence the value of y is the same at the points $x+\lambda$ and $x-\lambda$ as at the point x, hence λ is the distance apart of successive points in the wave which are in phase, and is therefore the wave-length.

The slopes of the displacement curves are given by differentiating the wave equation with respect to t and x respectively:—

$$\frac{\partial y}{\partial t}=-\left[a \cos \frac{2\pi}{\lambda}(x-vt)\right]\frac{2\pi v}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\frac{\partial y}{\partial x}=+\left[a \cos \frac{2\pi}{\lambda}(x-vt)\right]\frac{2\pi}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

these results confirm (8).

$\frac{\partial y}{\partial t}$ is the particle-velocity, so it follows that its value at any point is proportional to the slope of the displacement-time curve at that point. The horizontal portions of the y-t curve (Fig. 3.6) occur at the crests and troughs, so these are times of zero particle-velocity. At the points where it cuts the t axis, the curve has its maximum slope, and these are points of maximum particle-velocity. The rate of change of slope $=\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)=\frac{\partial^2 y}{\partial t^2}$, which is the acceleration. This is clearly zero at the points of maximum particle-velocity, and maximum at the points of zero particle-velocity, *i.e.* at the maximum points of the displacement-time curve. This is shown in the particle velocity-time curve which is reproduced below the y-t curve. Notice that the curves are 90° out of phase.

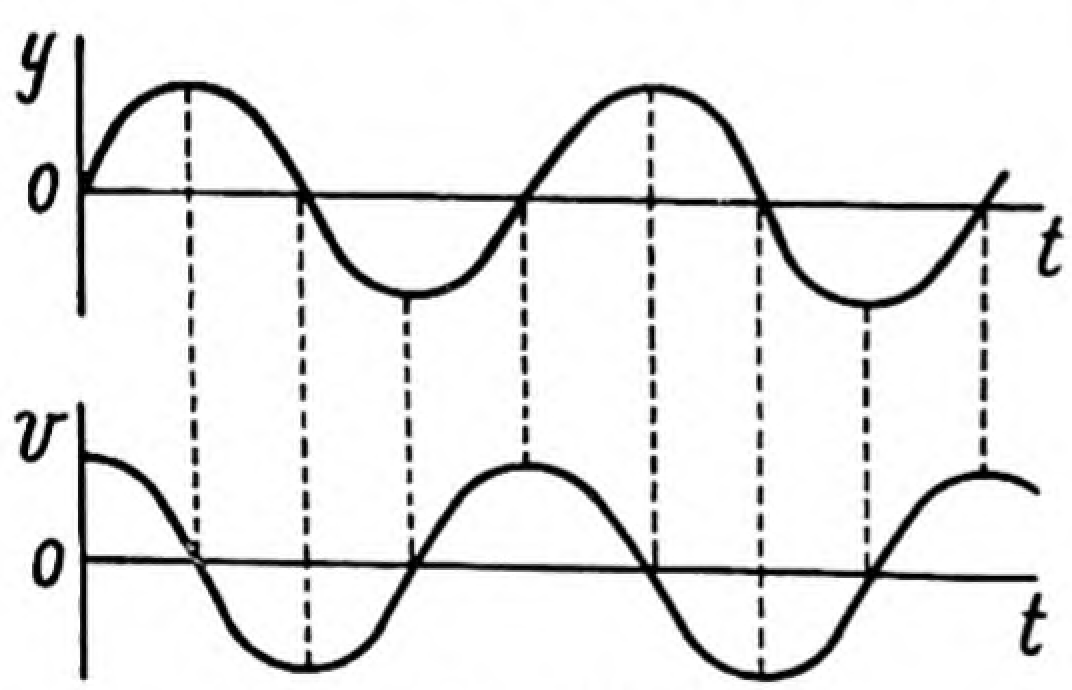


Fig. 3.6.

Application of these conclusions to the simple pendulum is helpful: when it is moving through its lowest point its velocity is maximum and its acceleration zero, but at its maximum displacement it has zero velocity, but the acceleration is maximum, which is correct, for in S.H.M. acceleration is proportional to displacement.

Let the displacement-distance or y-x curve in Fig. 3.7 represent a transverse wave train at a particular time, travelling with a velocity v in the positive direction, and the dotted curve the wave train after a short interval of time.

As $\frac{\partial y}{\partial t} = -v.\frac{\partial y}{\partial x}$ (from eqn. 8), when $\frac{\partial y}{\partial x}$ is negative, $\frac{\partial y}{\partial t}$ is positive, which means that the particles are moving upward on the forward side of each wave (the right-hand slopes in the diagram), and when $\frac{\partial y}{\partial x}$ is positive, *i.e.* on the left-hand slopes, $\frac{\partial y}{\partial t}$ is negative, *i.e.* the particle-velocity is downward.

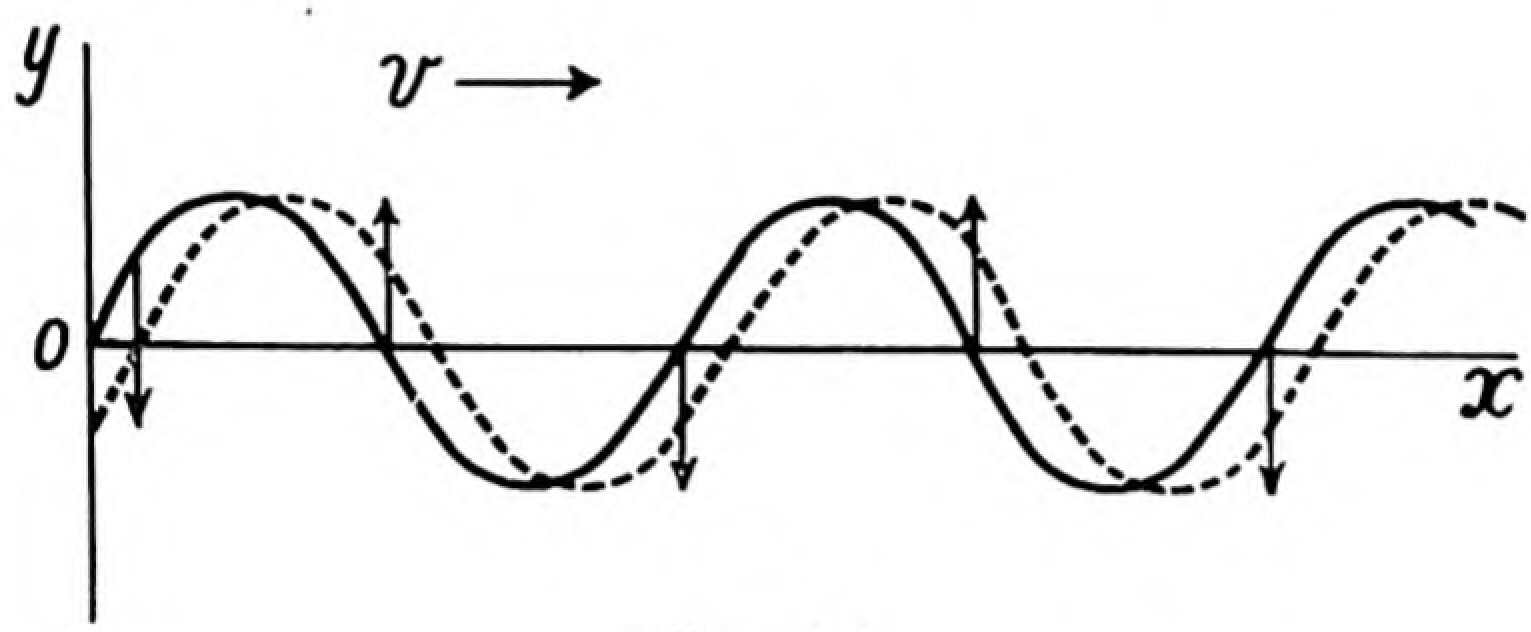


Fig. 3.7.

Now consider the curve to represent a train of sound waves in air; the ordinates now stand for the *longitudinal* displacement of the particles at a particular instant.

As before, when $\frac{\partial y}{\partial x}$ is negative, $\frac{\partial y}{\partial t}$ is positive, and the particles are moving in the positive direction, *i.e.* in the direction of the wave, when $\frac{\partial y}{\partial x}$ is positive, the particles are moving to the left. To determine

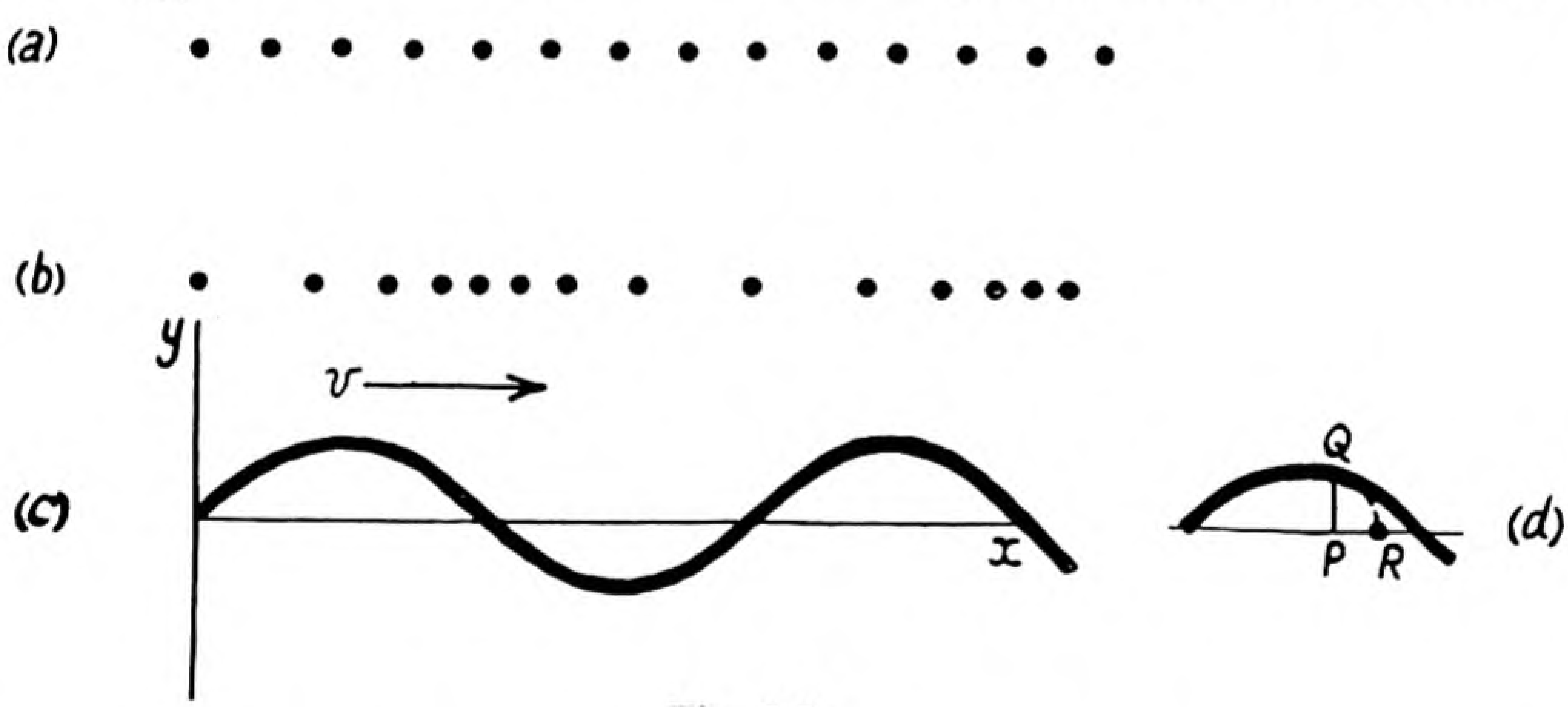


Fig. 3.8.

the actual position of the particle the ordinate *PQ* (Fig. 3.8*d*) at a particular point *P* is set off to the right of the equilibrium position if above the abscissa, *i.e.* to *PR*, and vice versa, a reversal of the procedure indicated on p. 23. Proceeding in this way the particles whose equilibrium positions are shown as dots in Fig. 3.8*a* actually occupy the positions as in (*b*). It follows that in a compression the particles are moving in the same direction as the waves, and in a rarefaction, in the opposite direction.

Energy of a particle moving in S.H.M.

As previously, let $y=a \sin \omega(cx-t)$ represent the S.H.M.; then if m is the mass of the particle its Kinetic Energy (K.E.) at any instant is given by

$$\tfrac{1}{2}m\dot{y}^2=\tfrac{1}{2}ma^2\omega^2 \cos^2 \omega(cx-t) \quad . \quad . \quad . \quad . \quad (17)$$

The maximum value of the K.E. is therefore

$$\text{K.E.}_{\text{(max.)}}=\tfrac{1}{2}ma^2\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

But since no energy is assumed to be dissipated by friction, etc., it follows from the conservation of mechanical energy that the total energy of the system at any time $=\tfrac{1}{2}ma^2\omega^2$, since P.E. is zero when K.E. is a maximum, and vice versa.

$$\therefore \text{ K.E.}+\text{P.E. (Potential Energy)}=\tfrac{1}{2}ma^2\omega^2 \quad . \quad . \quad (19)$$

Hence the P.E. of the particle at any instant

$$=\tfrac{1}{2}ma^2\omega^2-\text{K.E.}=\tfrac{1}{2}ma^2\omega^2[1-\cos^2 \omega(cx-t)]$$

$$=\tfrac{1}{2}ma^2\omega^2 \sin^2 \omega(cx-t) \quad . \quad . \quad . \quad . \quad . \quad (20)$$

But the expression (17) may be written as

$$\text{K.E.}=\tfrac{1}{2}ma^2\omega^2 \cos^2 \omega(cx-t)=\tfrac{1}{4}ma^2\omega^2(1+\cos 2\omega t) \quad . \quad (21)$$

The average value of $(1+\cos 2\omega t)$ over a *complete* cycle is unity.

Hence $\quad \text{K.E.}_{\text{(mean)}}=\tfrac{1}{4}ma^2\omega^2$

$$=\text{P.E.}_{\text{(mean)}} \quad \text{from (19).}$$

The total energy of the particle may also be expressed in terms of the frequency (n) of vibration, viz.

$$2\pi^2ma^2n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The above results signify that for a particle vibrating in S.H.M.

(*a*) The mean K.E. of the particle is equal to one-half of its maximum K.E.

(*b*) The total energy of the particle is on the average half kinetic and half potential.

(*c*) At a given frequency the energy of the particle is proportional to the square of its amplitude of motion.

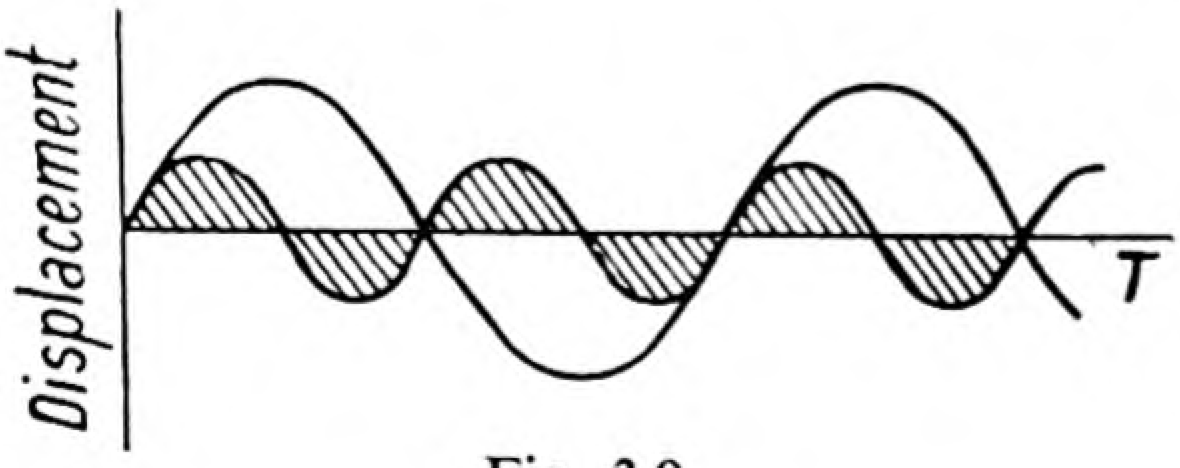


Fig. 3.9.

(*d*) If the frequency is variable, then for the energy of the particle to remain constant, a^2n^2 must not alter, *i.e.* the amplitude must vary *inversely* as the frequency (or directly as the period). The graph in Fig. 3.9 depicts two such vibrations of equal energy, the shaded curve referring to a vibration of double the frequency but half the amplitude of the other curve.

The above analysis may be extended to determine the energy per unit volume of a medium through which a plane wave is passing.

If ρ is the density of the medium, then from (19) it follows that the *energy density*, *i.e.* the total energy contained in unit length of the wave per unit area of the wave-front is given by

$$E_d=\tfrac{1}{2}\rho a^2\omega^2$$
$$=2\pi^2\rho a^2n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The *rate* at which energy flows through the medium will be equal to the energy contained in a cylinder of volume$=c$ times unit area of the wave-front, where c is the velocity of the wave.

Hence the energy flow per unit area of wave-front per unit time

$$=E_f$$
$$=\tfrac{1}{2}\rho ca^2\omega^2$$
$$=2\pi^2\rho ca^2n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

E_f is taken as a measure of the intensity of a sound wave (this is not, however, the same as the "loudness," see Chap. 10).

Obviously $\qquad E_f=c.E_d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$

E_f is sometimes termed *energy flux*.

CHAPTER 4

TRANSVERSE VIBRATIONS

Vibration of stretched strings

A sonometer or monochord consists of a uniform wire which passes over two fixed bridges B_1 and B_2 (Fig. 4.1), and is under tension which can be varied as required. A third bridge B which is slightly higher than the others is placed between them and is movable to

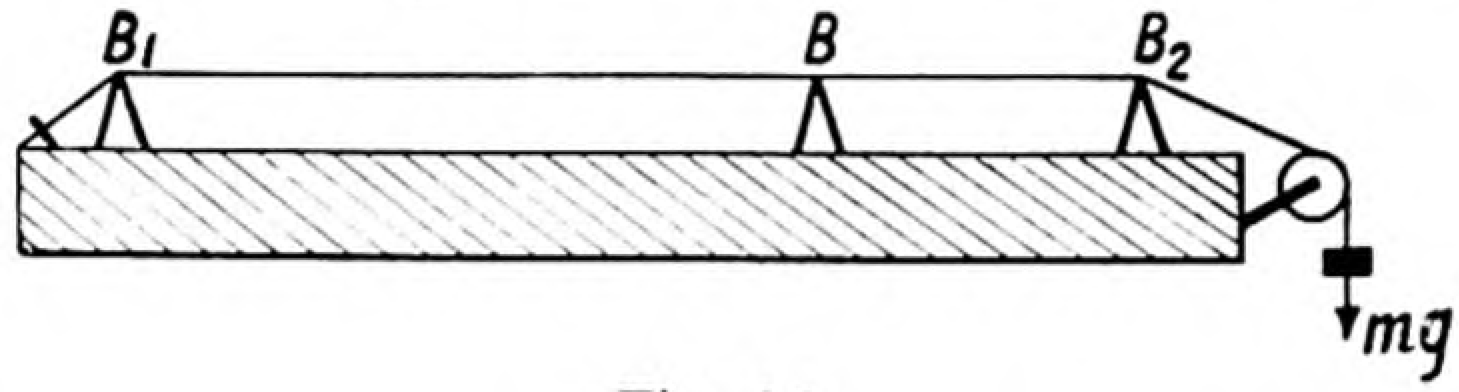


Fig. 4.1.

allow the vibrating portion to be altered in length. Sonometers are usually tuned by adjusting the tension of the wire so that the frequency of transverse vibrations is lower than that of a standard—usually a tuning-fork—and then moving the adjustable bridge until beats (p. 77) are heard. Continued adjustment makes the beats diminish in frequency and eventually they become imperceptible. The vibrating portion of the wire is then in tune with the fork. This is readily verified by placing a small paper rider astride the middle of the wire and putting the stem of the vibrating fork into contact with the movable bridge:

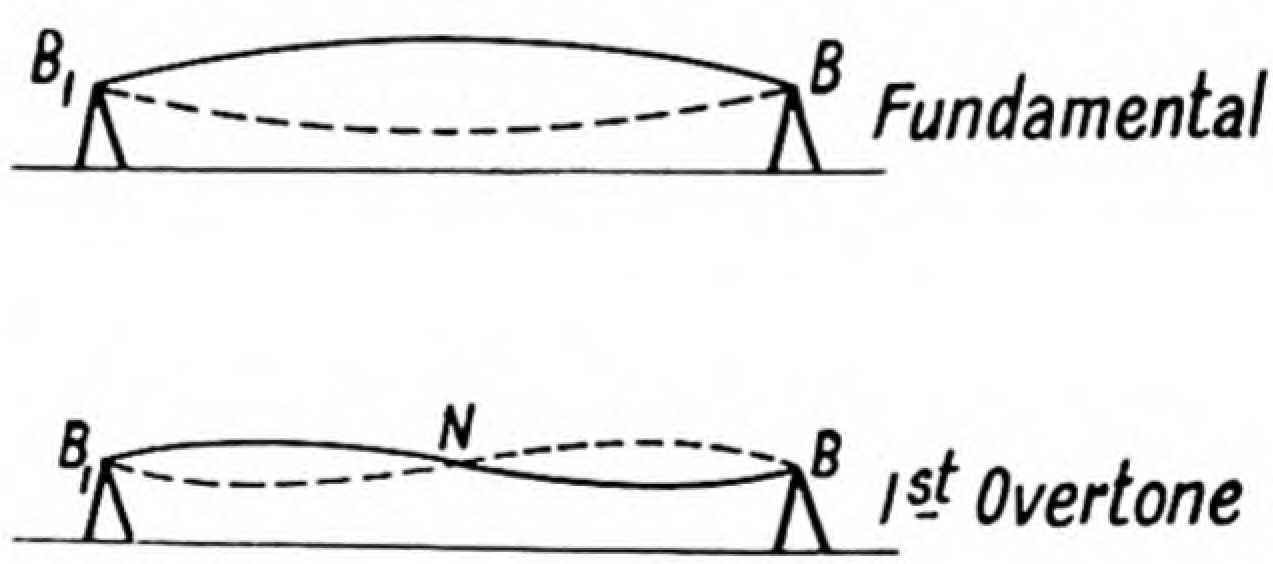


Fig. 4.2.

the wire is set into vibration by resonance through the motion of the bridge, and this agitates the rider, which may be thrown off.

An interesting experiment is as follows. A sonometer is tuned to a frequency of, say, 256 c.p.s., using a tuning-fork in the manner described above, and the setting tested by means of a paper rider. The rider is replaced and the test repeated with a tuning-fork of double the frequency of the first. The rider remains undisturbed. Two more riders are added, one midway between each end and the middle, and the vibrating fork again applied. The added riders are thrown off, but the original one remains. This means that the string

vibrates in two parts (Fig. 4.2) when the octave is elicited, and that the middle is a node; the ends of the parts, or loops, are nodes also. The bridge B is moved to coincide with the central node; the portion B_1B of the wire now responds to the octave, but *not* to the original fork. The octave and other overtones are usually present in a musical note, and are responsible for the particular quality by which the source of a note of a given pitch may be identified.

This experiment indicates one of the laws of stretched strings,* viz. that for a given tension, the frequency of transverse vibrations of an ideal string is inversely proportional to its length between two successive nodes. In symbols,

$$n \propto \frac{1}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{1st law.}$$

If the vibrating length of a wire is adjusted to vibrate in a single loop under a known tension of, say, 4 lb.-wt., the frequency being 256 c.p.s., the octave is elicited by increasing the tension to equal 16 lb.-wt., *i.e.* the frequency is doubled if the tension T is quadrupled. In general,

$$n \propto \sqrt{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{2nd law.}$$

Lastly, it can be shown that the frequency of a string of definite length and tension is decreased by increasing its mass per unit length, m, for

$$n \propto \frac{1}{\sqrt{m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{3rd law.}$$

The bass wires of a pianoforte are loaded by winding a copper wire closely on to each steel wire. This device is preferable to using thicker wires (reduction in tension of an ordinary wire being impracticable) as the mass per unit length is thus increased without sacrificing flexibility, and it has the added advantage of reducing non-harmonic vibrations. Alternatively, the frequency can be lowered by "overstringing" an upright piano. This consists in running the bass strings across the treble (Fig. 4.3), and allows longer strings to be used. The resulting tone resembles that of a grand piano, and such a piano is frequently termed an "upright grand."

The three laws are combined in the expression

$$n = \frac{1}{2l}\sqrt{\frac{T}{m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which n is the frequency in cycles per second, l the length in centimetres of the vibrating portion of the wire between two successive nodes, T the tension in dynes, and m the mass per unit length of the wire expressed in grams per centimetre. The reader should verify the form of the relationship by the method of dimensions. This formula may be proved conveniently in two stages. Firstly, the velocity V of propagation of a wave along a string is shown to be given by

$$V = \sqrt{\frac{T}{m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

* Usually known as Mersenne's Laws.

and then this is combined with the expression $V=n\lambda$ to give

$$n=\frac{1}{\lambda}\sqrt{\frac{T}{m}}=\frac{1}{2l}\sqrt{\frac{T}{m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

λ being the wave-length of the transverse vibrations.

This formula is strictly applicable only to an ideal string, *i.e.* one which is perfectly flexible, is perfectly uniform and does not suffer appreciable changes in length whilst vibrating. In a perfectly flexible string the stiffness is considered negligible, but when this condition does not hold a formula such as that due to Donkin should be employed, viz.:

$$n=\frac{N}{2l}\sqrt{\frac{T}{m}}\left\{1+\frac{\pi^2 N^2 r^4 E}{8l^2 T}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

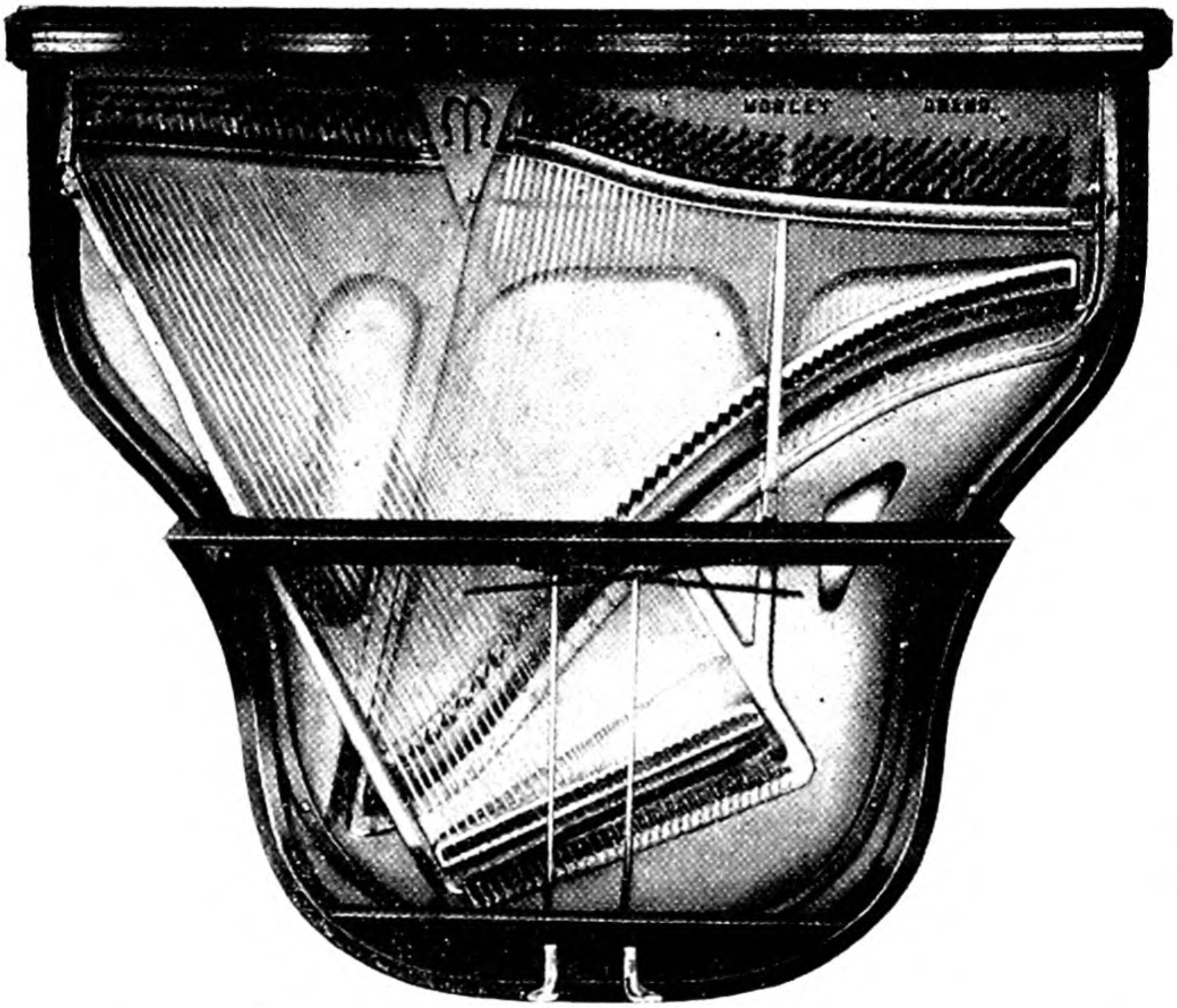


Fig. 4.3.
Piano frame—note the "overstrung" bass strings set in N.W. direction.

In this formula r is the radius of the circular section of a wire of total length l, and E is Young's modulus of elasticity of the wire. N assumes a value of unity for the fundamental and it is evident that the correction factor for the stiffness becomes increasingly important with the higher harmonics of the wire.

Before proceeding with the proof, it is essential that the reader should appreciate the fact that the wave which travels along a stretched string is reflected from the end, and that the outgoing waves are superimposed on reflected waves to form the standing wave system

of a vibrating wire. This may be difficult to visualise, but by considering the motion of a single pulse on a stretched string first, and then Melde's experiment, the problem should be clear.

Consider a rope AB in which a single transverse wave W is moving to the right (Fig. 4.4). Each particle of the rope repeats the motion of the particle to the left of it, *for in an undamped progressive wave each particle executes a S.H.M. of the same amplitude and period but in different phase.* Particles one wave-length apart will differ in phase by 2π radians. This wave propagation is conveniently demonstrated by means of a uniform rope 1 cm. in diameter and 15 m. long, secured

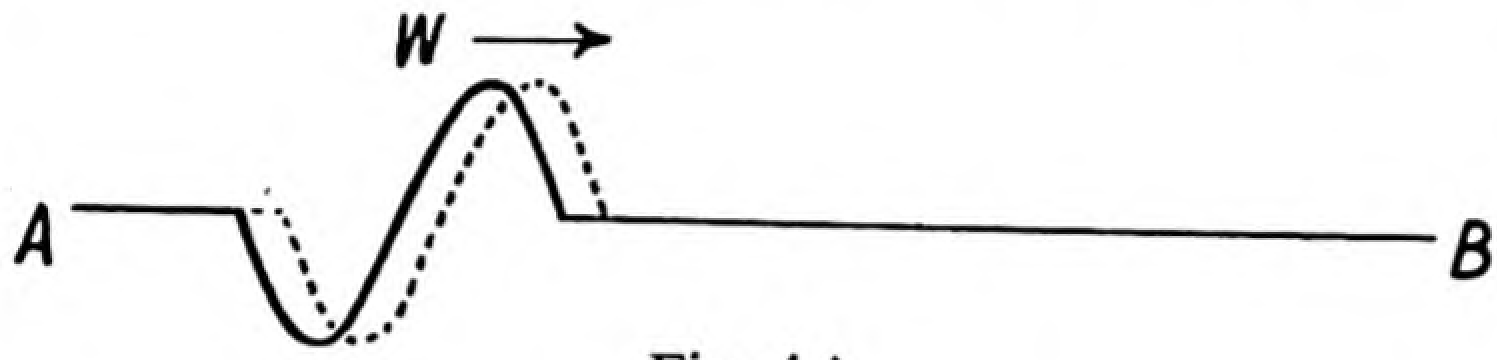


Fig. 4.4.

at one end to a wall and passing over a pulley near the free end. To this end is attached a load of about 4 kg. to maintain a uniform tension. If the horizontal portion of the rope is struck near to the pulley, a single wave-crest will be observed to move along the rope to the fixed end, from which it will be reflected, with amplitude reversed. Usually such a wave traverses the rope several times before becoming imperceptible, and by timing with a stop-watch it is easy to obtain the velocity. To verify experimentally the expression $V=\sqrt{\frac{T}{m}}$ for the rope, several values of V should be obtained with different loads.

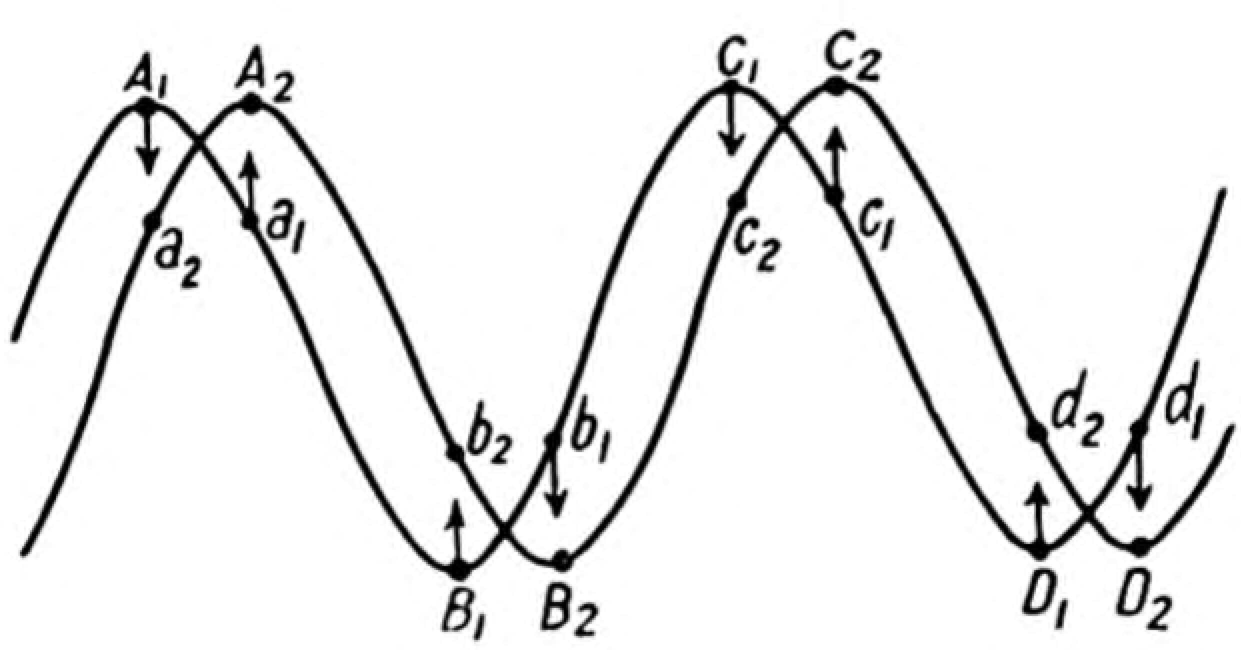


Fig. 4.5.

As m is constant a graph of V^2 against T should give a straight line, and from its slope m can be calculated. This can be checked by direct weighing.

Melde's experiment and standing waves

Melde's experiment resembles the one just described, but the periodic motion is maintained by an electric tuning-fork of between 20 and 50 c.p.s. to which the end of the string is attached. The load is gradually increased until resonance is obtained, which occurs when the string vibrates in a series of loops. The difference between the

experiment described above and Melde's experiment is that the wave-crest is caused to pass along the rope in the former, whereas in the latter *each* to and fro movement of the fork is responsible for sending

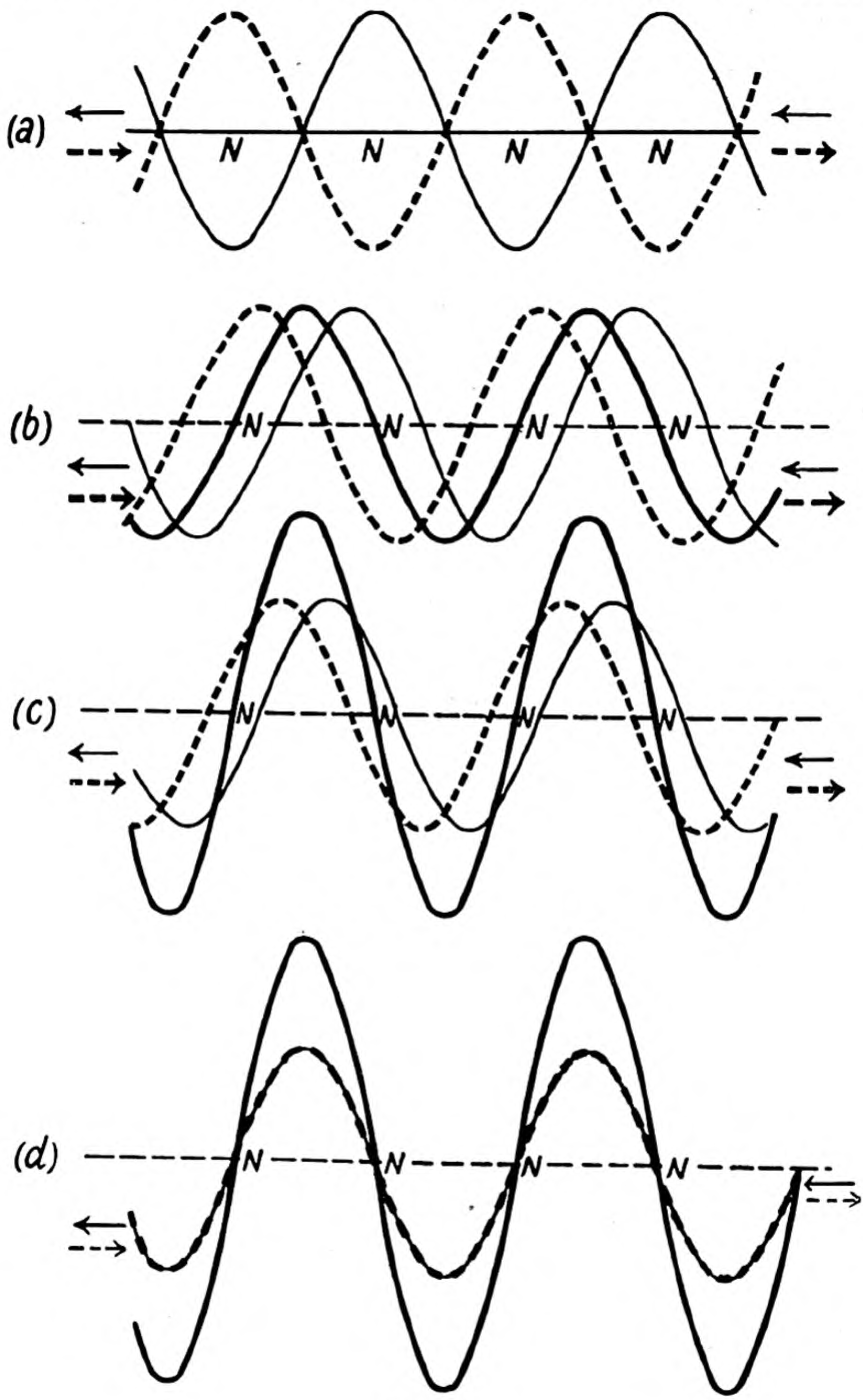


Fig. 4.6.

out a wave-crest. Each crest is reflected back from the pulley, and the returning system fits in with the outgoing system when resonance occurs, so that there may be other positions along the string besides

that at the pulley where the displacement is always zero. This superposition of similar waves moving in opposite directions causes *a standing wave system which is characterised by the presence of nodes*, and is more fully discussed on p. 106. A further increase in tension causes the string to be rearranged into a smaller number of loops, and finally there is one loop only. Now each loop ends in a portion of the string which is at rest, and therefore is a node; the point at which the string is attached to the fork, however, suffers a

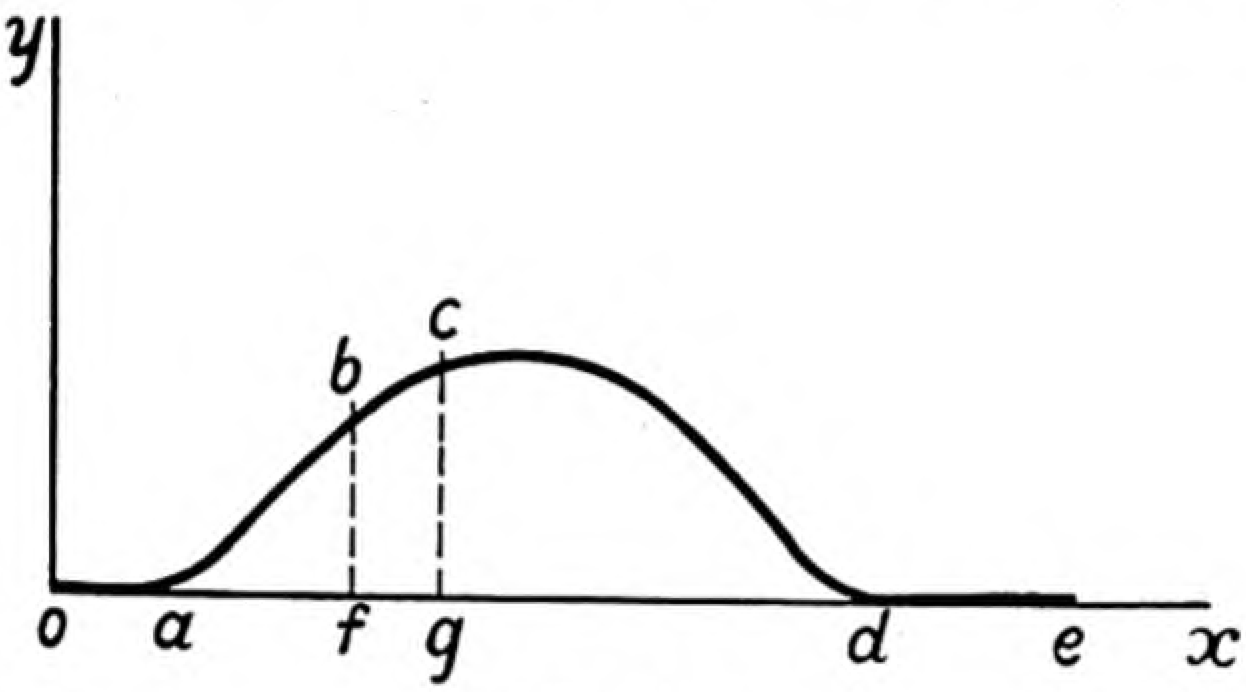


Fig. 4.7.

slight movement, and is termed an *approximate node*, which should be avoided in measurements. With two or more loops, the half wave-length of the vibration can be easily obtained.

Progressive waves and standing waves are shown diagrammatically in Figs. 4.5 and 4.6.

In Fig. 4.5 the wave $A_1B_1C_1D_1$ *progresses* to $A_2B_2C_2D_2$ owing to the *transverse* movement of the particle A_1 to a_2, a_1 to A_2, etc. This may be readily demonstrated by rotating a wire helix about its axis, and observing a point on the wire.

Fig. 4.6 shows two progressive waves moving in opposite directions; any point in their common paths is therefore subjected to two similar displacements. These are compounded and give rise to the thick curves which do not advance, hence the term *standing* wave. Notice that the nodes, marked N, are present at the same positions in each of the four successive curves, which are the resultants of the progressive waves. These waves are out of phase with each other in Fig. 4.6*a*, but are in phase in Fig. 4.6*d*. This case is easily demonstrated by rotating a wire sine-curve, of the form of the thick curve in Fig. 4.6*d* about the dotted line.

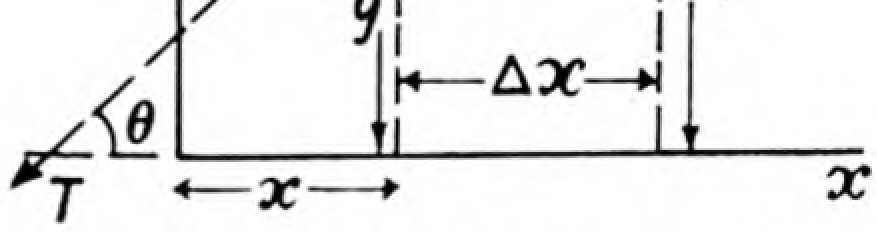


Fig. 4.8. *Note*, for ty read Δy.

Velocity of transverse waves along a stretched string. Consider a string *oabcde* (Fig. 4.7) to be displaced slightly from its position on the x-axis between a and d. The portion between b and c is enlarged in Fig. 4.8, and the tension is assumed to remain unchanged as the

displacements are small. The component of the tension acting perpendicular to the x-axis at b in positive direction of y is, at the instant under consideration,

$$=-T\sin\theta$$

$$=-T.\frac{\partial y}{\partial x} \text{ approx. when } \theta \text{ is small.}$$

The corresponding force at c

$$=T\sin(\theta+\delta\theta)$$

$$=T\sin\theta\cos\delta\theta+T\cos\theta\sin\delta\theta$$

$$=T\frac{\partial y}{\partial x}+T\delta\theta, \text{ approx.}$$

$$=T\frac{\partial y}{\partial x}+T\frac{\partial^2 y}{\partial x^2}\Delta x$$

Since

$$\theta\simeq\frac{\partial y}{\partial x}.$$

Therefore the resultant force on the element Δx in the y direction

$$=+T\frac{\partial^2 y}{\partial x^2}.\Delta x.$$

The equation of motion is

$$T.\frac{\partial^2 y}{\partial x^2}.\Delta x=m.\Delta x.\frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{m}.\frac{\partial^2 y}{\partial x^2}, \quad \ldots \ldots \quad (4)$$

i.e.

$$V=\sqrt{\frac{T}{m}}. \quad \ldots \ldots \quad (2)$$

If the tension T is expressed in dynes and m in grams per centimetre length, then the velocity of the pulse V is in centimetres per second.

The expression $n=\frac{1}{2l}\sqrt{\frac{T}{m}}$ is derived from $V=n.\lambda=n.2l=\sqrt{\frac{T}{m}}$, λ being the wave-length, and l the distance between successive nodes. If L stands for the complete length of the string in vibration and N the number of loops, $l=\frac{L}{N}$ and the expression may be written $n=\frac{N}{2l}\sqrt{\frac{T}{m}}$.

If $N=1$, the wire vibrates with its fundamental tone; if $N=2$, with the first overtone, and so on. Electrical engineers, in a similar case, refer to the first mode ($N=1$) as the first harmonic, the second mode as the second harmonic, and so on. This has the merit of simplicity, but to avoid complication in this book, the term overtone will be used, with the meaning indicated above.

By combining equation (4) above with equation (19) on p. 82, the differential equation of motion of a *stiff* wire, neglecting angular motion, is obtained, viz.:

$$\frac{\partial^2 y}{\partial t^2}=\frac{T}{m}.\frac{\partial^2 y}{\partial x^2}-\frac{Ek^2}{\rho}.\frac{\partial^4 y}{\partial x^4} \quad \ldots \ldots \quad (5)$$

where k is the radius of gyration of the cross-section of the wire about its neutral axis and the other quantities have their usual significance. It is to be noted that a stiff wire will act as a rigid bar for short wavelengths but as a flexible string for long waves.

The reader should perform Melde's experiment with the prongs of the fork moving at right angles to the string (Fig. 4.9), and adjust the tension to give an *even* number of loops. Next, turn the fork so that it vibrates towards the string. The number of loops is *halved*. The diagram (Fig. 4.10) is self-explanatory.

Fig. 4.9.

Two simple methods are available for maintaining a "metal string" at the frequency of, or a multiple of, that of an alternating current supply. In the one the current is passed through the wire (which need not be magnetic) and a permanent U-shaped magnet is placed astride the string at an antinode. In the other the alternating current is passed through an electromagnet which replaces the permanent magnet. The string must now be an iron wire and its fundamental frequency will be double that of the electrical supply.

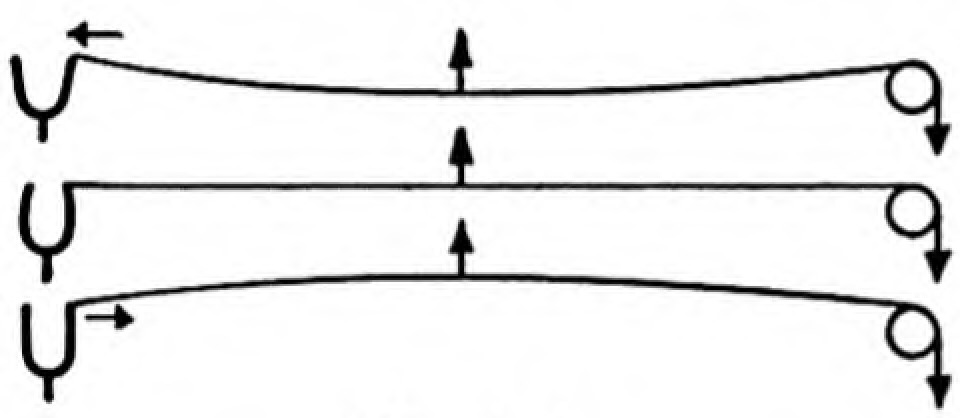


Fig. 4.10.

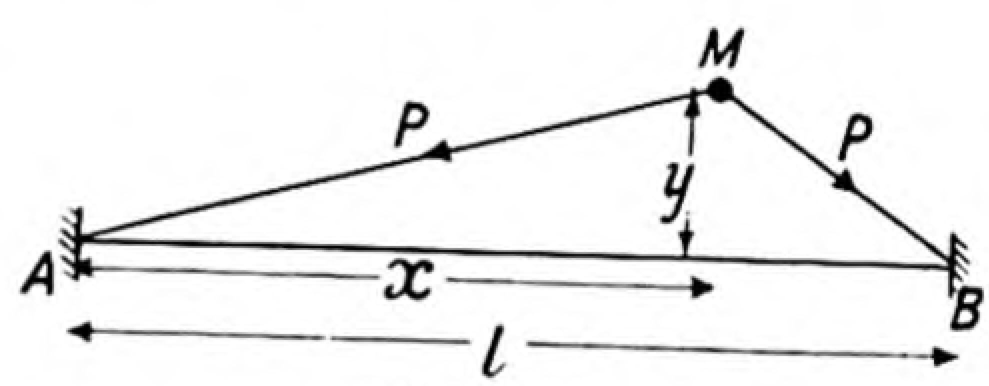


Fig. 4.11.

Frequency of vibration of a taut wire loaded at one point. M in Fig. 4.11 represents the load of mass M secured to the wire AB and displaced through a distance y. Actually y is small compared with l, and so the increase in tension due to displacement is negligible. Denoting the tension by P, and considering the resultant force in the y direction,

$$\frac{Py}{l-x}+\frac{Py}{x}=-M\ddot{y},$$

$$-\frac{\ddot{y}}{y}=\frac{Pl}{Mx(l-x)}, \text{ from which}$$

$$n=\frac{1}{2\pi}\sqrt{\frac{P.l}{Mx(l-x)}} \quad . \quad . \quad . \quad (6)$$

Clearly, n^2 is inversely proportional to $x(l-x)$, and so, by altering the position of the mass M the fundamental frequency can be adjusted, and the arrangement made to act as a frequency filter. The graph (Fig. 4.12) shows the calculated values for a thin wire of length 100 cm. with a load of 9·81 gm. and a tension of 10 kg. weight. The effect of moving the weight at the mid-point is small, but is considerable at about one-quarter of the length from the ends. Near the ends the effect of the difference in tension between each portion of the string

is no longer negligible, and so the values are unreliable. If the length x bears a simple ratio to $(l-x)$, *e.g.* 2 : 3, the string can vibrate in five loops (Fig. 4.13) with the mass M situated at a node, and the frequency will be that of the fifth harmonic or fourth overtone of the string (Fig. 4.13). This result is interesting in view of Young's law

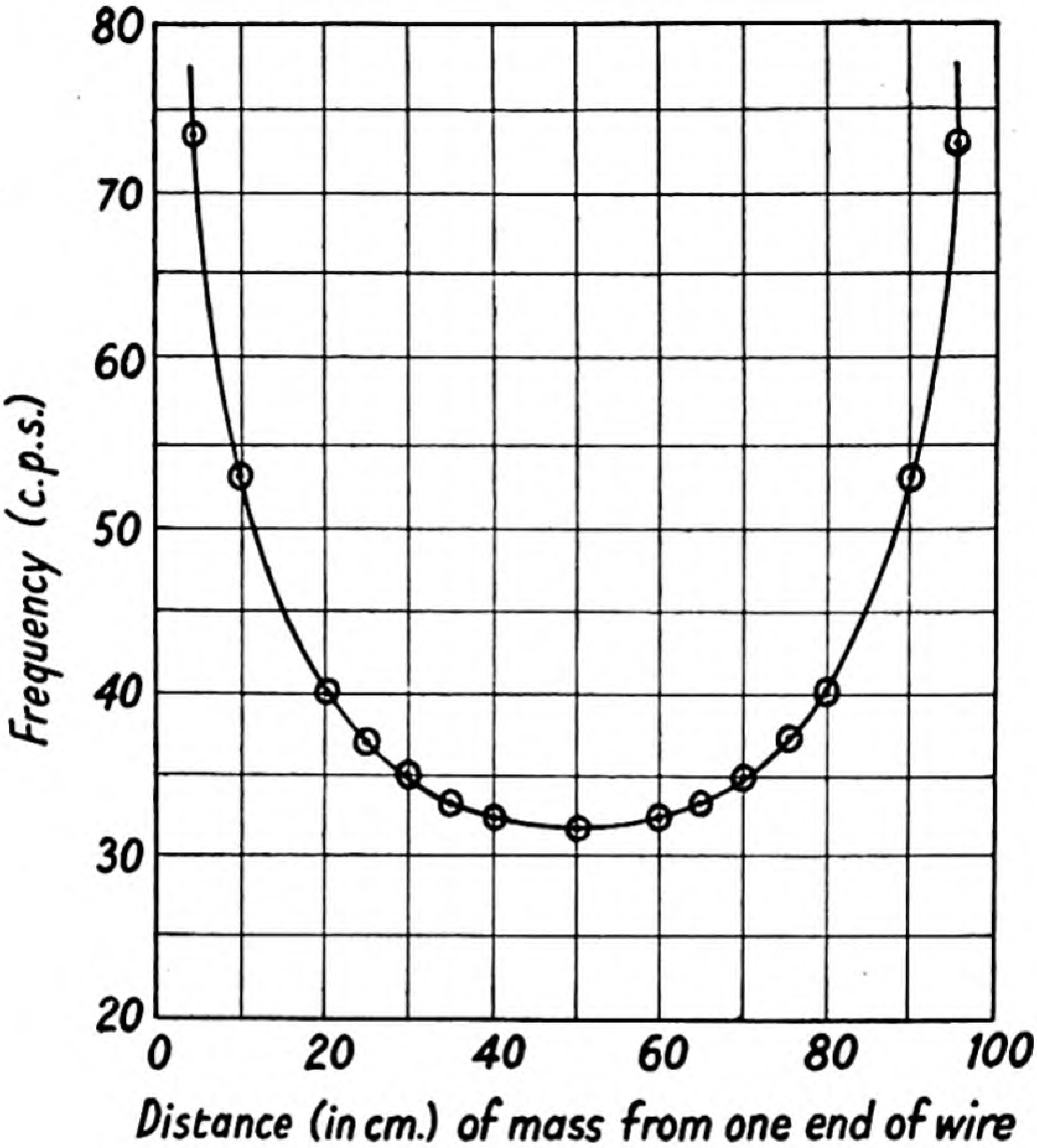


Fig. 4.12.

(1800): "No overtone can be present in a vibrating system which would have a node at the point of excitation," for it indicates a means of suppressing an undesirable overtone. Such an overtone is the seventh, and this is eliminated in a piano by making the striking point in the strings one-seventh of the length of each string from its end.

A drum skin is an example of a stretched *diaphragm*. The overtones are inharmonic, but their undesirable presence is suppressed in

Fig. 4.13.

tuned drums—kettledrums, *not* side-drums—by striking them at a point about one-quarter of the diameter from the edge; a distance which is a matter of experience, and which causes the overtones to be muffled. Diaphragms do not vibrate if limp, and for this reason they must be in tension, or of a rigid nature. This may be achieved by

pleating radially or corrugating concentrically as in an aneroid barometer, in which rigidity is combined with flexibility. Such corrugations introduce resonances which, however, may be minimised by making the corrugations follow a spiral path.

Wires, like diaphragms, must be rigid or in tension if they are to vibrate. For a wire to be rigid it must be short, and then it really constitutes a small rod or bar, and must be held at one or more points which are usually at the ends or at the middle. If secured (*a*) at the middle only, it is termed a free-free bar; (*b*) at one end only, a fixed-free bar; (*c*) at both ends, a fixed-fixed bar. Each form will vibrate transversely if struck, and longitudinally if stroked with a resined cloth. Torsional vibrations are possible if the bar is cylindrical, and are usually of high frequency, the squeak of non-lubricated wheels being a good example, the axle constituting the bar.

CHAPTER 5

PROPERTIES AND TYPES OF WAVE MOTION

Sound is the result of the movement of some material body, as for instance the parchment of a drum, or the strings of a violin, the reeds of a mouth-organ, or the performer's lips with a brass instrument, etc. Often the ear is able to directly associate a certain effect as the distinctive sound produced by the motion of a particular body under definite conditions, *e.g.* the characteristic "rustle" of a leaf in the wind. The movement giving rise to the noise, or sound, however, does not involve a bodily transfer of matter. This fact is at once made evident if a gas tap is "turned-on" in a room, for the "hiss" of the issuing coal-gas is heard at the remote corners of the room some finite time before the presence of the unburnt gas is detectable. The sound is conveyed across the room by a series of compressions and rarefactions, imparted to the immediately surrounding *air* by the issuing gas and then handed-on to neighbouring parts of the medium. Now the motion of the sound-producing body itself involves, in general, as for example in the case of the leaf cited above, a "to and fro" movement about the position of rest of the body. In other words, after the sound generator has received its initial displacement it must return towards its original position, in order that the vibration can be repeated for a sufficient number of times to be identified by the ear. This property, which controls the return of a body or material medium to its original physical state after being subjected to some form of distortion, is known as the elasticity of the body or medium.

Before, therefore, considering the various types of wave which may be transmitted through a medium, it will be necessary to become familiar with the various fundamental moduli of elasticity. Now if a body is acted upon by an external force, two effects are likely to occur—the body will tend to change its volume or its shape (or both), and hence at least two fundamental elastic constants are needed to describe the behaviour of the body.

The simplest type of deformation occurs when a body is subjected to compressive force acting equally in all directions, *e.g.* hydrostatic pressure at a point in a liquid, so that a change of volume is brought about without any change in the shape, of the body. The corresponding elastic modulus is known as the bulk or volume modulus, and it is a physical property of all matter, both solid and fluid. In point of fact it is the only type of deformation possible in fluids, as in general they do not offer any *permanent* resistance to changes of shape, although viscous liquids may resist changes of shape due to rapidly applied forces. A further qualifying statement with regard to liquids should be noted here, namely, that the *surface* of a

liquid exhibits resistance to change of shape, as is made evident by the phenomenon of surface tension. This latter force is the controlling factor in the so-called capillary waves which are propagated over the surfaces of liquids of shallow depth.

The magnitude of the force acting per unit area of a body is referred to as the applied stress, *i.e.* if F is the force in dynes acting over an area of A sq. cm., then the average stress is expressed as $\frac{F}{A}$ dynes per sq. cm. The strain due to the applied stress is measured by the fractional deformation produced, *e.g.* in the case of the bulk modulus it will be given by the ratio $\frac{v}{V}$ where v is the *change* in volume of a body whose total volume is V before the application of the stress. Now it has been shown experimentally by Robert Hooke that for *small* deformations of elastic solids (*i.e.* bodies which return to their original state after removal of small applied forces) the strain is proportional to the stress. The ratio $\frac{\text{stress}}{\text{strain}}$ then is constant and defines a modulus of elasticity, the particular modulus concerned being dependent upon the type of deformation. For example, the bulk (or volume) modulus K $\left(\frac{1}{K}\right.$ is known as the compressibility$\left.\right)$ is defined by the ratio,

$$\frac{\text{compressive (or tensile) force per unit area}}{\textit{change} \text{ in volume per unit volume}},$$

or in symbols by $K=-\frac{dp}{dv/v}$; the negative sign is inserted because an increase of pressure (dp) produces a decrease of volume (dv). Hooke's law does not hold for large values of applied stress, and the point at which the law begins to fail is known as the *elastic limit* of the body, and it will be made evident by a more rapid increase of strain with stress than holds over the elastic range (see Fig. 5.1). An experimental illustration of this effect may be conveniently carried out with a light spiral spring fixed at its upper end and loaded at the bottom. For small loads the stretch produced is seen to be proportional to the added load and on removal of the load the spring returns to its original length. If the stretching force becomes excessive, however, the spring may acquire what is known as a *permanent set*. The range of applicability of Hooke's law varies considerably both for different materials and for different types of deformation. For example, the range for steel is very large compared with that for lead, but the latter does behave as an elastic body within small limits, for it is this property which allows

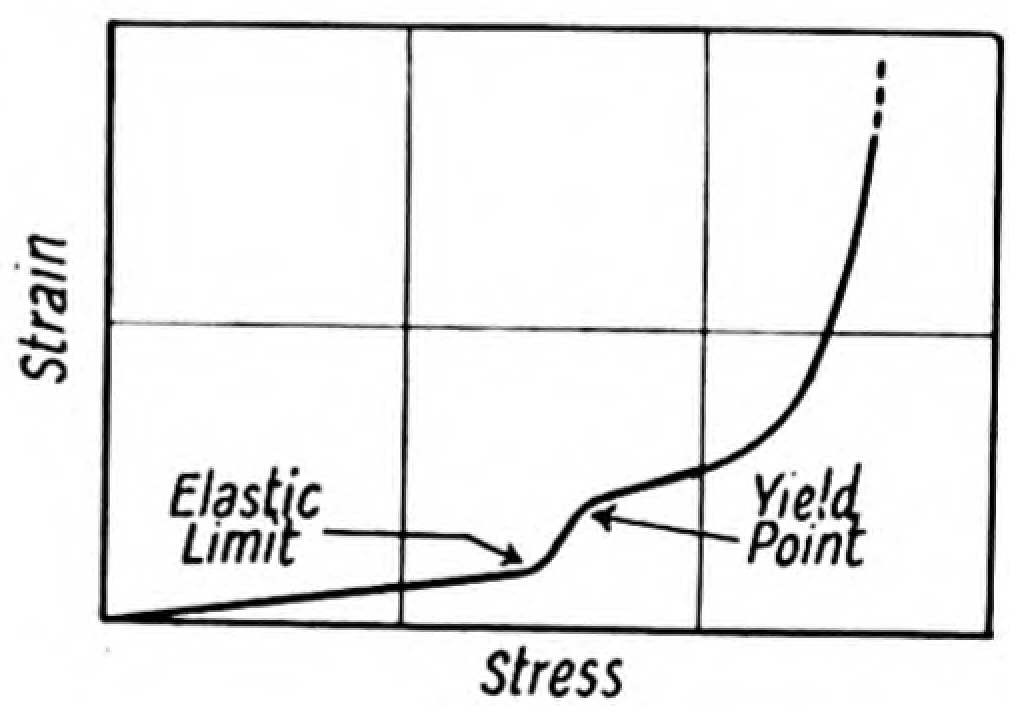


Fig. 5.1.

sound to be transmitted along lead pipes. The particular modulus of elasticity involved in the propagation of these longitudinal waves along a metal pipe or rod is known as Young's modulus. In this case the strain is measured by $\frac{l}{L}$ where l is the magnitude of the *change* in the length of a specimen of original length L, and the stress is expressed by $\frac{f}{A}$ where f is the total *tensile or compressive* force acting over the area of cross-section (A) of the material of the tube or rod. Hence Young's modulus is given by $E=\frac{f/A}{l/L}=\frac{fL}{Al}$ and its measurement for a specimen in the form of a wire may be carried out by the simple apparatus shown in Fig. 5.2. The specimen A under test is rigidly held in a clamp C, which is securely fixed to an overhead support such as a wooden beam. This wire carries a vernier V, and this moves over a scale S carried by a second wire B which itself is fixed similarly to A and is kept permanently taut by a weight P. By this arrangement any movement of the overhead support will affect both wires similarly and therefore does not become registered as a change of length, *i.e.* there is no movement of the vernier relative to the scale. The load W on the specimen A is varied and the corresponding changes of length recorded, care being taken that the initial loading is sufficient to render the wire taut. Readings should be taken with increasing and decreasing loads, and provided the elastic limit has not been reached a close concordance between the two sets should be attained. Any marked disagreement is probably due to removal of kinks in the wire, and in this case the cycle of readings should be repeated. A graph of extension against load (expressed in dynes) should be plotted, and from the slope of the straight line through the points the mean value of $\frac{f}{l}$ deduced. A micrometer gauge is used to measure the diameter (d) of the wire at a number of points along its length and in two directions at right angles, and so A is determined and finally L is measured. The value of E is then calculated and expressed in dynes per square centimetre; typical values for a variety of materials are shown in the table below.

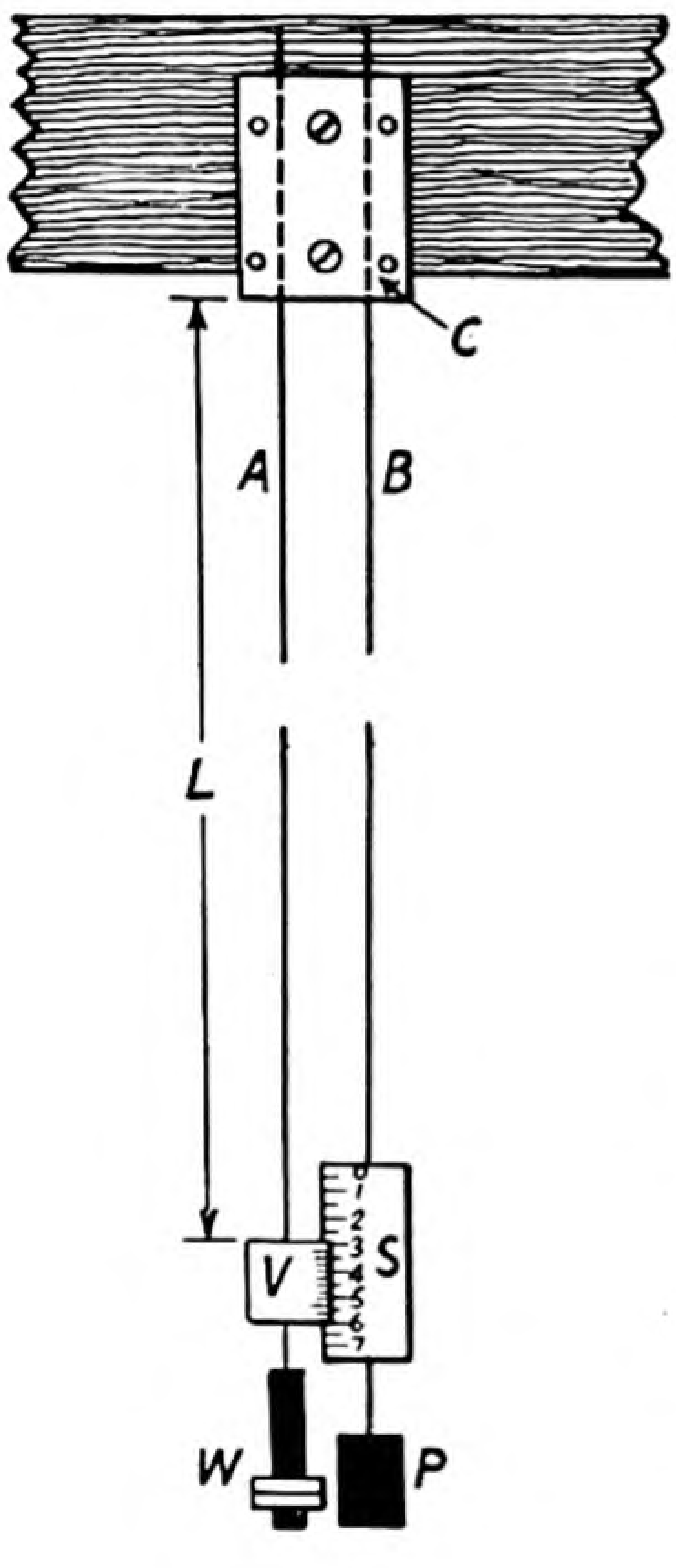


Fig. 5.2.

The range of values shown in certain cases is indicative of the variation of the elastic modulus with the state and purity of the material.

Approximate Values of Young's Modulus (E) for some Typical Materials
(dynes per square centimetre)

Aluminium	7×10^{11}	Oak	$1{\cdot}3 \times 10^{11}$
Copper	10 to 13×10^{11}	Quartz	8×10^{11}
Glass	5 to 8×10^{11}	Rubber	10^7 to 10^8
Lead	$1{\cdot}6 \times 10^{11}$	Steel	18 to 25×10^{11}

Now if a body is deformed a certain amount of work is performed on it and, if the strain is such that the elastic limit has not been reached, this energy will be stored in the body as potential energy, the amount being proportional to the square of the displacement (see p. 17). When the constraining force is removed there will be a gradual transformation of this stored energy from the potential to the kinetic form, and the change will be complete when the body reaches its original form or bulk. The velocities acquired by the particles of the body, by virtue of their kinetic energy, will, however, cause them to travel through the normal positions of rest (if free to do so) and in this way the body becomes distorted in a sense opposite to that of the initial deformation. The magnitude of this reverse deformation in general will be only slightly less than the original value, the difference resulting from energy expended in overcoming resistive forces experienced by the body during its motion. It should now be evident that any such elastic body (or medium), after suffering a small initial displacement, will perform "to and fro" motions under the appropriate elastic control. Since, by Hooke's law, the restoring force on any particle of the body (or medium) is proportional to the displacement of that particular particle from its rest position, then it follows from definition that the resulting motion is periodic and simple harmonic in type. Unless the energy loss due to friction, etc., is replenished by an external agency, *i.e.* unless the vibrations are *maintained* by a suitable stimulus, then the amplitude of the motion will gradually decay.

It should be noted here that some sound generators, *e.g.* flags when flapping in the wind, operate without recourse to an elastic control, but this cannot apply to the vibrations of the fluid medium (usually air, of course) by which ultimately the ear of the observer is affected. The last statement needs qualification, however, by noting that the ear may also become conscious of sound, not through aerial vibrations, but by "bone conduction," *i.e.* through the longitudinal vibrations of the facial bones, a process which may be experienced by holding the stem of a vibrating tuning-fork against a cheek bone or jaw, or a watch against the forehead.

The second essential property of a body or medium which is transmitting or generating mechanical vibrations is that it must possess inertia. This is the property by virtue of which a body is carried *through* its rest position when it is released from an initial displacement. If a medium did not possess appreciable inertia, any force applied to it would produce an effect instantaneously and hence the mechanical disturbance would be propagated with an infinite velocity.

To investigate in more detail the characteristics of the propagation of compressional waves it is convenient to consider the behaviour of a long spiral spring, say 5 or 6 ft. long, comprising approximately

one hundred coils of diameter 2 in. to 3 in., which is fixed at its upper end F and has a light body W attached to its lower end (Fig. 5.3). Suppose that W is suddenly moved from its equilibrium position, resting on a bench, say, at b, upwards to c and then back to b. The immediate effect will be for the coils nearest to W to become pressed against each other and then to rebound in consequence of the resistance offered by the elastic properties of the material of the spring. In this way this "condition" of compression will be handed on by contiguous coils upwards along the spiral. This propagation of a condition in a medium is what is meant by a wave motion, so it can be stated that a wave of compression has been propagated along the spiral spring. Now in the above experiment the body W was finally brought back to its original position b, *i.e.* the total length of the spring was unaltered. Hence it follows that if some portion of the spring is being compressed, then at the same instant another portion must become elongated. In other words, the wave of compression will be followed by one of rarefaction. The time taken for either condition, *i.e.* compression or rarefaction, to move unit distance along the spring will be a measure of the speed of propagation of compressional waves along the spring, and its value will depend upon the elastic constants of the material of the spring and upon its dimensions.

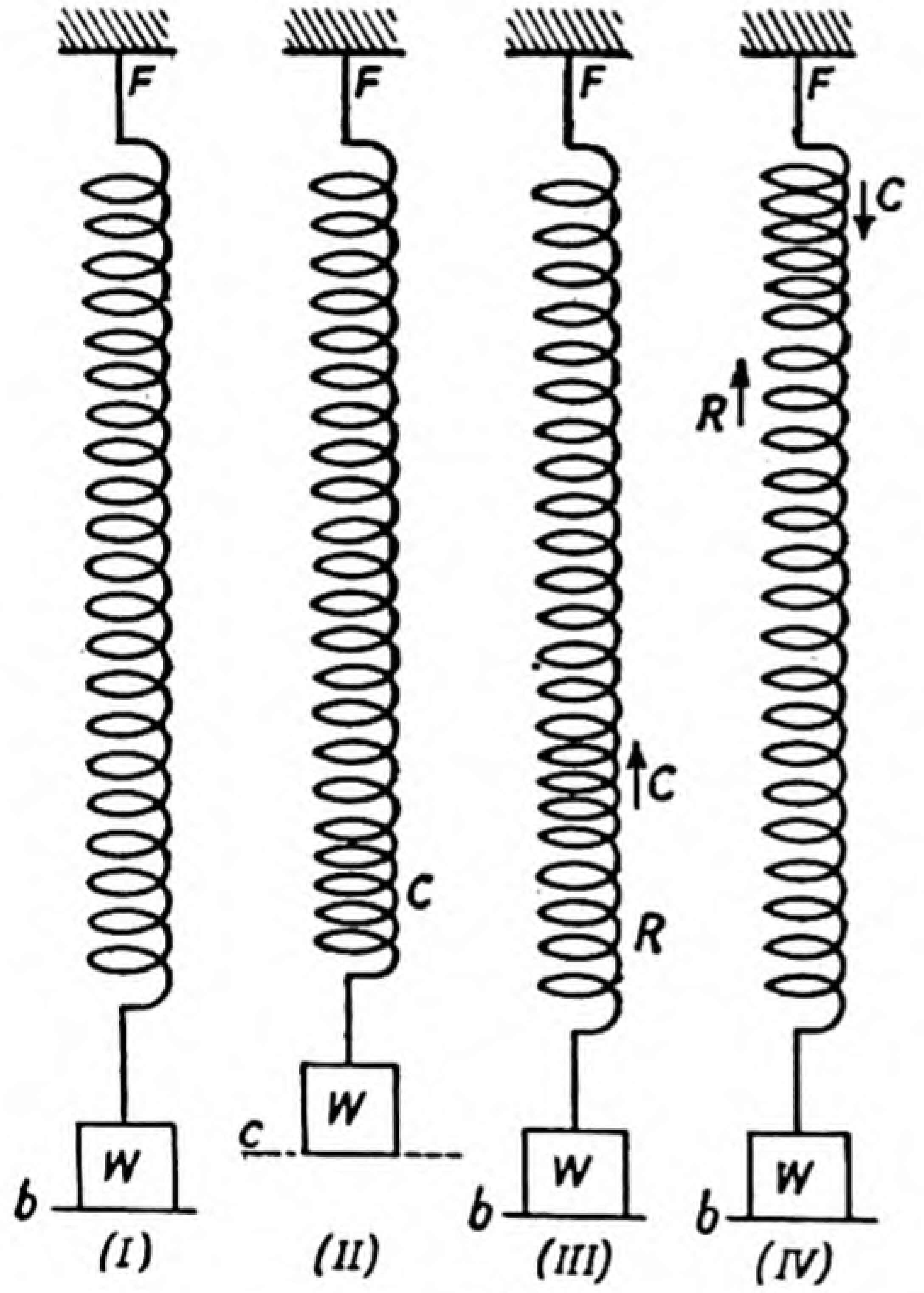


Fig. 5.3.

Now the problem just considered is analogous in some respects to that of a stretched string (Chap. 4) and if, at any instant, the *displacement* of each coil of the spring from its equilibrium position is plotted against this position along the axis of the spring, then the curve obtained will be similar to the displacement curve (Fig. 3.1). In the case of a plucked violin string it must be remembered that it is a transverse wave which is transmitted along the string and that it results from the resistance offered by the stretched string to a change of *shape*. In the case of the spring, however, the actual coil displacements are in the direction of the axis of the spring, *i.e.* in the direction in which the compression travels. Such a wave motion is referred to as compressional or longitudinal, and is moreover brought about by the resistance of a body (or medium) to a change of *size*. When the disturbance on the string reaches a fixed end it should be evident that there is a reflected disturbance which is 180° out of phase with the incident one, *e.g.* a wave-crest will be reflected as a trough, a natural consequence of the fact that the *resultant* displacement at a fixed point

must be zero at all times. Such a condition must also hold for the compression along the spring when it arrives at the upper fixed point, but the necessary equal and opposite motion of the particles in the oppositely-travelling reflected wave must obviously be associated with a compression (often termed condensation) in that wave as for the incident one. Hence at a fixed end a compression will be reflected as a compression (see Fig. 5.3 (IV)) and a rarefaction as a rarefaction. At a free end, *e.g.* if the end W is attached to the load, which is no longer supported on the bench, the incident condensation would become a reflected rarefaction and vice versa, for W is now free to move. It is evident therefore that, with both ends of the spring fixed as in Fig. 5.4, the compression will travel up and down until the energy expended in the original work of compression has been completely dissipated in overcoming the frictional resistance to the motion. Suppose, however,

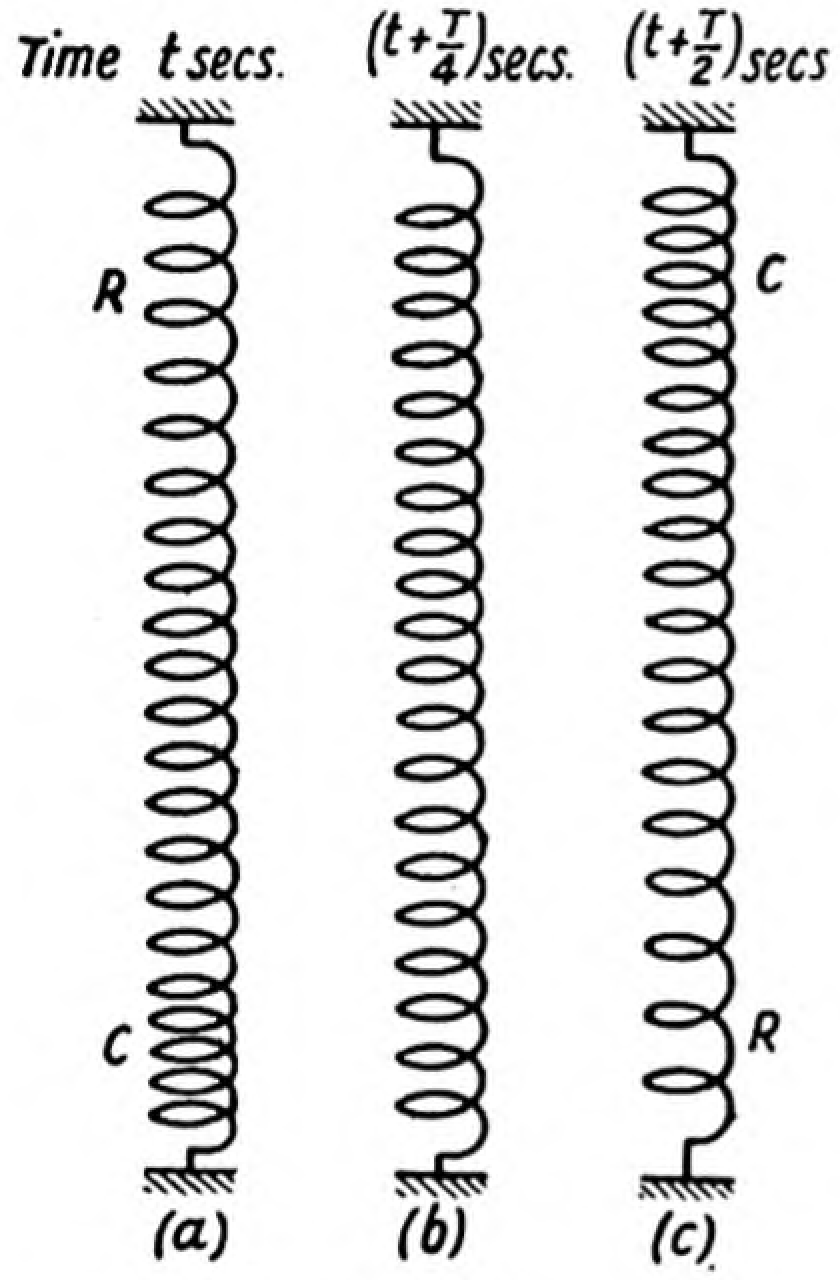


Fig. 5.4.

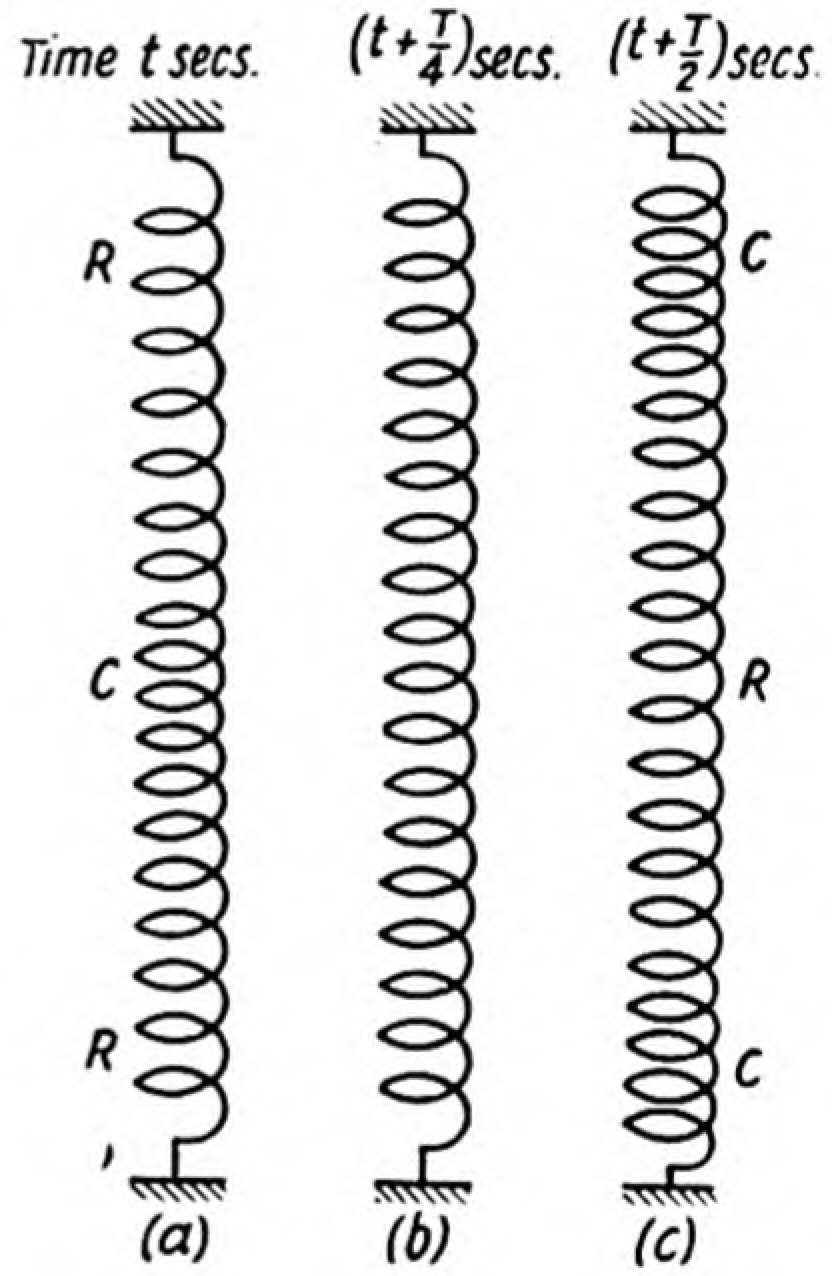


Fig. 5.5.

that just as the compression reaches W, after its first excursion up and down the spring, the load is moved to c and back to b (Fig. 5.3), and the movement is repeated every T sec., which is the time for the compression to travel twice the length of the spring. Then it should be evident that the motion of the spring will be maintained and a so-called *standing* wave system (p. 49) set up, but for the motion to be simple harmonic in type the load should move between limits a and c which represent extreme positions on either side of but equidistant from b. If ω is the pulsatance associated with this S.H.M., then $T=\frac{2\pi}{\omega}$ and the applied force F at any time will be given by $F=F_0 \sin \omega t$, where F_0 is the amplitude of the force. The appearance of the spring at different times during its fundamental period (T) of vibration is shown in Fig. 5.4 *a*, *b* and *c*. Standing wave systems will also be set up on the spring

if the frequency of the applied force is an integral multiple of the fundamental frequency $\frac{\omega}{2\pi}$, and Fig. 5.5 *a*, *b* and *c* illustrate the case of the first overtone.

Another aspect of the motion of the spring is obtained by focusing attention on a particular coil as the transmitted wave passes along; it will be seen to perform a similar movement to that imparted to the end of the spring, but each coil will begin its motion a little later than the coil immediately below, *i.e.* there is a continuous change of phase in the direction of travel of the wave. This "handing-on" of vibrational energy may also be illustrated by having a dozen or so similar steel ball bearings in contact with one another inside a horizontal glass tube of slightly larger diameter than that of the balls. If an additional ball is rolled into the tube to strike one of the end balls, then it will be found that the other end ball will move away, but the remainder will stay at rest. The mechanism at work, as in the case of the spring, is that the kinetic energy imparted to the first ball has been transferred almost completely from ball to ball down the line in the form of elastic energy of compression until the last ball has been reached. Like the end coil at a free end of a spring, this last ball is free to move, so that its potential energy due to compression is transformed into kinetic energy of motion, which will be slightly less than that possessed by the ball initially projected on to the end of the line.

It should be noted, however, that a finite time elapses before the end ball moves away, *i.e.* there is a phase lag, so that the transfer of energy does not occur instantaneously, which implies that the velocity of propagation of the compressive strain is not infinite.

Suppose now that in the case of the spring the number of turns per centimetre be increased and the coil diameter be reduced (or the wire diameter increased), then it is evident that the system will approximate finally to a solid rod of the material of the spring. Now such a rod may also be excited, like the spring, to transmit longitudinal waves, for example, by clamping the rod at its centre and drawing a "resined" cloth in a longitudinal direction over a short distance at one end (see Kundt's tube experiment, p. 137). Alternatively the rod may be struck by a hammer on an end face and a compressional (*i.e.* longitudinal) wave propagated along its length.

Torsional vibrations. Torsional vibrations arise from the resistance of a *solid* medium to shearing forces which are such as produce a deformation of a body without changing its volume. The corresponding elastic modulus which controls such strains is known as the coefficient of rigidity (n) and it is defined by the ratio $\frac{\text{shear stress}}{\text{shear strain}}$. The meaning of this coefficient may be illustrated by means of Fig. 5.6 where a block of solid material, shown in section as *abcd*, is subjected on its upper and lower parallel faces to equal tangential forces T, the rotation of the body being prevented by applying forces T' as indicated so that clockwise and anti-clockwise couples are equal. If the block be considered as built up of a series of planes parallel

to the upper and lower faces, then the effect of applying the shearing forces is for each layer to move horizontally relative to its neighbour (Fig. 5.6*b*), in the manner of a pack of playing cards. In the aggregate effect the body is said to be sheared through an angle θ (Fig. 5.6*c*) and the

$$\text{Coefficient of rigidity } n = \frac{\text{shear stress}}{\text{shear strain}} = \frac{T/A}{\theta}$$

where A is the area of the upper and of the lower face.

In the case of a circular rod (Fig. 5.7*a*) subjected to a torsional couple G acting about the vertical axis through C, the solid element *abcdefgh* (Fig. 5.7*b*) becomes sheared through an angle θ to the position $a'b'cde'f'gh$, the lower end of the rod being fixed. The angle of shear θ may be expressed in terms of the angle of twist ϕ, and the coefficient of rigidity is then given by $n = \dfrac{G}{\dfrac{\pi a^4}{2} \cdot \dfrac{\theta}{l}}$, where a is the radius and l the length of the wire or solid cylinder.

The setting-up of torsional waves in a cylinder of material is indicated in Fig. 5.9. Torsional (and also bending) waves are prominent means

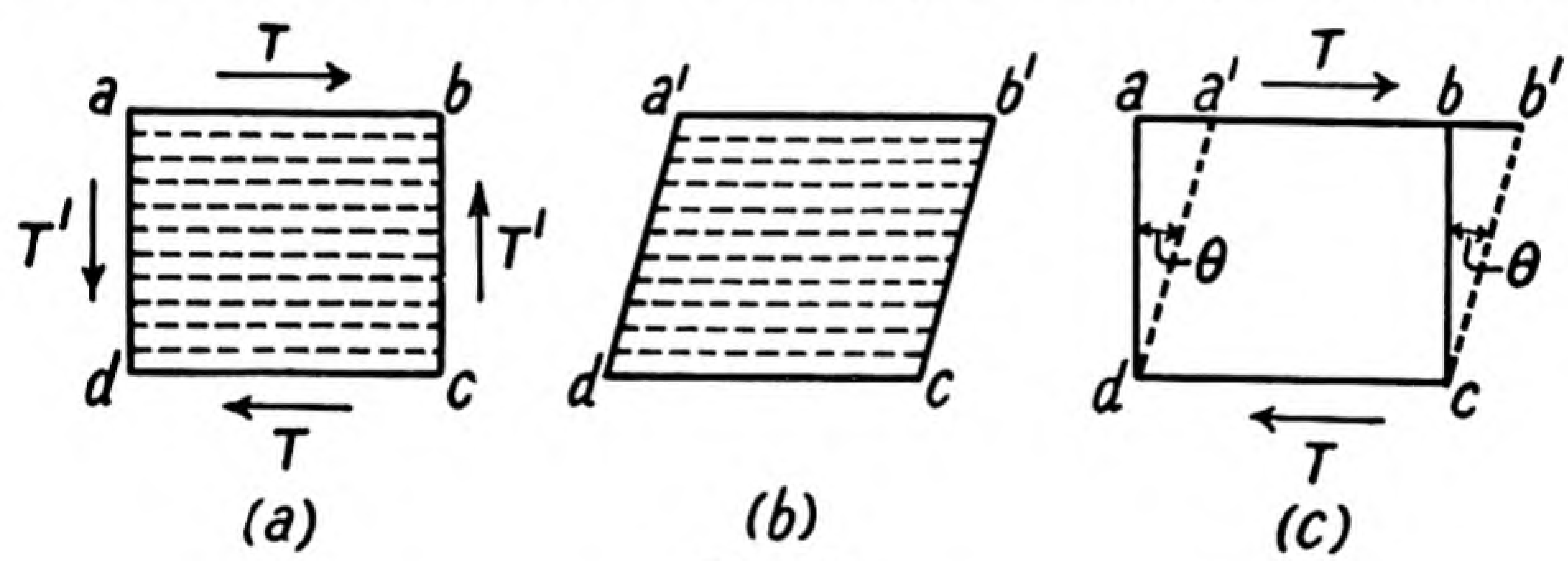


Fig. 5.6.

whereby sound is propagated through the structures of buildings, the velocity of propagation being less than that of longitudinal waves in the same material. Particular attention has to be directed to the existence of torsional vibrations in all types of machinery with fluctuating torque, in the motion of the wings of aeroplanes and also in certain types of acoustical apparatus. An example of the latter occurs in the moving system of an electro-magnetic cutter used for sound recording, where a rubber rod is chosen of such a length that the torsional vibrations transmitted along it become so attenuated that the amplitude of the wave reflected from the far end may be neglected (*cf.* a correctly terminated electrical transmission line).

The velocity of propagation of transverse vibrations through an elastic solid is given by $v_t = \sqrt{\dfrac{n}{\rho}}$, n being the coefficient of rigidity and ρ the density of the medium, whereas the corresponding expression for longitudinal waves in an infinite medium is (see Appendix) $v_l = \sqrt{\dfrac{K + \frac{4}{3}n}{\rho}}$, where K is the bulk modulus of the material. Either type of wave may be propagated alone in a homogeneous medium

but the existence of boundary surfaces or surfaces of discontinuity will, except in the case of normal incidence, give rise to a partial conversion into the other type. In the elastic solid theory of light, the light waves were identified with the transverse wave motion and the elastic solid medium with the so-called ether. Based on these assumptions Fresnel worked out correctly the quantitative laws regarding reflection and transmission coefficients, but the difficulty created by the existence of longitudinal motion within an elastic solid was the important factor that led to the acceptance of the electromagnetic theory of light, which will only permit transverse motion.

Particle motion

In this paragraph attention will be directed to the motion of the actual elements or particles of the disturbed medium, for the type of

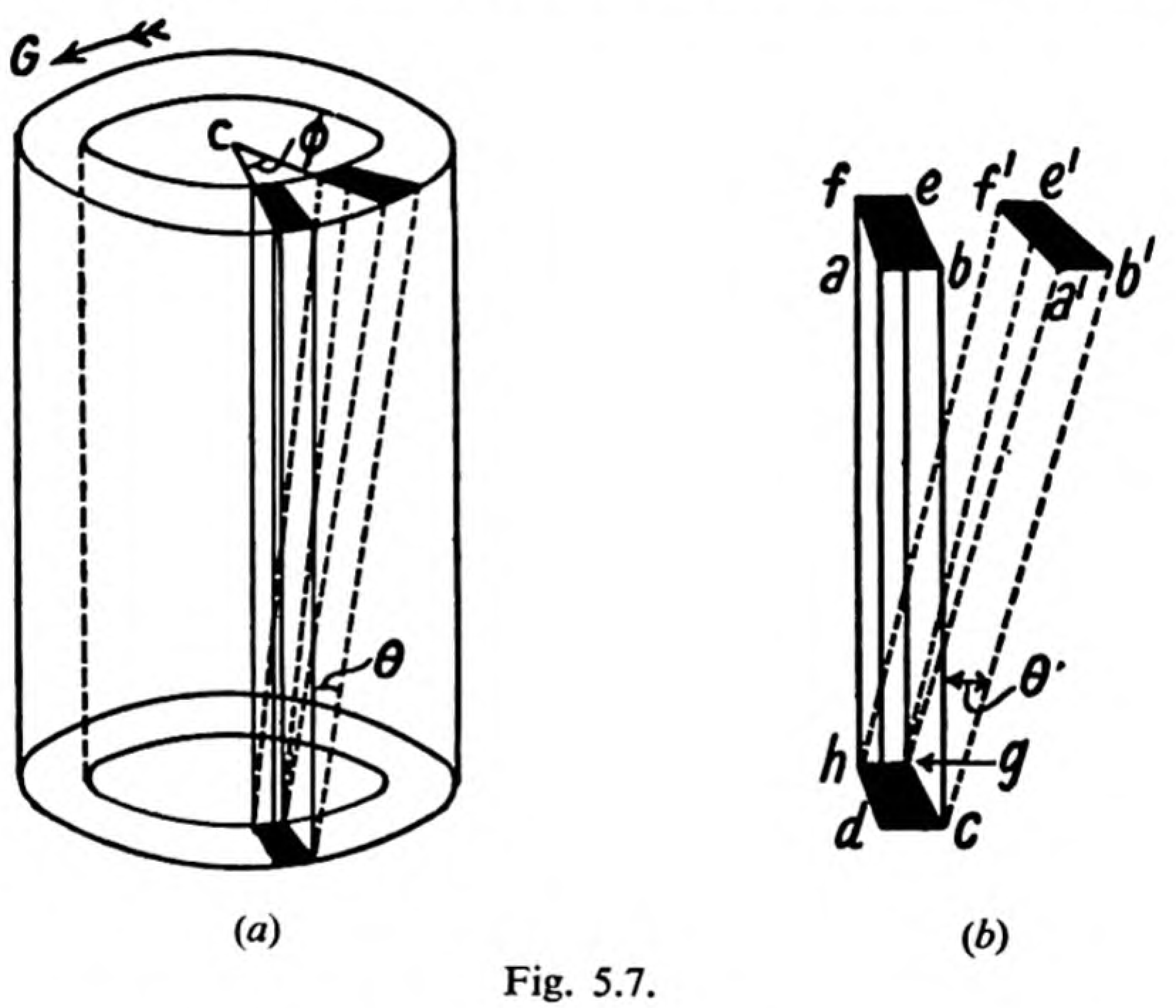


Fig. 5.7.

wave propagated is fundamentally dependent upon the form of the orbits of those particles. The linear motion of a particle is the simplest type of orbital vibration and this may occur (i) in a plane perpendicular to the direction of propagation of the wave, as in a string vibrating transversely, or (ii) in a plane containing the direction of propagation, as for compressional waves in a fluid or solid. The upper diagram of Fig. 5.9 indicates how planes of particles, equidistant from each other in the undisturbed *fluid*, become displaced during the passage of a compressional wave due to the applied force $F=F_0 \sin \omega t$. In the lower diagram the passage of a compressional wave through a solid of finite lateral dimensions, *e.g.* a rod, is seen to give rise to a *small* lateral movement, shown greatly exaggerated in the figure. Waves which are propagated over the *surface* of a liquid are mainly controlled by the surface tension of the liquid-air interface, and in this case the water particles perform circular orbits with their planes parallel to the

direction of propagation (see later). In contrast the propagation of torsional waves in a rod, due to an applied alternating couple, involves the ultimate material particles in circular, or part-circular, vibrations

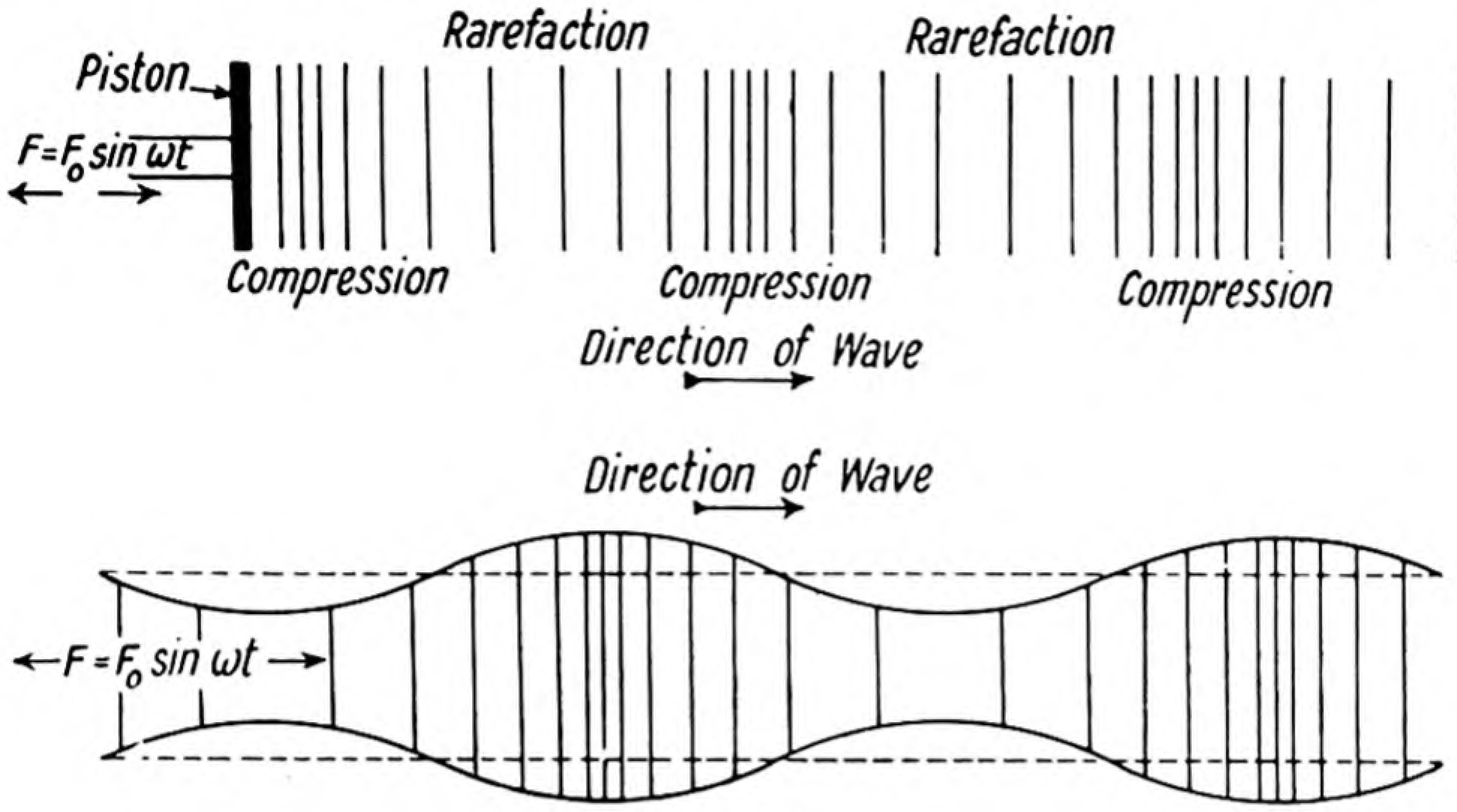


Fig. 5.9.

which take place in planes *perpendicular* to the direction of propagation (see Fig. 5.9). If *all* the particles of the medium through which the wave is travelling perform identical orbits, although the vibrations of each lag behind those of neighbouring particles nearer the source of disturbance, then the wave is spoken of as being of *constant type*.

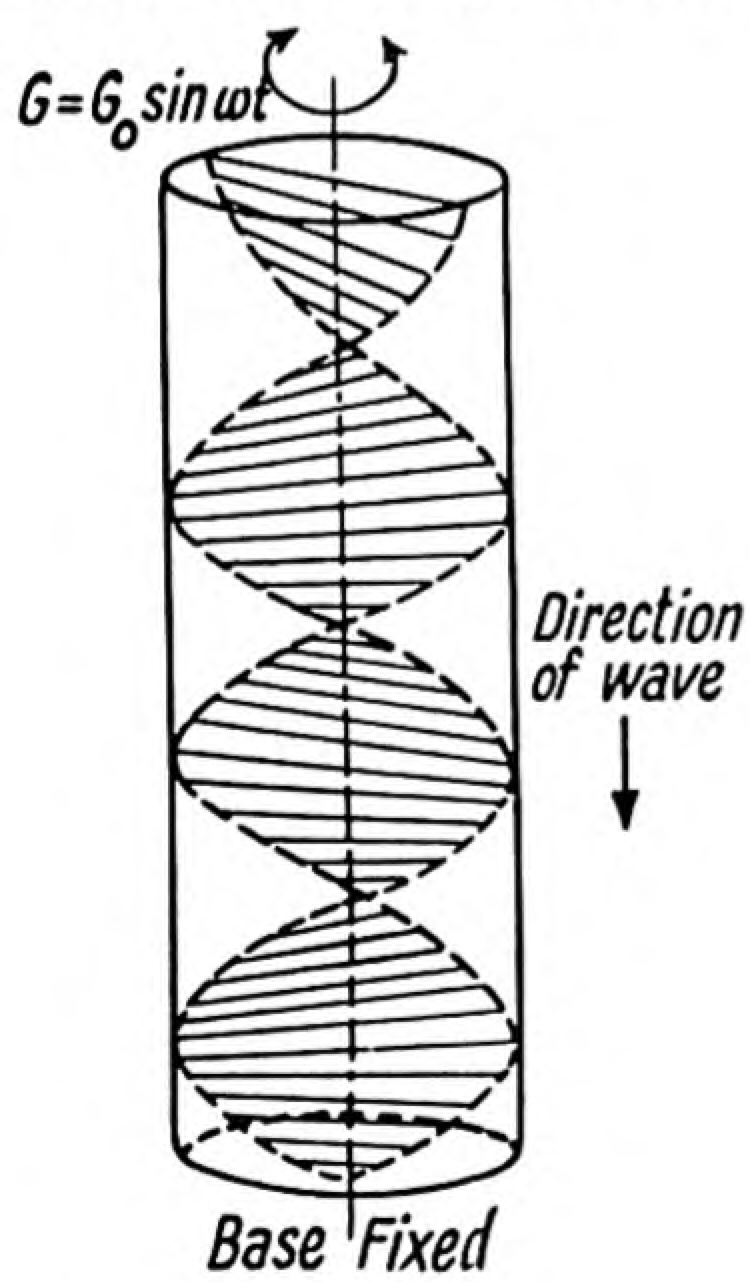


Fig. 5.8.

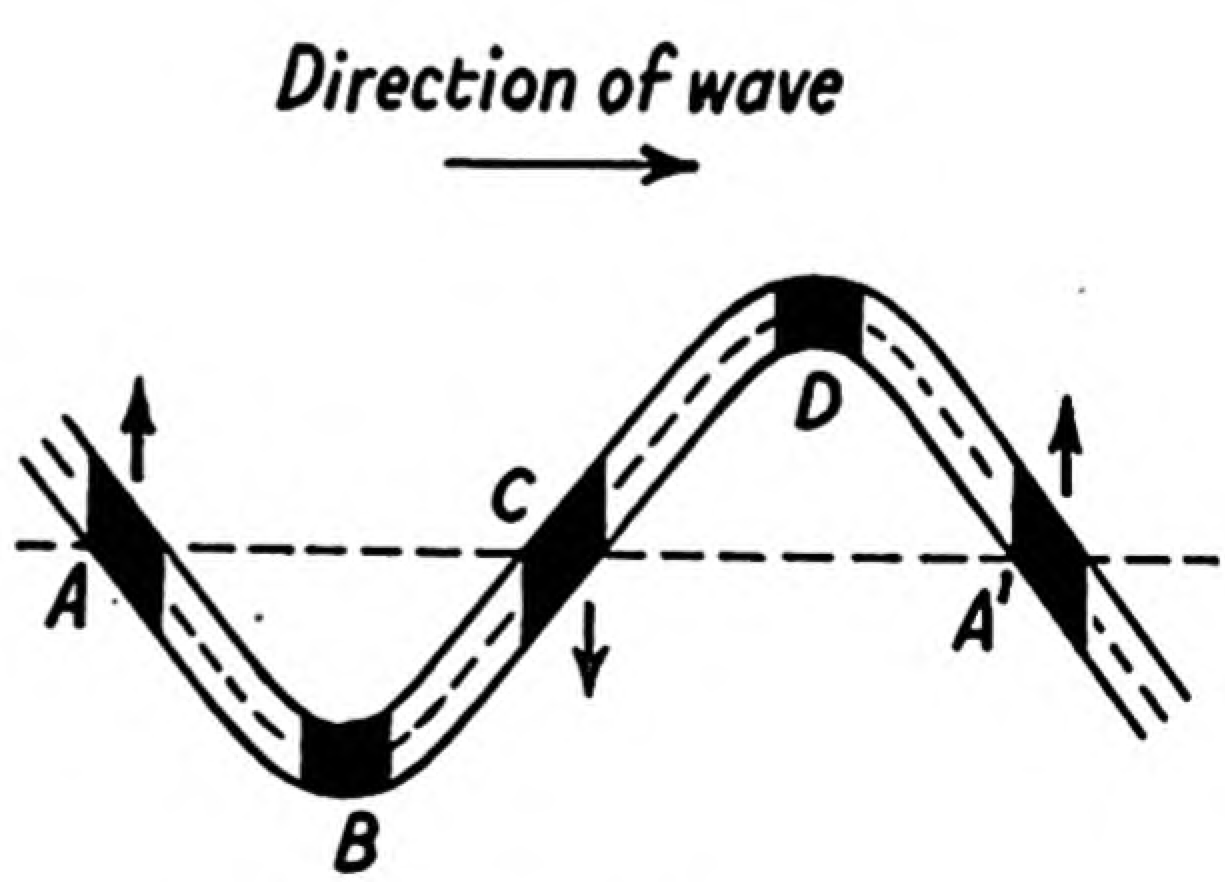


Fig. 5.10.

The wave motion depicted in Fig. 5.10 shows a transverse wave propagated in an infinite solid medium, the control being exercised by shearing forces. It will be noted that a very small element of the

medium at the crest D or trough B of the wave is momentarily stationary and undistorted, and hence both its kinetic and potential energies will be zero. Those elements, however, which are passing through their equilibrium positions A, C, and A' suffer the maximum distortion and their potential and kinetic energies will be a maximum.

The further the wave motion spreads out from the disturbance centre the larger the mass of the medium set into vibration, and since for S.H.M. the kinetic energy of motion is proportional to the product of the mass and the square of the amplitude of the particle vibration (p. 42), then the amplitude will decrease with the distance. In the case of plane waves which are propagated in the form of a beam of *constant* cross-section, any diminution of particle amplitude at increasing distances from the source will be due solely to energy losses brought about by viscous resistance, etc. With cylindrical wave propagation the spreading of the waves is in a horizontal plane only, so that the *attenuation* due to the larger masses set into vibration can be calculated as follows. Let a_1 and a_2 be the respective amplitudes at distances r_1 and r_2 from the source, then if the corresponding masses of the medium set into vibration are M_1 and M_2, it is evident that

$$\frac{M_1}{M_2}=\frac{2\pi r_1}{2\pi r_2}=\frac{r_1}{r_2}.$$

Hence, assuming no energy dissipation due to viscous resistance, $KM_1a_1{}^2=KM_2a_2{}^2$ from equation (22) (p. 42) where K is a constant, and so

$$\frac{a_1{}^2}{a^2}=\frac{r_2}{r_1},\quad \text{i.e.}\quad \frac{a_1}{a_2}=\sqrt{\frac{r_2}{r_1}},$$

or the amplitude varies inversely as the square root of the distance. A similar argument applied to spherical waves shows that in this case the amplitude is proportional to the inverse of the distance.

Lord Rayleigh has shown ("Theory of Sound," Vol. II) that, owing to viscosity and heat conduction, the intensity of sound falls off more rapidly than follows from the above geometrical considerations, and it is the high-pitched sounds which are most strongly attenuated. In the case of plane sound waves Rayleigh deduced the formula $d=\frac{(8800)c^2}{n^2}$, where d is the distance from the source of sound, at which the amplitude of the waves is reduced to $\frac{1}{e}$, *i.e.* $\frac{1}{2{\cdot}718}$, of that of the source. c and n are respectively the velocity and the frequency of the sound. For a note of frequency 1000 c.p.s. the distance d is approximately $1{\cdot}05\times10^7$ cm., while for a frequency of 10,000 c.p.s. it is reduced to $1{\cdot}05\times10^5$ cm. Knudsen has since shown the effect of humidity and temperature on the absorption of sound in air, and has found that there is a critical value of the humidity, which varies with the frequency, at which maximum absorption occurs.

Particle-velocity and wave-velocity

The velocity attained by any particle of a medium subjected to a disturbance must be carefully distinguished from the actual wave-velocity, *i.e.* the rate at which energy is being propagated through the

medium. Even though the particle may be executing several thousand vibrations (n) per second, its amplitude (a) of motion in a normal sound wave is quite small, being of the order of a few thousandths of a millimetre, hence the maximum particle-velocity $v=2\pi na$ (from eqn. (7), p. 15) is seen to be extremely small compared with that of the wave. In the case of surface water waves of very large amplitude, however, it is possible for the water particles, as they execute circular or elliptical orbits, to attain a horizontal velocity equal to that of the waves.

The relation between the particle-velocity (v) and wave-velocity (c) may be conveniently deduced by reference to a wave-displacement diagram. The wave form AP_1P_2BCD (Fig. 5.11a) represents, at

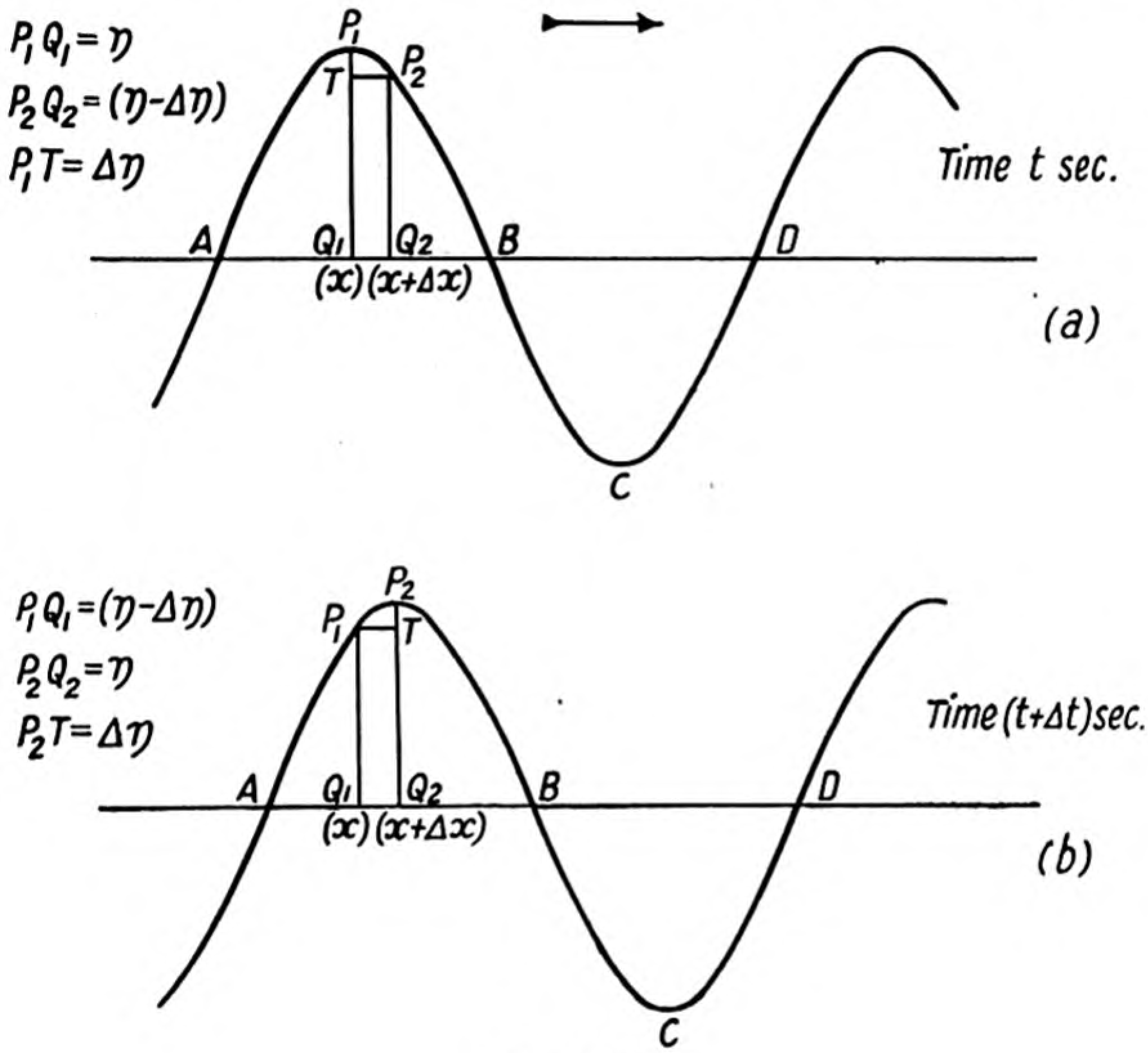


Fig. 5.11.

time t, the particle displacements (longitudinal or transverse) of a wave travelling from left to right. The displacements at points x and $(x+\Delta x)$ are taken as η and $(\eta-\Delta\eta)$ respectively, and for convenience Q_1P_1 is assumed to be the maximum displacement. At a later instant $(t+\Delta t)$, the particle at Q_2 will have attained its maximum displacement (Fig. 5.11b), and its velocity will be given by $v=\frac{\eta-(\eta-\Delta\eta)}{\Delta t}=\frac{\Delta\eta}{\Delta t}$, or in the limit when Q_1Q_2 is infinitesimally small,

$$v=\frac{d\eta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

During this time interval of Δt sec. the maximum of the wave will therefore have moved from P_1 to P_2, *i.e.* a distance of Δx in the

direction of the wave propagation, and hence the wave-velocity

$$c=\frac{\Delta x}{\Delta t}, \text{ or, in the limit } c=\frac{dx}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

as Q_1Q_2 is made infinitesimally small. The ratio of $\frac{\text{particle-velocity}}{\text{wave-velocity}}$ is therefore given by .

$$\frac{v}{c}=\frac{\frac{\Delta\eta}{\Delta t}}{\frac{\Delta x}{\Delta t}}=\frac{d\eta}{dx} \text{ in the limit} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The ratio $\frac{v}{c}$ may now be expressed in terms of the bulk modulus of a fluid medium when the displacement will necessarily be longitudinal.

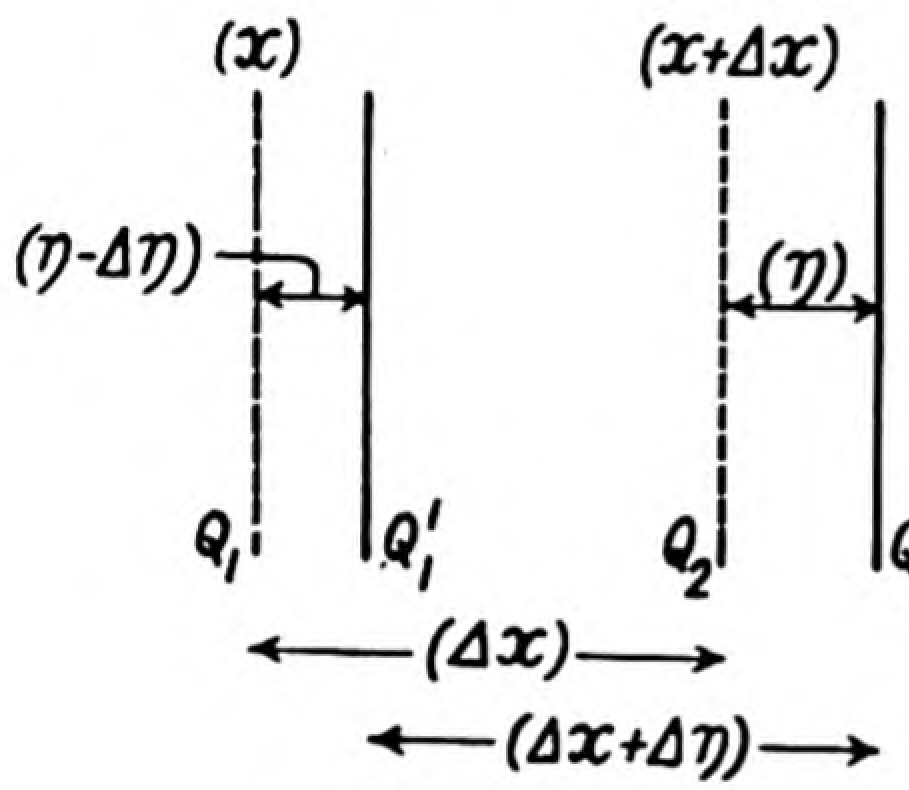


Fig. 5.12.

For, considering unit area of cross-section of the medium and supposing lines Q_1' and Q_2' (Fig. 5.12) to represent the displaced positions of the planes of particles normally situated at Q_1 and Q_2 respectively. Then it is easily seen that the volume Δx of the medium between planes Q_1 and Q_2 has become, as a result of the passage of the wave, a volume $(\Delta x+\Delta\eta)$. Hence the change in volume is given by $\Delta\eta$, and if Δp is the corresponding excess pressure (in this case it will be negative), it follows from definition that the bulk modulus (K) of the medium may be expressed as

$$K=\frac{-\Delta p}{\frac{\Delta\eta}{\Delta x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By combining equations (3) and (4) the *numerical* ratio of particle- and wave-velocities may be written as

$$\frac{v}{c}=\frac{\Delta p}{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Similar reasoning to the above will give the corresponding expression

$$\frac{v}{c}=\frac{\Delta p}{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

for the case of longitudinal waves in a solid rod, where E is Young's modulus and Δp is the applied stress.

Condensation in a compressible medium. The condensation (s) of a medium at any point is a measure of the change of density there due to the passage of a sound wave, and it is defined by $\rho=\rho_0(1+s)$ where ρ_0 is the undisturbed density of the medium.

Hence $$s=(\rho-\rho_0)/\rho_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now the mass of the medium between planes Q_1 and Q_2 (Fig. 5.12)

is the same as that between the displaced positions Q_1' and Q_2' respectively,

i.e.
$$\rho_0 \Delta x = \rho(\Delta x + \Delta\eta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Therefore from (7) and (8) it follows that

$$1+s=(1+\Delta\eta/\Delta x)^{-1} \quad \text{or} \quad s=-\frac{\Delta\eta}{\Delta x} \text{ approx.} \quad . \quad (9)$$

Substituting in (4) the following expression is obtained

$$K=\Delta p/s \quad \text{or} \quad \Delta p=Ks \quad . \quad . \quad . \quad . \quad . \quad (10)$$

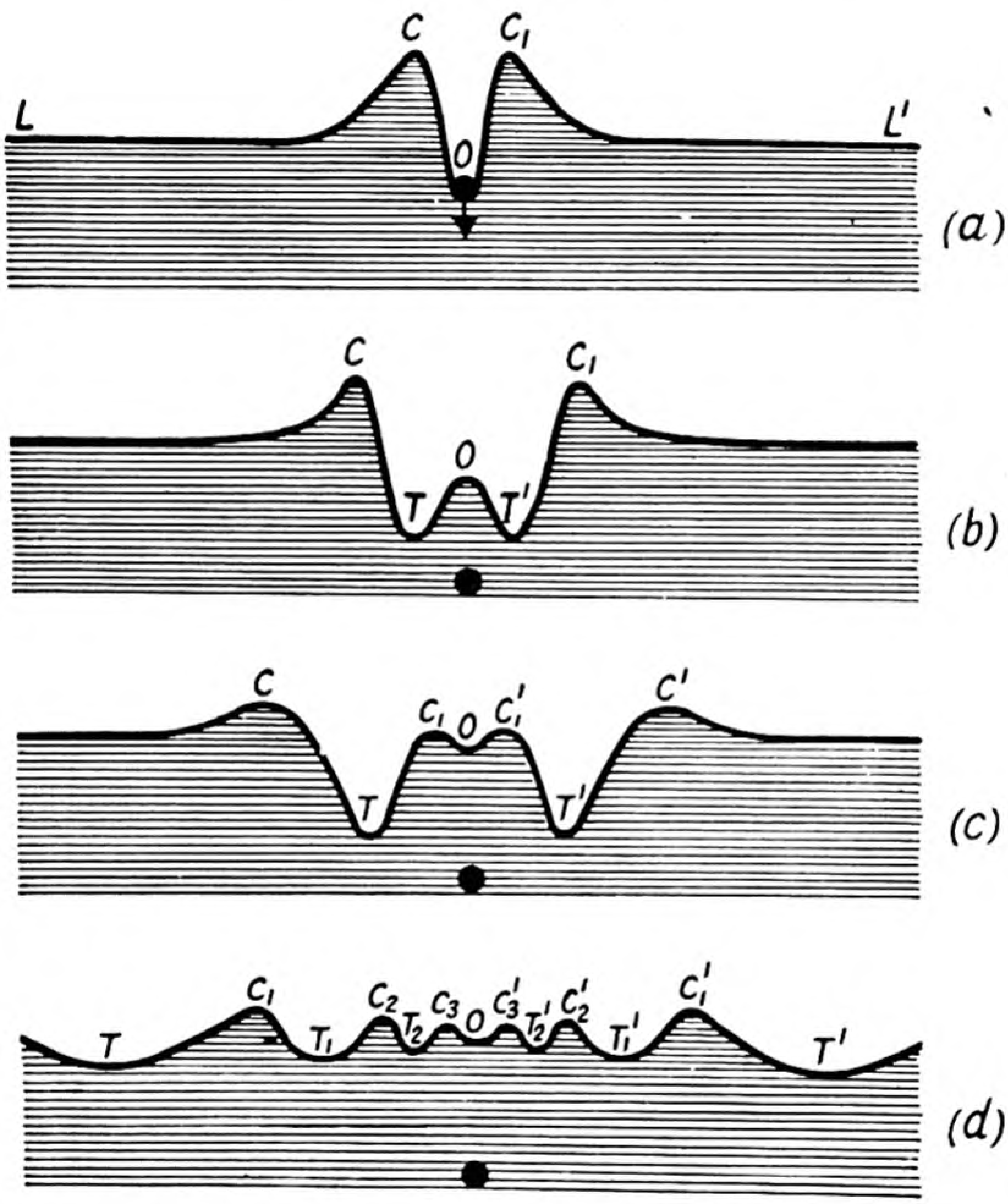


Fig. 5.13.

Water waves

These waves will now be considered in more detail, since they form an instructive means of seeing, as distinct from inferring, many of the properties associated with wave motion.

When a pebble or similar object O (Fig. 5.13) is dropped on to a water surface LL' it tends to pull the water particles near the point of incidence downwards with it, but the initial effect will actually be a "heaping up" as indicated in Fig. 5.13*a*. The sides of the trough created are thus subjected to an increased fluid pressure which tends

to fill the cavity, but owing to its inertia the water is carried past its equilibrium position and the surface assumes the form shown in Fig. 5.13*b*. The continued repetition of this process results in the generation of a small group of waves travelling outwards from *O* (Fig. 5.13 *c* and *d*), the centre of disturbance, with a velocity which is independent of the amplitude. The dependence or otherwise of the velocity of a water wave upon either or both wave-length and depth of water is determined by the conditions prevailing (see below). The waves described above are chiefly controlled by gravity, but a closer

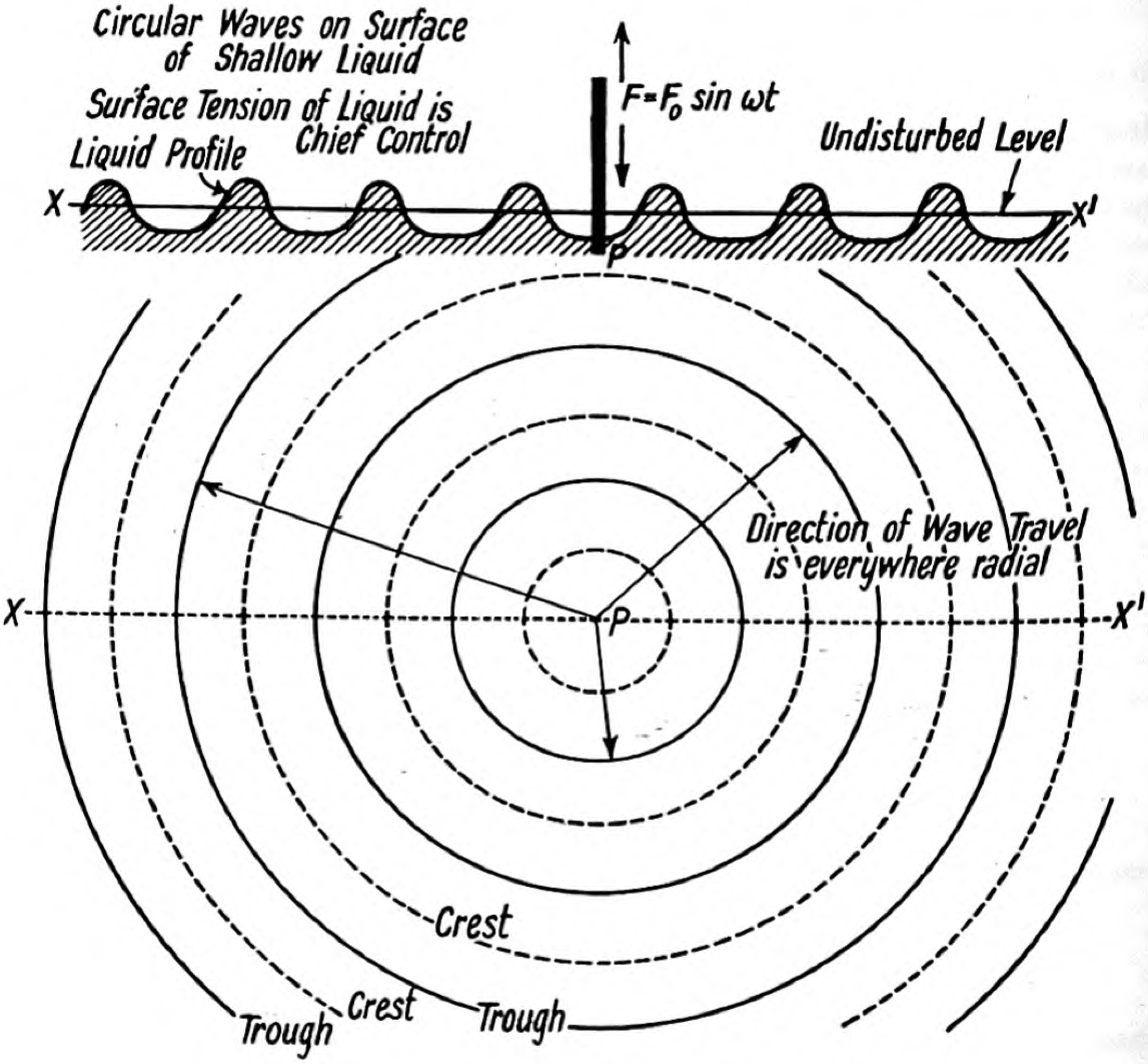


Fig. 5.14.

inspection of the liquid surface would reveal the formation of a number of smaller waves of much smaller amplitude, which are generated immediately the object *O* *touches* the water surface. These *capillary* waves, as they are termed in distinction from the *gravitational* waves. are principally controlled by the surface tension of the liquid-air interface. Fig. 5.14 shows in profile and plan the system of circular waves generated on the surface of a shallow liquid by a thin rod excited simple harmonically to vibrate up and down in the surface. The control of these waves will be essentially due to surface tension, and if they are viewed stroboscopically, *i.e.* by intermittent light of

the same frequency, their wave-length may be measured and eqn. (13), p. 70, used to deduce the surface tension of the liquid.

All surface water (or liquid) waves will be subjected to both gravitational and surface tension control in proportions which vary according to the length of the waves.

If the slope of the water wave and the ratio of its elevation to wave-length are both small everywhere, then the displacements may be considered as simple harmonic functions of the time and horizontal distance. The orbits of the water particles have been stated (p. 65) to be circular, as are shown by the dotted circles in Fig. 5.15. The time taken for the particle at C, the position of a crest, to move down from C to C', the trough position, is one half of the period of the wave and represents the time taken for the wave to advance one half wave-length, *i.e.* half the distance CQ between two crest positions. Furthermore, since the troughs and crests pass periodically any point in the path of the wave, it necessarily follows that the periodic time of the horizontal oscillations of a water particle is the same as that of the vertical oscillations. These water waves possess high and narrow crests with broad and flat troughs.

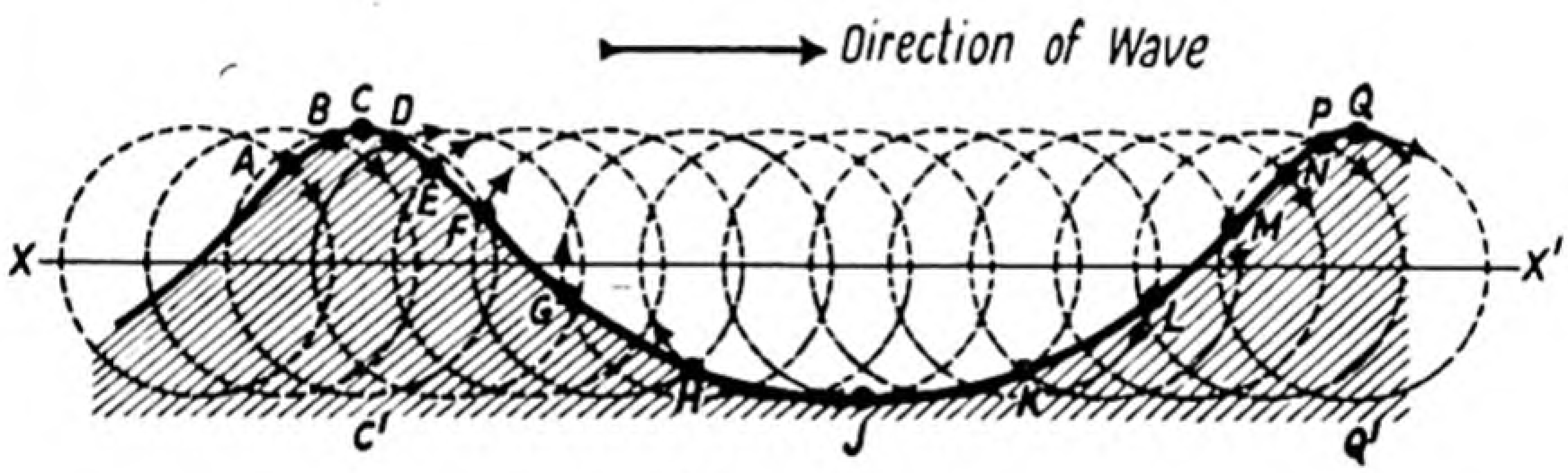


Fig. 5.15.

It is evident from Fig. 5.15 that the continuity of the liquid is preserved by the movement of the water particles from a hollow to a crest, a fact which is borne out by a swimmer who, in a rough sea, experiences a forward urge on the crests but tends to be carried backwards in the troughs. The extent to which particles, initially at rest in a horizontal plane beneath the surface, are disturbed, decreases exponentially with the depth of the plane below the surface; this is indicated qualitatively by the various surfaces b_1b_2, c_1c_2, etc., in Fig. 5.16. Actually the particle amplitude at a depth of half a wave-length is only about 4 per cent. of that at the surface, while at a depth of one wave-length it is reduced to 0·2 per cent.

The classical theory of hydrodynamics* gives the following expression for the velocity (c) of a wave on the surface of a liquid:

$$c^2=\left(\frac{g\lambda}{2\pi}+\frac{2\pi S}{\rho\lambda}\right)\tanh\frac{2\pi h}{\lambda} \qquad . \quad . \quad . \quad . \quad . \quad (11)$$

where S is the surface tension, ρ the density and h the depth of the liquid, and g the acceleration due to gravity.

* See for example Milne-Thomson, L. M.: "Theoretical Hydrodynamics." Macmillan, 1938.

Deep-water waves. Now $\tanh\left(\frac{2\pi}{\lambda}\right)h=0{\cdot}995$ if $\frac{2\pi h}{\lambda}=3{\cdot}0$, i.e. if $\frac{h}{\lambda}=\frac{1}{2{\cdot}1}$ or $h>\lambda/2$, so that the above expression may be then written simply as

$$c^2=\frac{g\lambda}{2\pi}+\frac{2\pi S}{\rho\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which indicates that under these conditions the velocity of propagation is independent of the depth.

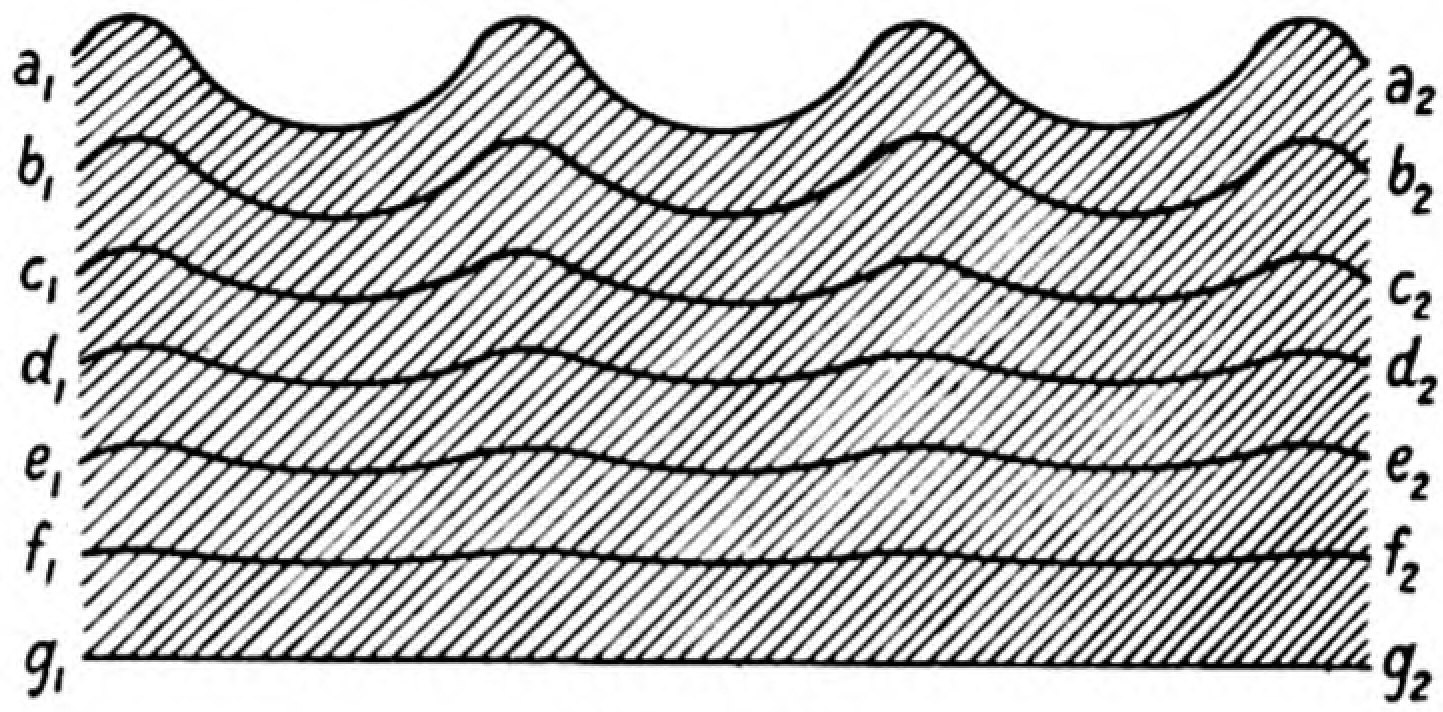


Fig. 5.16.

Shallow-water waves. These are propagated under conditions defined by $\frac{h}{\lambda}$ being small, so that $\tanh \frac{2\pi}{\lambda}h$ may be taken to be $\frac{2\pi}{\lambda}h$ without serious error. This means that $\frac{2\pi}{\lambda}h$ is of the order of 0·1 or 0·2 for tanh 0·1=·0997, and tanh 0·2=·1974. Equation (11) now becomes

$$c^2=\left(g+\frac{4\pi^2 S}{\rho\lambda^2}\right)h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

since $\tanh \frac{2\pi h}{\lambda}$ reduces to $\frac{2\pi h}{\lambda}$.

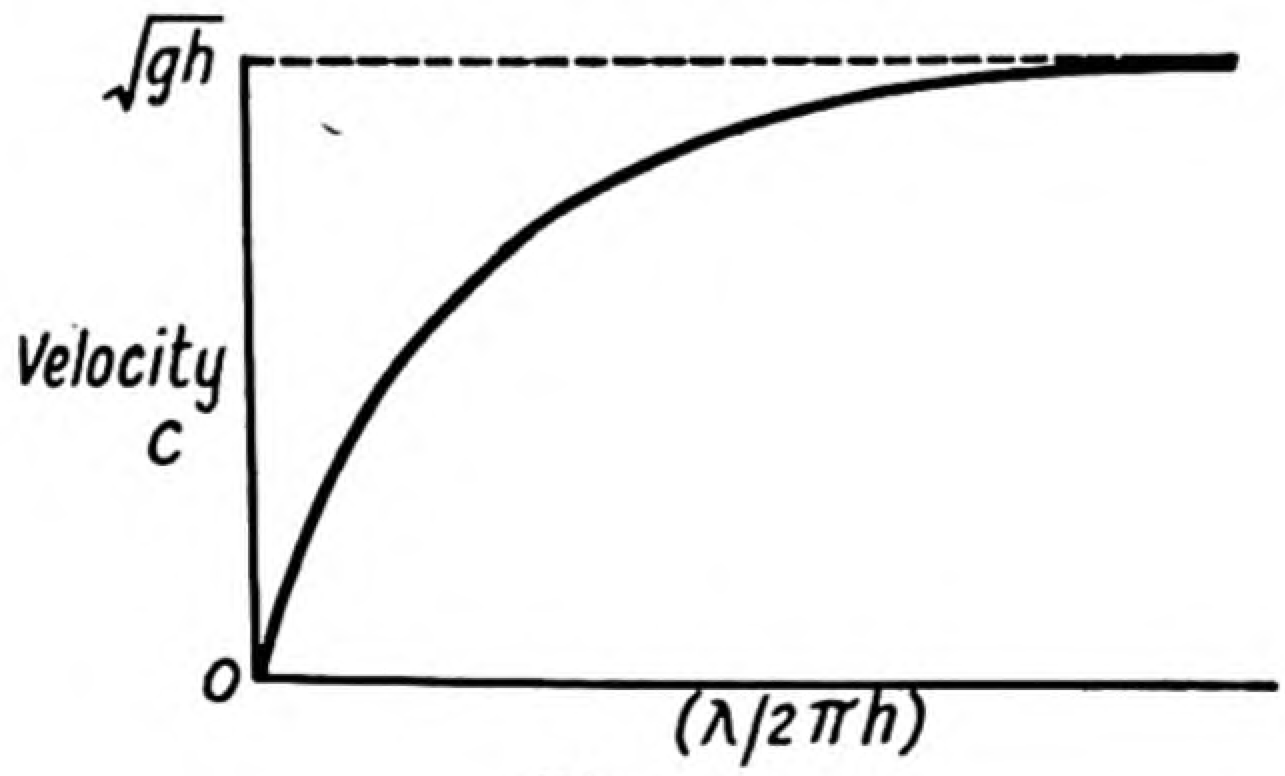


Fig. 5.17.

If, furthermore, λ is greater than a few centimetres in length the second term in the above expression may be neglected and $c=\sqrt{gh}$. These results are shown (for constant h) in Fig. 5.17. Since the velocity of the so-called "long waves" is independent of wave-length they are analogous to sound waves, but it is to be noted that the velocity is

proportional to the $\sqrt{\text{depth}}$. This dependence upon depth is due to the presence of a solid bottom, where the vertical motion of the particles is prevented from taking place and so brings about a modification of the orbits of the neighbouring water particles in the layers above. The penetration of this effect towards the surface will be the greater the smaller the total water depth. Furthermore, the orbits of the water particles become elliptical, with their major axes horizontal, owing to the reduction of their vertical velocities, and at the bottom of the liquid the particles will merely oscillate "to and fro" in a horizontal plane (Fig. 5.18). An extreme example of long waves are tidal currents, in which the horizontal velocity may amount to several feet per second, whereas the vertical velocity may be only a few inches per hour.

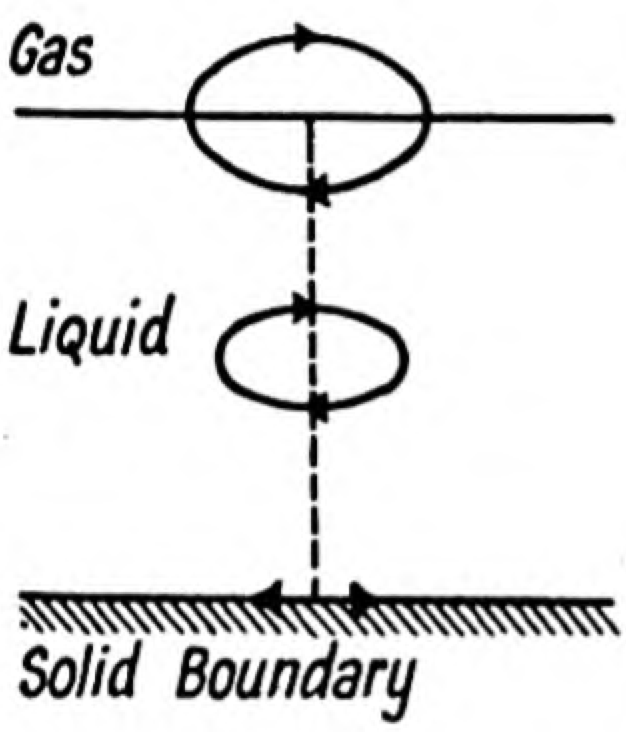


Fig. 5.18.

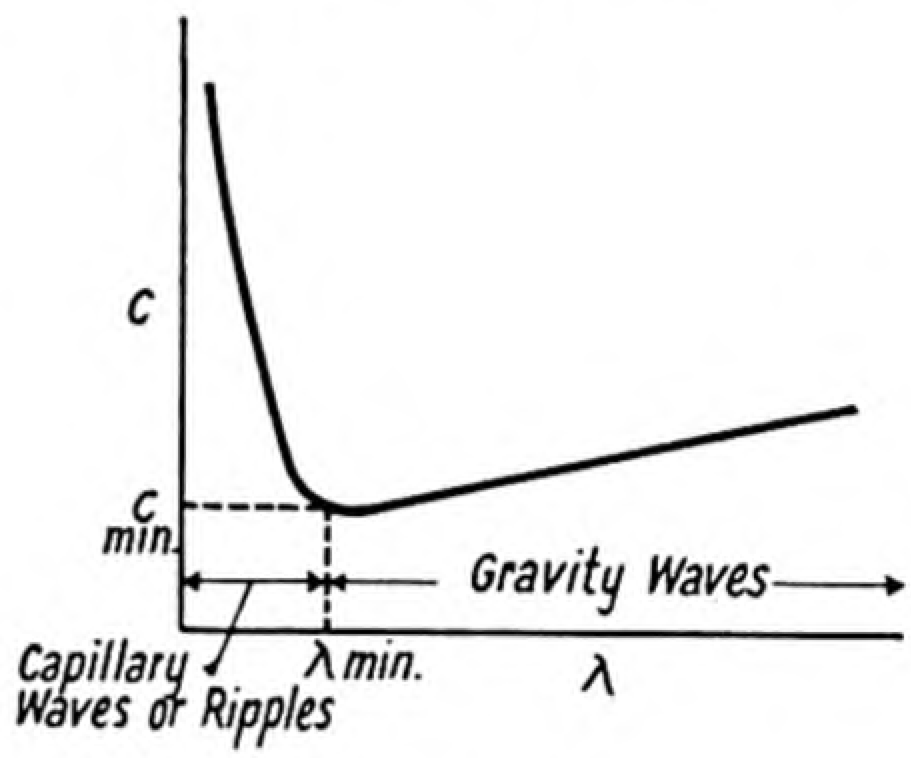


Fig. 5.19.

It follows from (12) that there is a minimum velocity of propagation of surface waves on a liquid surface, which is obtained by differentiating c^2 with respect to λ and equating to zero. This leads to the condition for a minimum that $\lambda^2 = \dfrac{4\pi^2 S}{\rho g}$ and

$$\therefore \quad c^2_{\text{min.}} = \frac{2\pi S}{\rho \lambda_{\text{min.}}} + \frac{g\lambda_{\text{min.}}}{2\pi} = \frac{2\pi S}{\rho} \cdot \frac{\sqrt{\rho g}}{2\pi\sqrt{S}} + \frac{g \,.\, 2\pi\sqrt{S}}{2\pi\sqrt{\rho g}} = \sqrt{\frac{4Sg}{\rho}}$$

$$\text{or} \quad c_{\text{min.}} = \left(\frac{4Sg}{\rho}\right)^{\frac{1}{4}}.$$

In the case of water $\lambda_{\text{min.}} = 1{\cdot}73$ cm. and $c_{\text{min.}} = 23{\cdot}2$ cm. per sec., *i.e.* 0·52 m.p.h. Hence for a wind to *maintain* waves on a water surface its speed must *exceed* 0·52 m.p.h., since there will be dissipation of energy due to viscosity of water.

It is to be noted that surface tension and gravity exercise equal controls at the minimum velocity and the relation between velocity and wave-length is shown graphically in Fig. 5.19. For the shorter wave-lengths defined by $\lambda < 2\pi\sqrt{\dfrac{S}{\rho g}}$ the term *ripples* as suggested by Kelvin is used, while waves whose $\lambda > 2\pi\sqrt{\dfrac{S}{\rho g}}$ are known as *gravity waves*.

For water waves of large amplitude the velocity of propagation becomes greater than the value indicated by classical theory, and the

water particles in the wave actually show a forward movement, *i.e.* their paths are no longer closed. In such cases the profile of the waves is not sinusoidal but may be considered as following the trochoidal curve which would be traced out by a point at a distance a from the centre of a rolling circle, where a is the amplitude of the waves. This circle (Fig. 5.20) has a radius $\frac{\lambda}{2\pi}$ and rolls on the underside of a horizontal line at a height $\frac{\lambda}{2\pi}$ above the mean level of the surface.

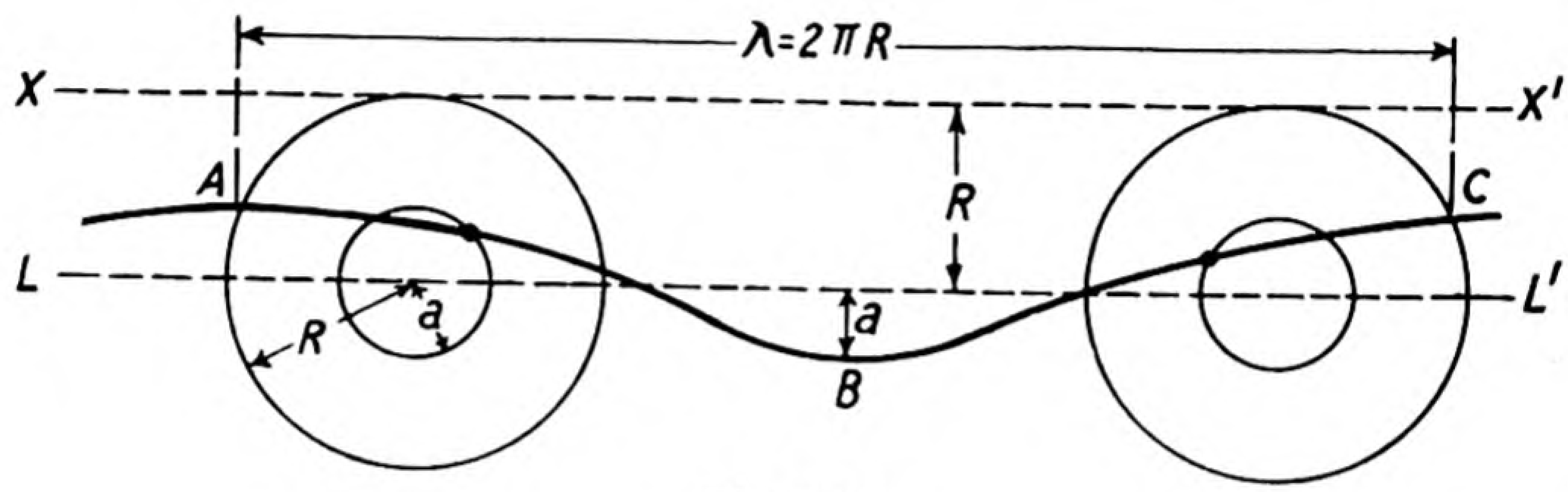


Fig. 5.20.

Group- and wave- (or phase-) velocity

If an ocean wave is observed to approach a sea wall at a slight angle so that the wave-front reaches one end of the wall a short time before it arrives at the other end, the "splash" set up at the point of contact of the wave-crest with the wall is seen to travel along the latter at a speed apparently much greater than the speed of propagation of the incoming waves. This rate of travel of the disturbance along the wall is an example of a *phase*-velocity, for it measures the rate of travel of a certain phase, here the crest, and the reader acquainted with the ultra short-wave technique will recognise the similarity of the above illustration with a certain aspect of the problem of the propagation of electromagnetic waves along a wave-guide. The dependence of the

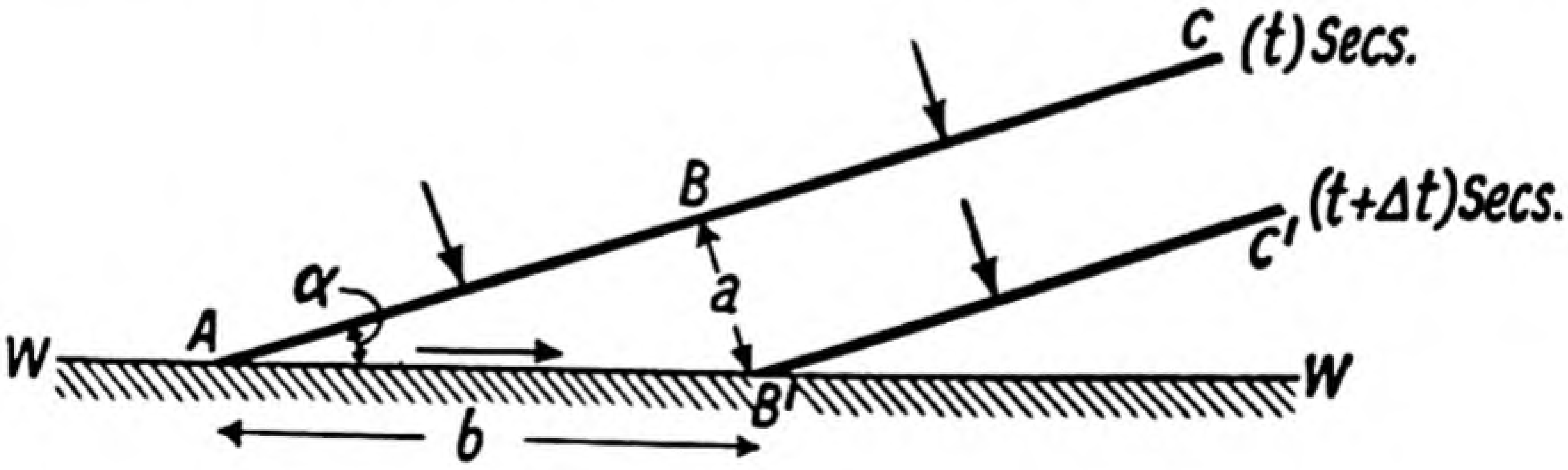


Fig. 5.21.

phase-velocity of the water wave upon the angle between the wave-front and the wall may be easily calculated as follows.

Let ABC and $B'C'$ (Fig. 5.21) represent successive positions of the wave-front at times t and $t+\Delta t$ respectively, and a the perpendicular distance between the two wave positions. Then the velocity of approach of the wave-front towards the wall is given by $U=\frac{a}{\Delta t}$ and

the phase-velocity by $V_p=\frac{b}{\Delta t}$ so that $\frac{V_p}{U}=\frac{\frac{b}{\Delta t}}{\frac{a}{\Delta t}}=\frac{b}{a}=\frac{1}{\sin\alpha}$ which shows that V_p increases as α gets smaller. These two velocities V_p and U correspond respectively to the phase- and group-velocities in an electromagnetic wave-guide, and it is evident that the phase-velocity becomes infinite and the group-velocity (in direction WW) zero when the wave-front is parallel to the wall.

When the surface of still water is disturbed by the dropping of a stone the wave system generated (see Fig. 5.13) can be analysed into a number of simple harmonic components of different wave-lengths. The resultant wave surface (see Fig. 5.22) due to the compounding of two S.H.Ms. of equal amplitude but different wave-lengths is similar in form to that of Fig. 5.13. Now the velocity of propagation depends upon the wave-length so that, in course of time, there will be a grouping of the waves into *sections* of approximately the same wave-length. In observing such a complex wave motion, the attention is focused on the velocity, wave-length, and period of the *resultant* disturbance due to the combined action of the constituent waves. The rate of advance of, say, the maximum of the combined effect measures what is known as the *group*-velocity, which in the case of *gravity* water waves is less than that of the individual waves. Hence if attention be fixed on one of the latter it will be seen to advance through, and to gradually die out as it approaches the front of, the group. The gap thus created is occupied successively by the other waves advancing from the rear. By way of illustrating the difference between group- and wave-velocities, H. Skilling cites the case of a caterpillar which may be considered to move across a surface at a group-velocity. The speed at which the ripples travel along the back of the caterpillar from its tail to its head corresponds to the phase-velocity, which here, as for the gravity waves, is greater than the group-velocity.

This difference between group- (U) and phase- (c) velocities arises from a variation of the velocity of wave propagation with frequency, *i.e.* wave-length, and if this law of variation is known the relation between the group- and wave-velocities for a given system of waves may be deduced.

Diagrams (I) and (II) of Fig. 5.22 show two wave trains *defg* and *DEF* respectively, the latter possessing the longer wave-length. The resultant form at time $t=0$ of the group comprising these two waves is shown in diagram (III), while the form after a time t when the individual waves have advanced by ct and $(c+\Delta c)t$ is shown on the right-hand side of the Fig. 5.22. In diagram (IV) of Fig. 5.22 the wave trains (I) and (II) are shown together on the same time-base and the resultant wave-form in diagram (V). At Dd and Fg the crest D and d and F and g of the two systems coincide, and after a certain time interval τ the crest at e will have caught up with E as indicated by their new positions $e'E'$ on the right-hand side of diagram (IV). The graph wave-form at this juncture is shown below the latter diagram. Obviously the distance Dd and $E'e'$ represents the distance the maximum

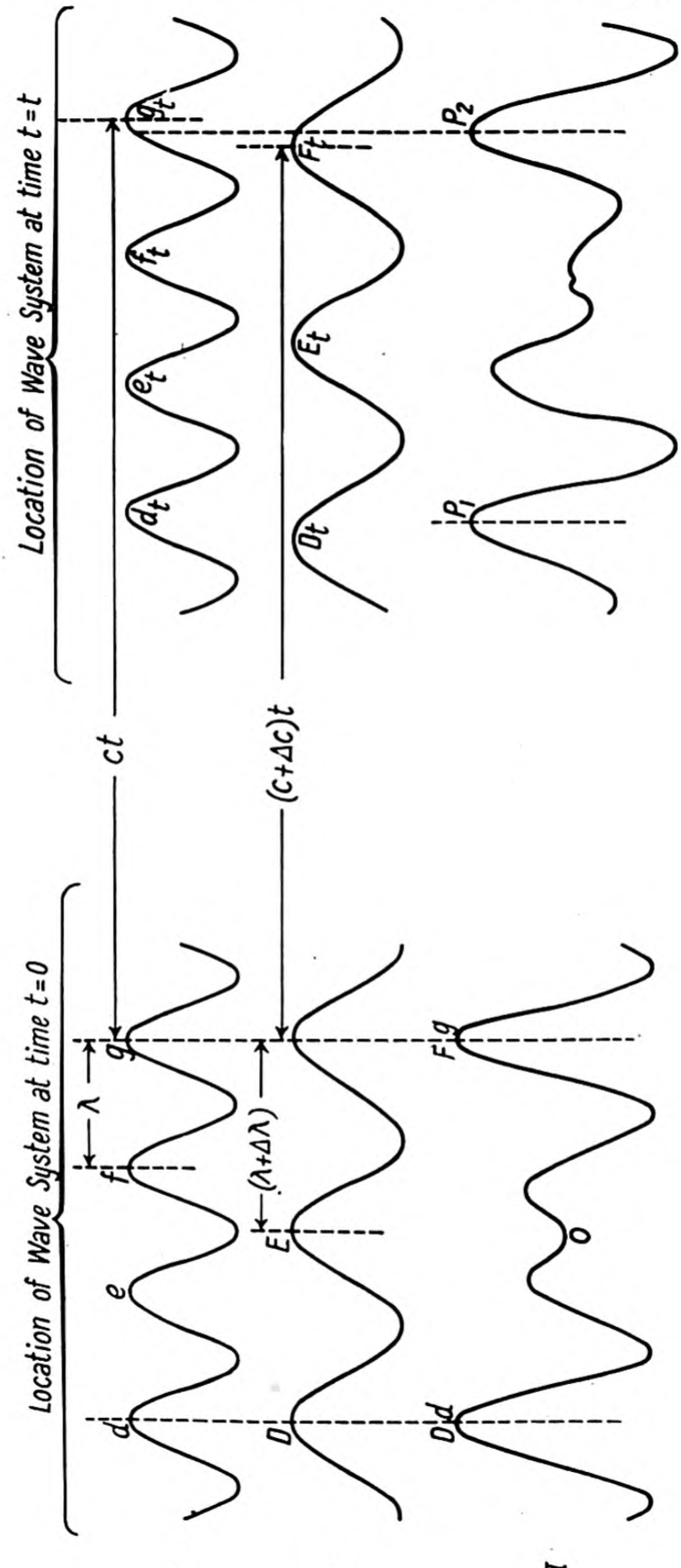
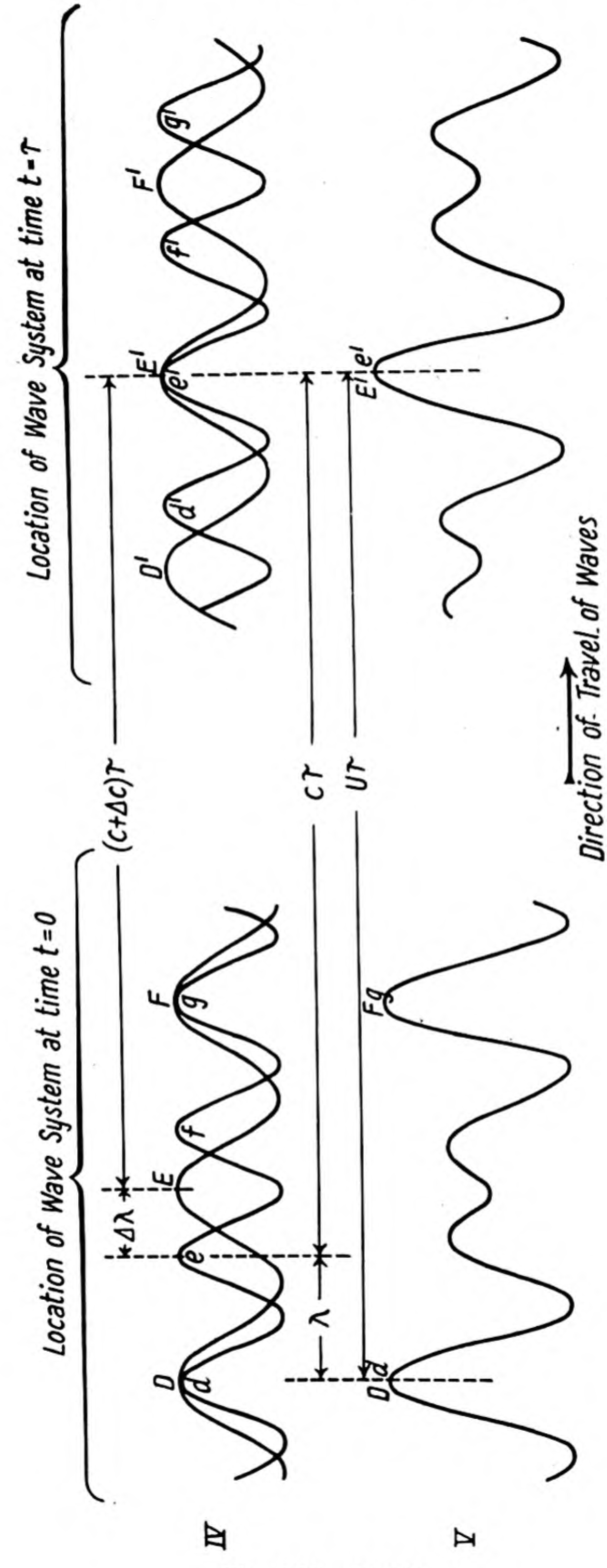

Location of Wave System at time t=t
Location of Wave System at time t=0
ct
(c+Δc)t
λ
(λ+Δλ)
I
II
III

Fig. 5.22. (I-III.)



Location of Wave System at time t=0
Location of Wave System at time t-τ
(c+Δc)τ
cτ
Uτ
Δλ
λ
Direction of Travel of Waves
D
d
e
E
f
F
g
D'
d'
e'
E'
f'
F'
g'
Fg
E'e'
IV
V

Fig. 5.22. (IV–V.)

disturbance of the group has travelled in τ sec., i.e. $U\tau$ cm. where U is the group-velocity in centimetres per second. But from the figure it is easily seen that $U\tau=c\tau+\lambda$ where c is the velocity of the wave of length λ. Also it follows from the diagram that $c\tau=(c+\Delta c)\tau+\Delta\lambda$ where $(c+\Delta c)$ is the velocity corresponding to the wave-length $(\lambda+\Delta\lambda)$. Hence from these two relations it is found that $U=c-\lambda\frac{\Delta c}{\Delta\lambda}$ or in the limit when $\Delta\lambda$ and therefore Δc are very small, we obtain Rayleigh's equation

$$U=c-\lambda\frac{dc}{d\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In the above diagram the phase-velocity has been shown to decrease with wave-length, i.e. $\frac{dc}{d\lambda}$ is negative, and so the group-velocity U is greater than the phase-velocity, which is the state of affairs holding for capillary waves.

If the relation between wave- (or phase-) velocity and wave-length can be written as $V=k\lambda^p$ where k and p are constants, then Rayleigh's equation may be rewritten as $U=(1-p)c$. A few typical examples of the values of p and U for different waves are given below.

Sound waves ..	$p=0$	$U=c$
Deep gravity waves	$p=\frac{1}{2}$	$U=\frac{c}{2}$
Capillary waves ..	$p=-\frac{1}{2}$	$U=\frac{3c}{2}$
Flexural waves ..	$p=-1$	$U=2c$

In the case of light waves, it is not possible to observe the progress of any individual wave in a group and hence all *direct* methods of measuring the velocity of light will necessarily give only the group velocity. It is the group-velocity which determines the rate of transference of energy by a wave train of finite length, and it is this rate which is actually observed in such experiments.

If V_0 and V be the respective *wave*-velocities of light in vacuo and in an optical medium of refractive index μ, then $\mu=\frac{V_0}{V}$ and hence $\frac{dV}{d\mu}=-\frac{V_0}{\mu^2}$. It follows that $\frac{dV}{d\lambda}=\frac{dV}{d\mu}\cdot\frac{d\mu}{d\lambda}=-\frac{V_0}{\mu^2}\cdot\frac{d\mu}{d\lambda}=-\frac{V}{\mu}\cdot\frac{d\mu}{d\lambda}$, and so (14) becomes

$$U=V\left(1+\frac{\lambda}{\mu}\cdot\frac{d\mu}{d\lambda}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

From his early direct measurements of the velocity of light in carbon disulphide Michelson deduced that the refractive index was 1·758, which was considerably in excess of the value 1·635 as obtained in ordinary refractive index measurements. The discrepancy was later seen to be due to the fact that it was not μ which had been calculated, but the ratio of the group-velocities $\frac{U_0}{U}$ which is equal to $\frac{V_0}{U}$, since wave- and group-velocities are identical in vacuo. The application

of (15) showed that the two values were actually consistent with each other. The distinction between group- and wave-velocities is also of considerable importance in the electrical transmission of signals along "lines" or in the atmosphere, the dispersion in the latter case being due to the presence of electrons in the ionised layers.

A very convenient graphical method of deducing the group-velocity when the relation between wave-velocity and λ is known may be noted here, taking as reference Fig. 5.23 which represents rather qualitatively the variation of the velocity of gravity water waves with their length (see also Fig. 5.13). It is easily seen from the diagram that the intercept (TO) on the y axis of the tangent $\left(\frac{dV}{d\lambda}\right)$ to the curve at the point P will be a measure of the group-velocity V_p, corresponding to a length λ_p of the waves under consideration.

If the two waves of Fig. 5.22 differ only slightly in length, but are still of equal amplitude, then the form of the resultant wave becomes

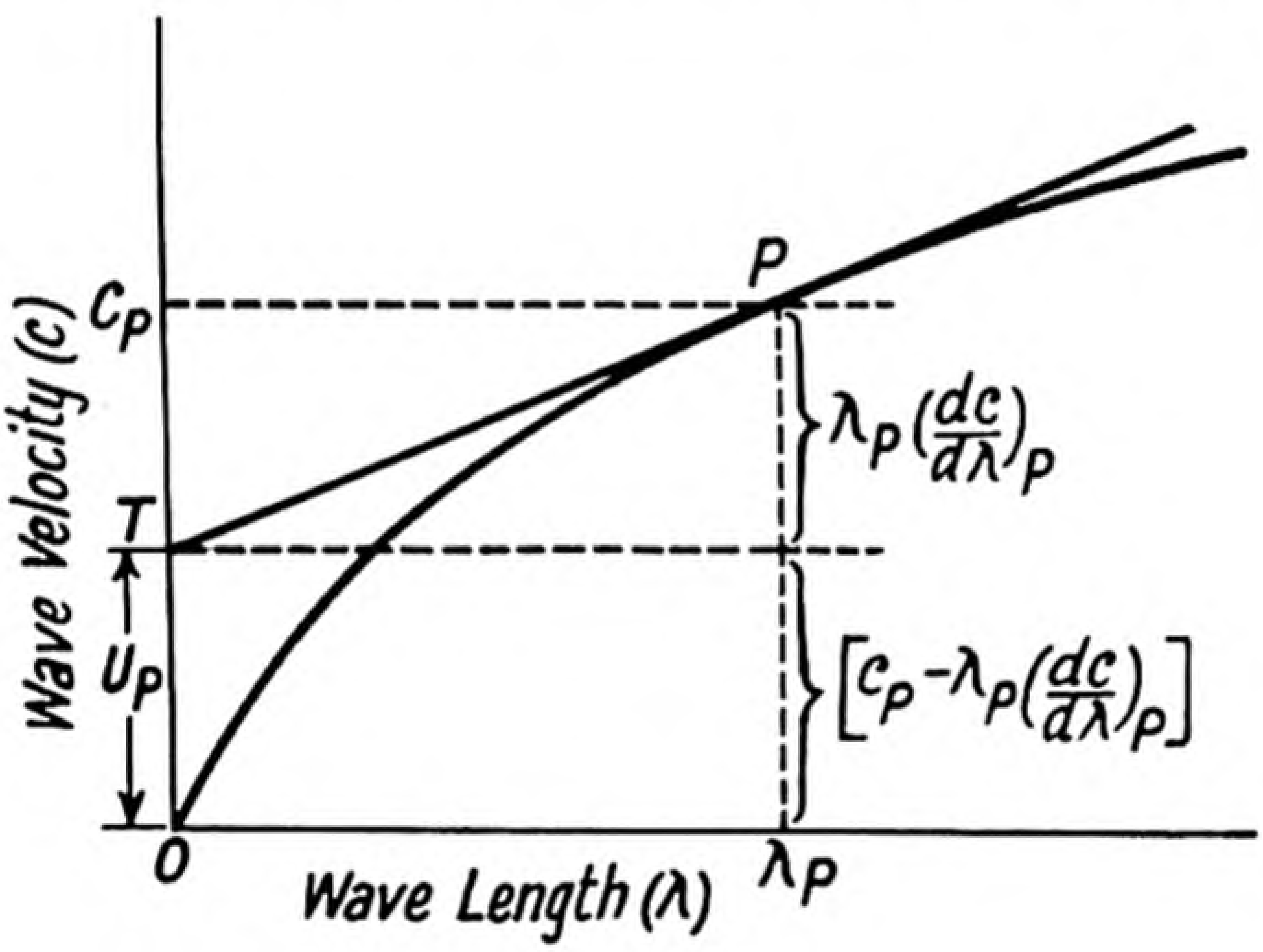


Fig. 5.23.

similar to that shown in Fig. 5.24. The envelope of this wave-form may be recognised as the beat variation of intensity between two sounds of equal amplitude, but slightly different frequency (∴ wave-length), while the corresponding phenomenon in radio is that of the modulated wave analysed into its two side-band frequencies and carrier frequency.

An alternative method of deriving equation (14) will now be given based upon the wave-system depicted above. Suppose that the disturbance is represented by two Fourier components given by

$$\eta_1 = a \sin(kx - \omega t) \quad \text{and} \quad \eta_2 = a \sin\{(k + \Delta k)x - (\omega + \Delta\omega)t\}$$

where Δk and $\Delta\omega$ are very *small* and have their usual significance. Then by the principle of super-position the resultant disturbance is

$$\eta = \eta_1 + \eta_2 = 2a \cos \tfrac{1}{2}(x\Delta k - t\Delta\omega) \sin(kx - \omega t),$$

i.e. $\eta = \beta \sin(kx - \omega t)$ which can be regarded therefore as representing a progressive sine wave of slowly varying amplitude given by

$$\beta = 2a \cos \tfrac{1}{2}(x\Delta k - t\Delta\omega).$$

The dotted line in Fig. 5.24 will be seen to constitute an envelope to the actual wave-form and represents a "wave of amplitude" of length $\lambda' = \frac{4\pi}{\Delta k} = \frac{2\lambda^2}{\Delta\lambda}$.

An instantaneous picture of the wave system will show appreciable lengths of calm stretches interspersed with groups of waves since β will only vary slowly with x at constant t, for Δk and $\Delta\omega$ are supposed extremely small. In the general case of more than two components therefore, the places of maximum β will obviously mean that a certain range of λ's are nearly in phase at these points, brought about by the fact that their different λ's have been compensated by their different speeds of propagation. Hence this particular group of waves must have moved with a group-velocity U such that the component waves, in phase at the origin, are also in phase

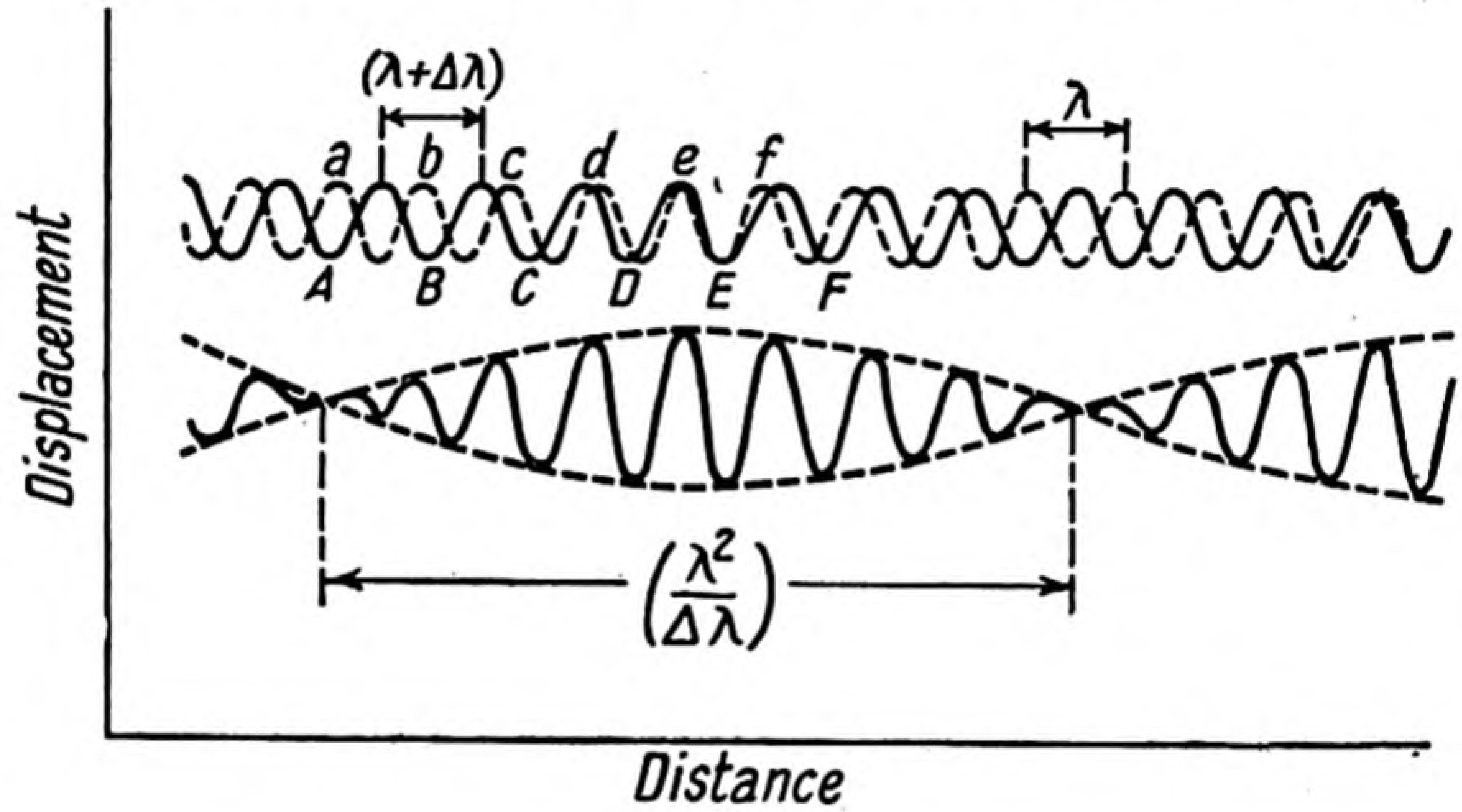


Fig. 5.24.

at a distance x, for the particular instant under consideration. This implies that:

$$kx - \omega t = (k+\Delta k)x - (\omega + \Delta\omega)t$$

or

$$U = \frac{x}{t} = \frac{\Delta\omega}{\Delta k},$$

i.e. in the limit the group velocity $U = \frac{d\omega}{dk}$.

But

$$\omega = kc,$$

$$\therefore\quad U = c + k\frac{dc}{dk} = c + k\left\{\frac{dc}{d\lambda}\cdot\frac{d\lambda}{dk}\right\} = c - \lambda\frac{dc}{d\lambda}.$$

Before closing this section it is opportune to make a brief reference to an example of wave phenomena occurring in atomic physics.

When a beam of electrons impinges on a thin crystalline substance C (Fig. 5.25) it suffers diffraction and gives rise to a pattern on a suitable screen S, in the manner of that produced by light or sound waves after incidence upon a regularly spaced grating. In other words, the electron beam exhibits wave-like properties, and the associated wave-length λ can be deduced from the spacing of the rings in the diffraction pattern D (Fig. 5.25). The wave-length is found to vary *inversely* as the particle velocity V, the latter being suitably varied in an experiment by adjustment of E, the electrical

potential difference through which the electrons (of mass m and charge e) are accelerated. The value of V is calculated from the equation $\frac{1}{2}mV^2=eE$. In the case of a "packet" of electron waves of different lengths, the speed of travel of the centre of the group, the group-velocity V_g, is defined by $V_g=\frac{d\nu}{d\left(\frac{1}{\lambda}\right)}$ where ν is the frequency. It is easily deduced* that for *free* electrons $V_g=V$.

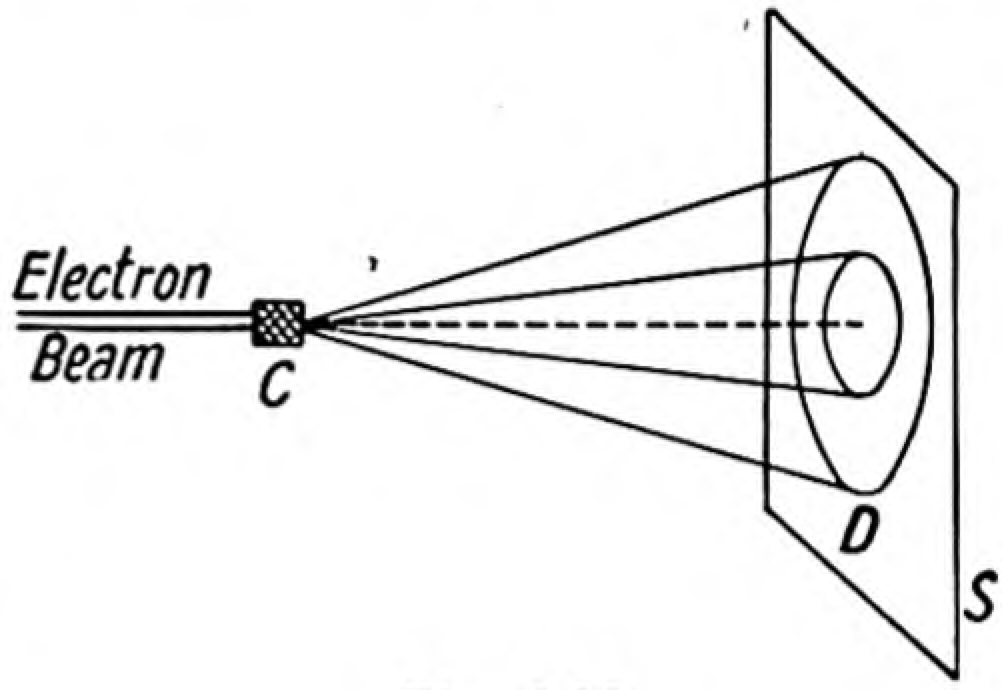


Fig. 5.25.

Flexural vibrations

For a thin wire the diameter is very small compared with its length, and in the elementary treatment of the transverse vibrations of a taut wire stiffness is neglected and only its tension is considered. On the other hand, as the ratio of diameter to length increases, a condition is reached when the stiffness becomes all-important, and the body is then termed a rod or bar, and for the theoretical treatment of this case it is the effect of tension which is neglected.

The restoring forces in the transverse or flexural vibrations of a bar (Fig. 5.26) are due to bending so that, as in the case of longitudinal vibrations, it will be Young's modulus of elasticity which is involved. The equations of motion in the two types of vibration, however, are notably different.

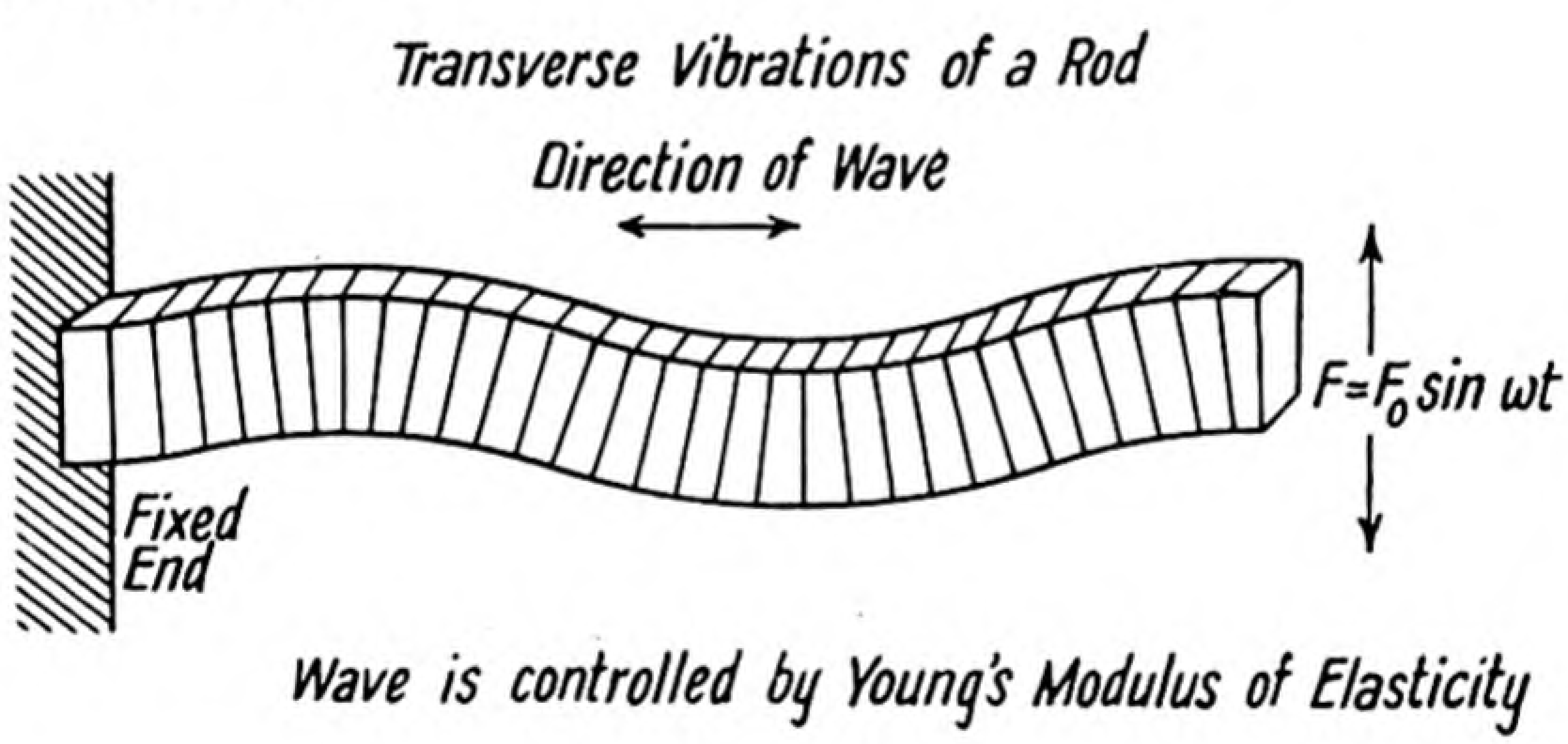


Fig. 5.26.

In order to appreciate the nature of the forces involved the static equilibrium of a uniform bar of regular cross-section, bent by equal forces F applied at the ends (Fig. 5.27), will be considered. In consequence of the applied couple, the longitudinal fibres in the lower part of the bar become compressed and are shortened, while those in the upper part are put in tension and extended. Hence it is evident that since the distortion produced is due to simple extensions and

* Quantum theory states $\lambda=h/mV$, $h\nu=E_T$ and $E_T=\frac{1}{2}mV^2+mc^2+W$, where h is Planck's constant, E_T the total energy, c the velocity of light and W the constant potential energy of a *free* electron. The phase velocity $V_p=\nu\lambda\simeq c^2/V$.

compressions, then the extent of the deformation will be governed by the Young's modulus E of the material concerned. It will be noted that one plane N_1N_2 of the *bent* bar remains unstrained, and this is termed the *neutral surface*. The longitudinal axis of this surface is termed the *neutral axis*, and it passes through the so-called centre of gravity of any cross-section of the rod, the *radius* of curvature of this axis being denoted by R in Fig. 5.27(*a*), and the centre of curvature by C. C is the common centre of curvature for all longitudinal filaments, and the line through C perpendicular to the plane of the paper is termed the axis of bending.

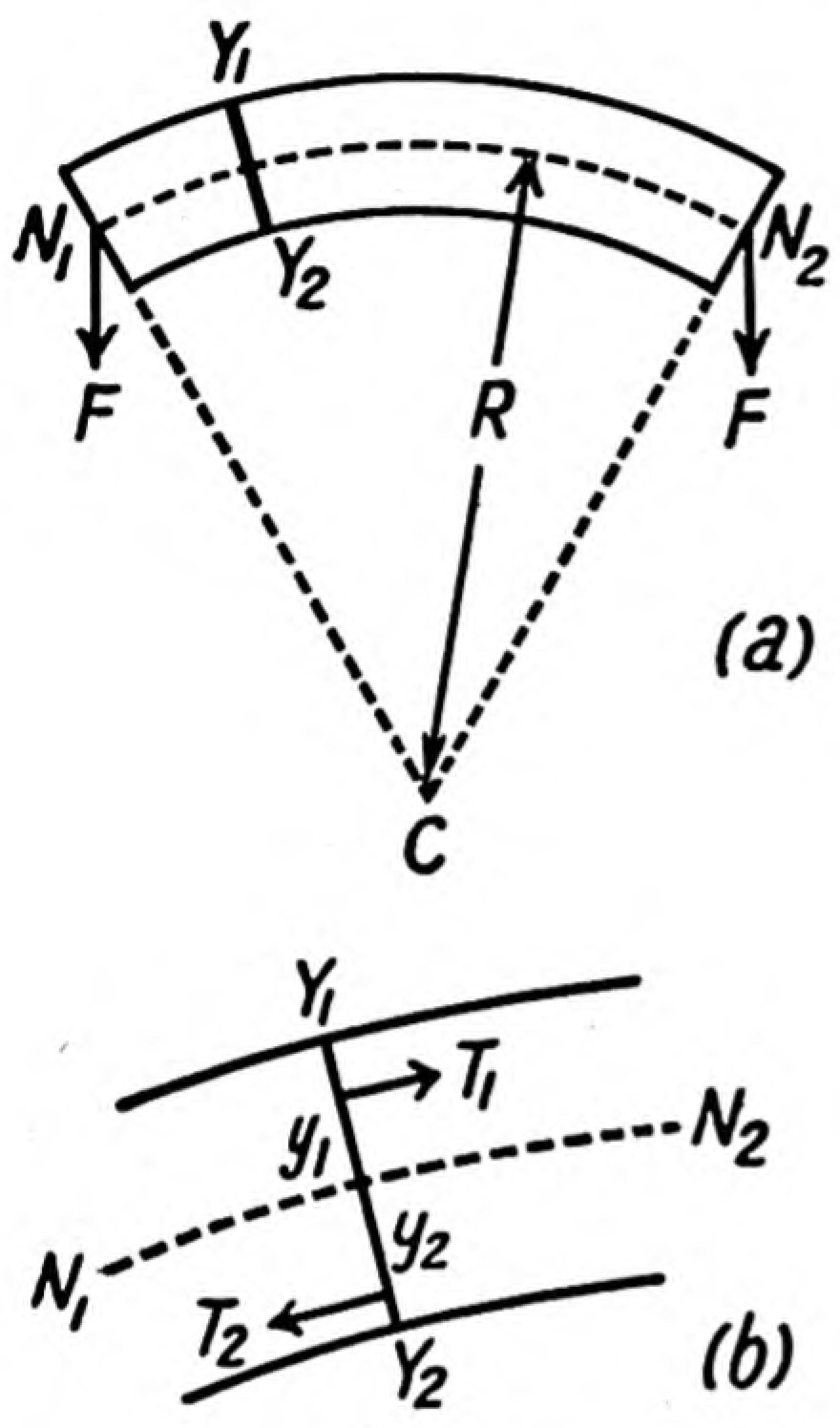


Fig. 5.27.

The internal forces which are brought into play by the straining of the filaments, say at Y_1Y_2 in Fig. 5.27, must constitute a couple equal and opposite to that due to the external forces. The forces above and below N_1N_2 will be tensile and compressive respectively as mentioned above, and the bending moment (G) exerted at Y_1Y_2 will be given by $G=\Sigma Ty$, where y is the distance from N_1N_2 of the filament under stress T. The variation of T across the section of the bent bar may be deduced by considering the strained and unstrained length of the longitudinal filaments lying in the plane X_1X_2. The unstrained length will obviously, by definition, equal those lying in the neutral surface, viz. $N_1N_2=R\theta$, and the strained length (Fig. 5.28) will be given by $X_1X_2=(R+y_1)\theta$.

Hence by definition

$$E=\frac{\text{longitudinal stress}}{\text{longitudinal strain}}=\frac{\frac{T}{a}}{\left(\frac{X_1X_2-N_1N_2}{N_1N_2}\right)}=\frac{\frac{T}{a}}{\frac{y_1\theta}{R\theta}}=\frac{\frac{T}{a}}{\frac{y_1}{R}},$$

where a is the area of cross-section of the *filament*.

It follows that $T=\dfrac{Ey_1a}{R}$ and hence the bending moment at Y_1Y_2 is

$$G=\sum_{\substack{\text{over}\\ \text{cross-section}}} Ty_1 =\Sigma\frac{Ey_1{}^2a}{R}=\frac{E}{R}\Sigma ay_1{}^2 \quad . \quad . \quad . \quad . \quad (16)$$

The quantity $\Sigma ay_1{}^2$ is denoted by I, and it usually signifies a moment of inertia, but in this instance a mass does not enter into the expression, and the meaning here is that of a second moment of *area* of cross-section about the axis in which the neutral surface cuts it. Alternatively I may be written as Ak^2 where A is the area of the uniform

cross-section of the bar and k is the radius of gyration of the section about the above-mentioned axis. For small bending the expression for the radius of curvature may be written approximately as $\frac{1}{R}=\frac{d^2y}{dx^2}$, where (x, y) is the location of any point with respect to the centre of the bar on the neutral axis. x is measured along the bar and y vertically downwards.

Hence equation (16) may be written as

$$G=EI\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Attention is now focused upon an element of the bar between x and $x+\delta x$ (Fig. 5.29), the forces which act on any section being represented by a couple and a single shearing force acting in the

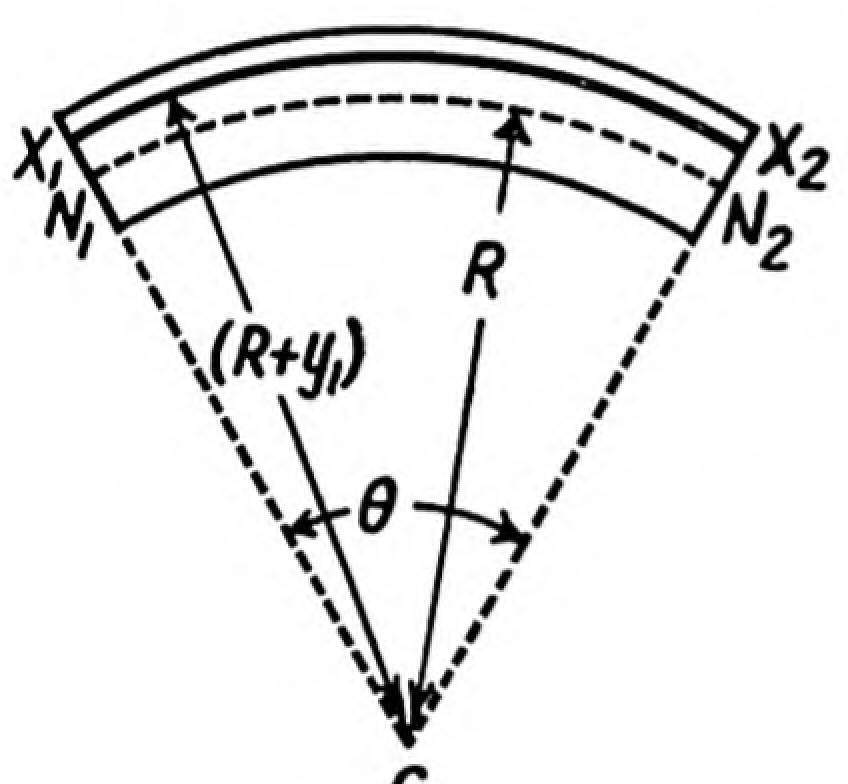


Fig. 5.28.

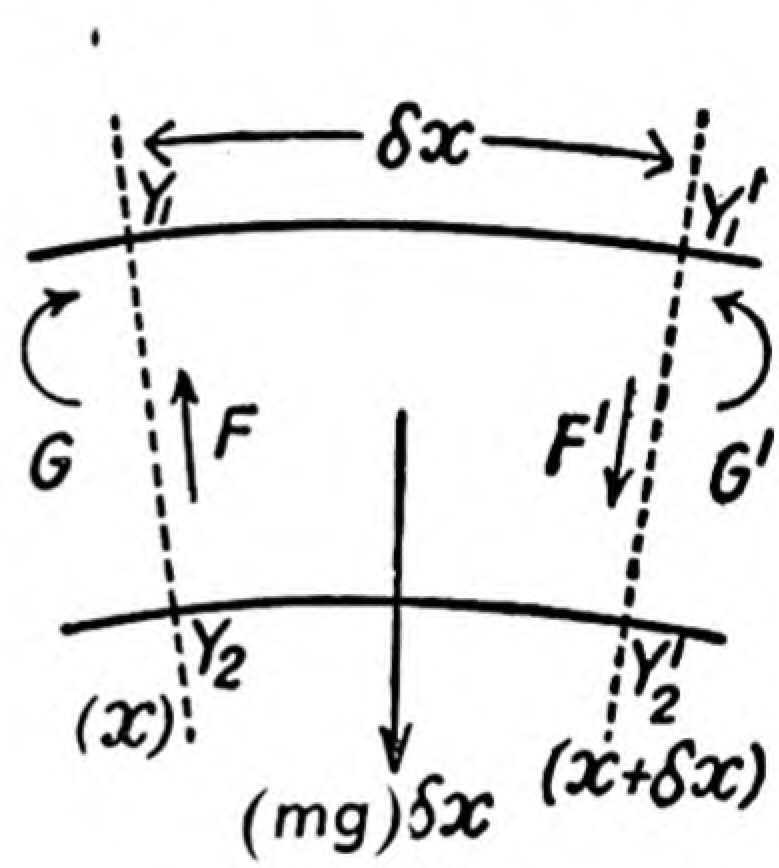


Fig. 5.29.

plane of the cross-section. If m is the mass per unit length of the bar, then taking moments about Y_1' it follows that for equilibrium

$$G+F\delta x=G'+\frac{mg(\delta x)^2}{2}$$

or

$$G'-G=\left(\frac{\partial G}{\partial x}\right)\delta x=F\delta x-\frac{mg(\delta x)^2}{2}.$$

Hence in the limit, assuming the weight of the element to be negligible,

$$\frac{dG}{dx}=F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Also the resultant *lateral* force on the element of the beam is given by $F'+mg\,\delta x-F=F'-F=\left(\frac{\partial F}{\partial x}\right)\delta x$ in the limit,

i.e.

$$\text{lateral force}=\left(\frac{\partial^2 G}{\partial x^2}\right)\delta x=EI\left(\frac{\partial^4 y}{\partial x^4}\right)\delta x$$

from (17) and (18).

It follows that on the release of the impressed forces the resulting acceleration, due to the above lateral forces called into play by the bending, will be given by

$$\frac{\partial^2 y}{\partial t^2}=\frac{-EI\left(\frac{\partial^4 y}{\partial x^4}\right)\delta x}{\rho A\delta x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

or

$$\left(\frac{\partial^4 y}{\partial x^4}\right)+\frac{1}{c^2k^2}\left(\frac{\partial^2 y}{\partial t^2}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where ρ is the density of the material, and $c=\sqrt{\frac{E}{\rho}}$ is the velocity of propagation of longitudinal waves in the medium.

In the above analysis the small angular rotation of the bar caused by bending is neglected.

The constant k^2c^2 occurring in equation (20) does not possess the dimensions of (velocity)^2, since k is a length, and hence the velocity of propagation of a train of harmonic waves along the bar cannot depend solely on elasticity and density, but must depend also on the wave-length. Let the equation of a simple harmonic wave propagated with velocity v in one direction along the bar be given by

$$y=\beta\cos\frac{2\pi}{\lambda}(vt-x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

On substituting from (21) in (20), the latter equation is satisfied if

$$\left(\frac{2\pi}{\lambda}\right)^4\cos\frac{2\pi}{\lambda}(vt-x)-\frac{\left(\frac{2\pi v}{\lambda}\right)^2}{c^2k^2}\cos\frac{2\pi}{\lambda}(vt-x)=0, \text{ i.e. } v=\frac{2\pi ck}{\lambda}.$$

Hence the velocity of propagation of transverse vibrations varies inversely as the wave-length, so that the laws governing these vibrations of an elastic rod are much more complex than those of a string. It follows that in the propagation of a non-harmonic wave the resultant disturbance would be the sum of those due to its harmonic components, each of which, however, would travel with a different velocity owing to their different wave-lengths.

Just as in the case of strings, *standing* lateral vibrations can be set up by the interference of waves reflected from the ends of the bar, and the possible modes of these vibrations depend on the method of supporting or clamping the bar. Now in a standing transverse wave system the nodes are points at which the *displacement* is always *zero*, but the *change of slope* is a *maximum*; it follows that an end fixed in direction is not a true node. Again anti-nodes are places where the largest displacements occur, but where the body (string or bar) is *always* moving *parallel* to its original position, *i.e.* there are no changes of slope involved; it is evident, therefore, that a free end is not a true anti-node. Furthermore, the point of zero lateral motion nearest to a free end is not a true node since the change of slope there is smaller than at the free end itself.

End conditions for a bar of finite length

(i) *Fixed:* for all values of time it follows that

$$y=0 \quad \text{and} \quad \frac{dy}{dx}=0.$$

(ii) *Free:* y and $\frac{dy}{dx}$ are quite arbitrary, but both the bending moment G and the shearing force F must be zero as there is no material beyond the end to produce a couple or transmit a force. It follows therefore from (17) that $\frac{d^2y}{dx^2}=0$, and from (17) and (18) that $\frac{d^3y}{dx^3}=0$.

(iii) *Supported end* (or point along bar) as provided by a rigid knife edge prevents displacement so that $y=0$ always, but any slope may be assumed. The external force imposed by the constraint must of necessity be applied at this point and therefore exerts zero moment about it, i.e. $\frac{d^2y}{dx^2}=0$ from (17). The toy known as a harmonicon consists essentially of a series of thin metal bars, resting on supports, which are set into vibration by striking with a small hammer.

The full solution of equation (20) for the various end conditions is rather beyond the scope of this book and the reader is referred to treatises such as Lamb's "Dynamical Theory of Sound." However, the general results of the problem showing the dependence of the frequency of the lateral vibrations upon the dimensions and material of the bar may be easily deduced as follows.

Let d be distance between two true nodes on the vibrating bar, then the frequency of vibration (f) will be given by $f=\frac{v}{\lambda}=\frac{v}{2d}$ or $f\propto\frac{v}{d}$, but from (p. 82) $v=\frac{2\pi ck}{\lambda}\propto\frac{ck}{d}$, hence it follows that

$$f\propto\frac{kc}{d^2}\propto\frac{k}{d^2}\sqrt{\frac{E}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

For a rectangular bar $k^2=\frac{t^2}{12}$, where t is the thickness of the bar, *i.e.* the dimension *in* the direction of the lateral vibrations. The expression (22) therefore becomes for a rectangular bar

$$f\propto\frac{t}{d^2}\sqrt{\frac{E}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If two bars are vibrating in the same mode, then d is the same *fraction* of the length l of the bar in each case and hence their frequencies are $\propto \frac{t}{l^2}\sqrt{\frac{E}{\rho}}$; this law has received considerable experimental verification.

Summary of properties of lateral vibrations of a solid bar

(*a*) The frequency (f) is $\propto\sqrt{\frac{E}{\rho}}$, *i.e.* to the velocity of propagation of longitudinal vibrations in the bar.

(*b*) The velocity of propagation is dependent on wave-length or frequency, since $f \propto \frac{1}{l^2} \propto \frac{1}{\lambda^2}$, then $v = f\lambda \propto \frac{1}{\lambda}$.

(*c*) The frequency is proportional to the thickness (t) but independent of the width of the bar.

(*d*) The frequencies of the successive modes of vibration of a free-free bar, *i.e.* a bar unclamped at the ends, for example, are $\propto 3^2, 5^2, 7^2, 9^2$, etc., hence the overtones are not harmonic in contrast to

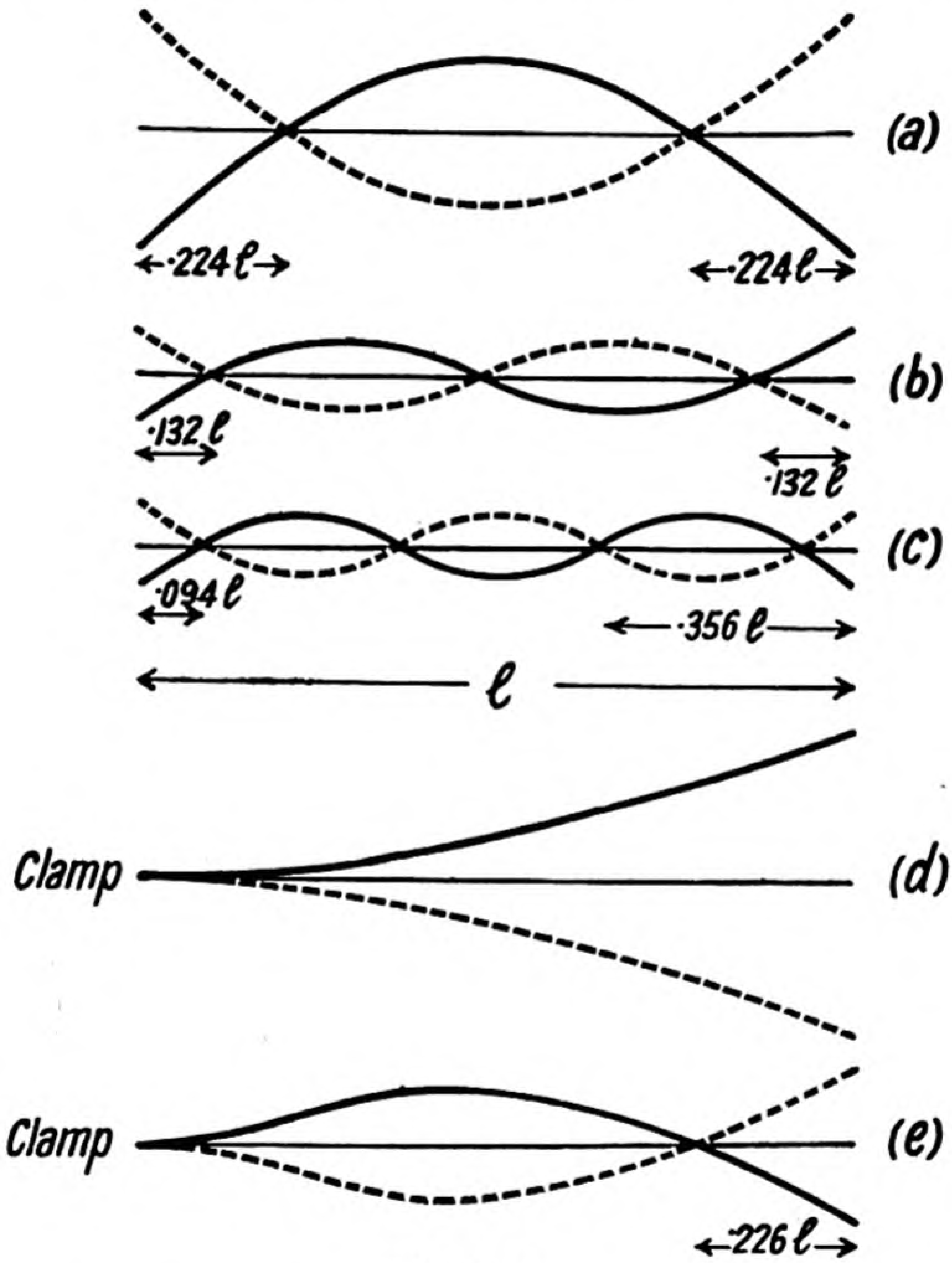


Fig. 5.30.

those of a vibrating flexible string. As in the case of the string, however, the partials actually present in a vibration depend on its method of excitation and on the point of application.

The form of the curves assumed by a free-free bar vibrating transversely in its first three modes are shown in Fig. 5.30 *a*, *b*, and *c* respectively, while *d* and *e* refer to the first two modes of a clamped-free bar.

If a free-free bar is bent, then the two nodes (denoted by crosses in Fig. 5.31) of the fundamental mode of vibration move towards one another, and this tendency increases with the degree of bending; the limiting case in the figure showing two parallel limbs corresponds

to that of a tuning-fork. Chladni found that for a bent bar the mode of vibration with three nodes was impossible, and the first three modes of a tuning-fork are indicated in Fig. 5.31*b*. In practice, a tuning-fork is invariably thickened at the centre of the bend, where the stem is attached, and it has been argued that it is preferable to regard *each* prong as being equivalent to a straight bar fixed at the stem-end but free at the other end. The overtones of a tuning-fork, as to be expected, are not harmonic, but if generated they become less important compared with the fundamental as the amplitude of vibration gets smaller.

The stem is not completely at rest, having a small component vibration in a vertical direction which can be communicated to a table by direct contact. Frequently large forks are mounted on resonance boxes; these consist of boxes open at one end and of suitable volume (Fig. 5.32) for resonance. Amplification resulting from contact with a table is due to the greatly increased area in vibration, despite the much smaller amplitude; the vibrations will, however, die away much more rapidly. A similar argument applies to amplification by a horn.

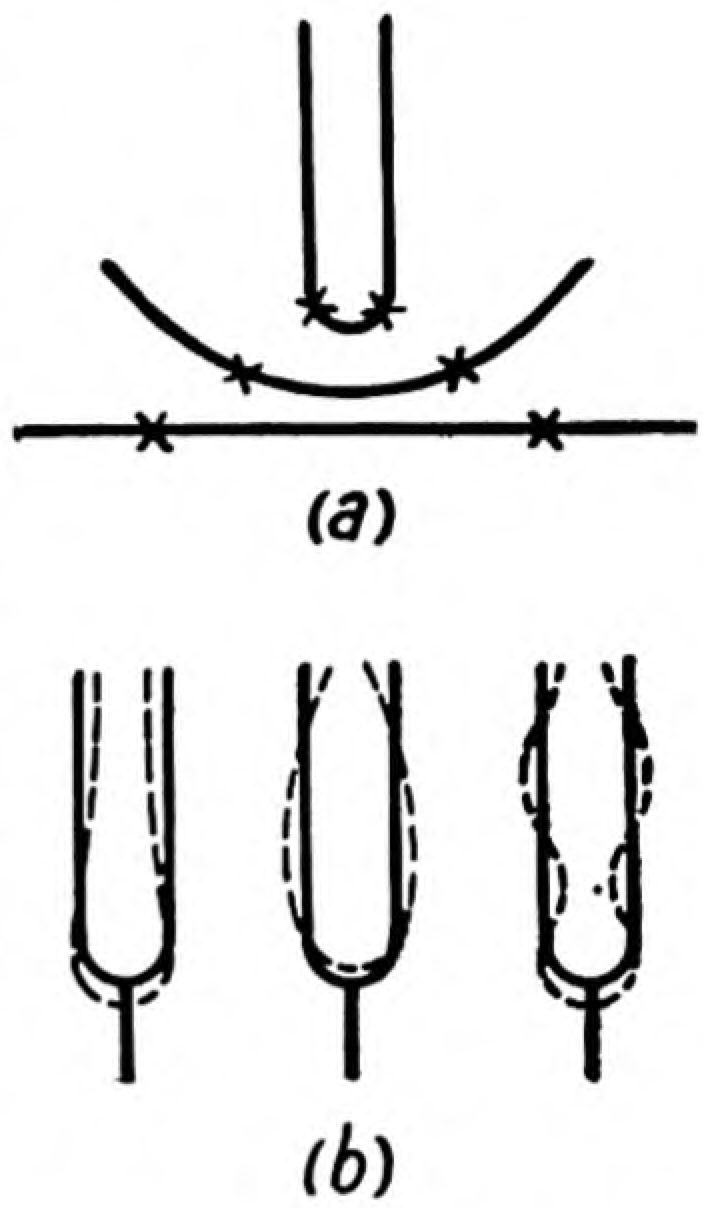


Fig. 5.31.

Occasionally the frequency of response of the supporting table appears to be the octave of the vibrating fork. This has been explained by assuming the prongs to be accurately parallel; such prongs have a small downward component as they move inward from the central position, which is reversed as they move outward (Fig. 5.33). After passing the central position, the components are downward, but on the return of the prongs the components are upward. These components are transmitted to the table via the stem, giving a frequency double that of the fork. The effect is eliminated by inclining the prongs towards each other.

The frequency of a tuning-fork may be adjusted by a small sliding weight on each prong which can be fixed in any desired position. The load increases the inertia and lowers the frequency. This also occurs if one prong only is loaded, but the motion is eccentric.

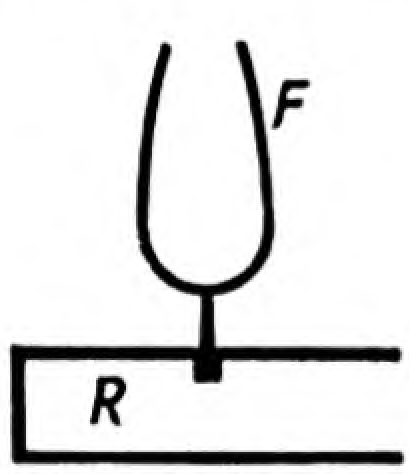


Fig. 5.32.

The vibrating membrane

The ideal membrane is assumed to be a uniform and infinitesimally thin solid lamina possessing perfect flexibility, so that its motion is essentially controlled by the applied tension, and is practically independent of the elastic constants of the material. A good approximation to an ideal membrane clamped around its periphery is provided by the vibrating soap film shown in Fig. 5.34, the film in this case covering the opening of a resonator. The modes of vibration of the film when set into motion by aerial vibrations are shown in Fig. 5.34 *a* and *b* and refer

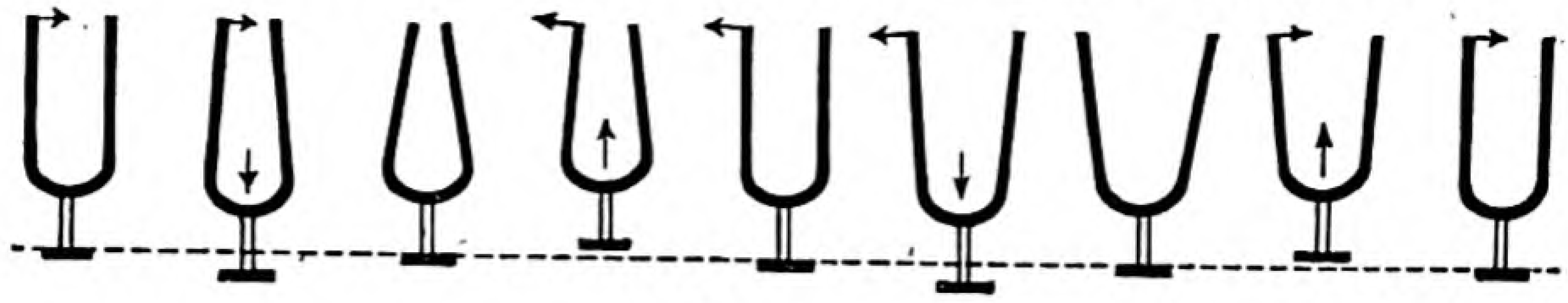


Fig. 5.33.

to frequencies of 1200 and 650 c.p.s. respectively. The drum is a notable example of a membrane, the stretching frame providing the necessary tension, while the human ear affords an interesting case of a membrane performing *forced* oscillations under the impact of sound waves. In the kettledrum the membrane closes a hollow hemispherical resonator, and the tension adjustment is by means of screws around the periphery, which permits a range of pitch control of nearly a fifth.

The ideal properties assumed for a membrane suggest that it may be regarded as an assembly of strings parallel, say, to x_1Ox_2 (Fig. 5.35), but the film is also, in general, curved in a perpendicular direction and hence there is the additional effect of a series of filaments parallel to y_1Oy_2 to be considered. In fact, if the thickness of the membrane is neglected the problem of its dynamic equilibrium may be solved by reference to the analogous case of a soap film. Let the rectangular element $ABCD$ of the membrane be chosen sufficiently small so that the forces $T\delta y$ and $T\delta x$ may be regarded as acting at the centres of the sides and tangential to the membrane, T being the uniform tension (measured, like surface tension, as a force per unit length) which is stretching the membrane. It follows from surface tension theory that these tangential forces are together equivalent to a normal force on the membrane equal to $T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\delta x\,\delta y$, where R_1 and R_2 are the principal radii of curvature at O, *i.e.* the radii of curvature in the vertical planes containing x_1Ox_2 and y_1Oy_2 respectively. If m be the mass per unit area of the membrane whose acceleration at any instant is given by $\frac{d^2z}{dt^2}$, then the equation of motion of the membrane may be written as (*cf.* vibrating string, p. 50)

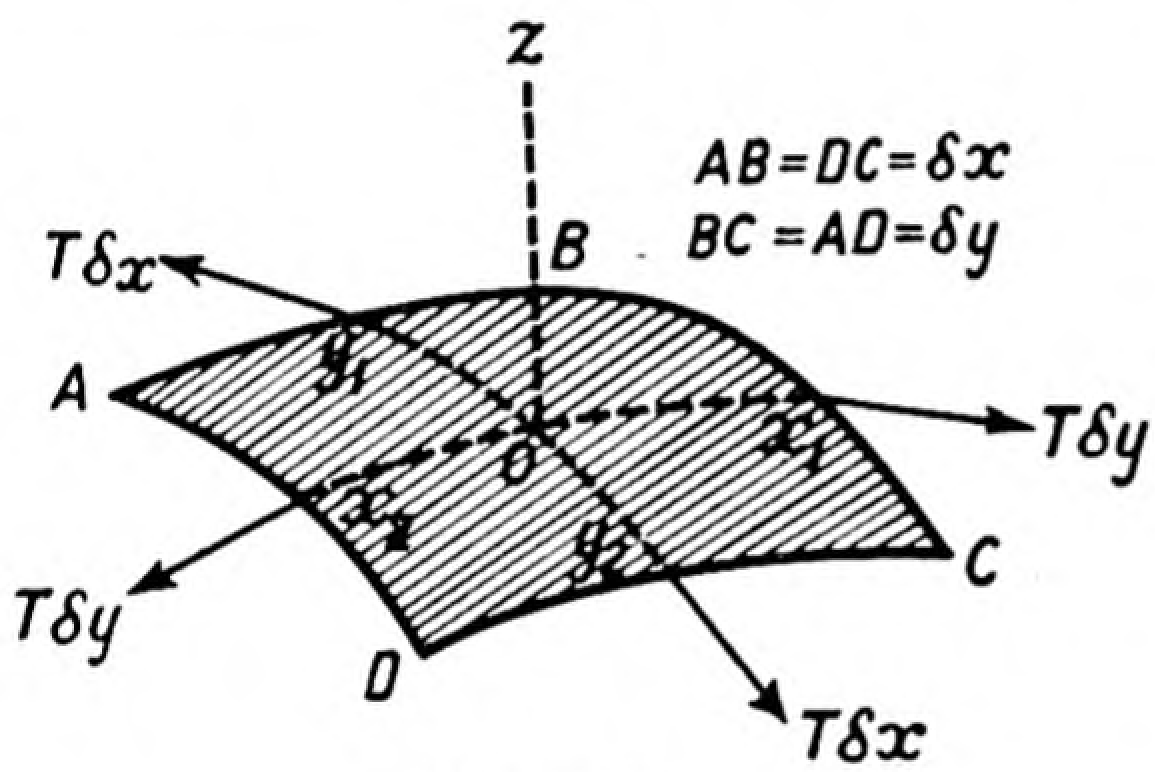
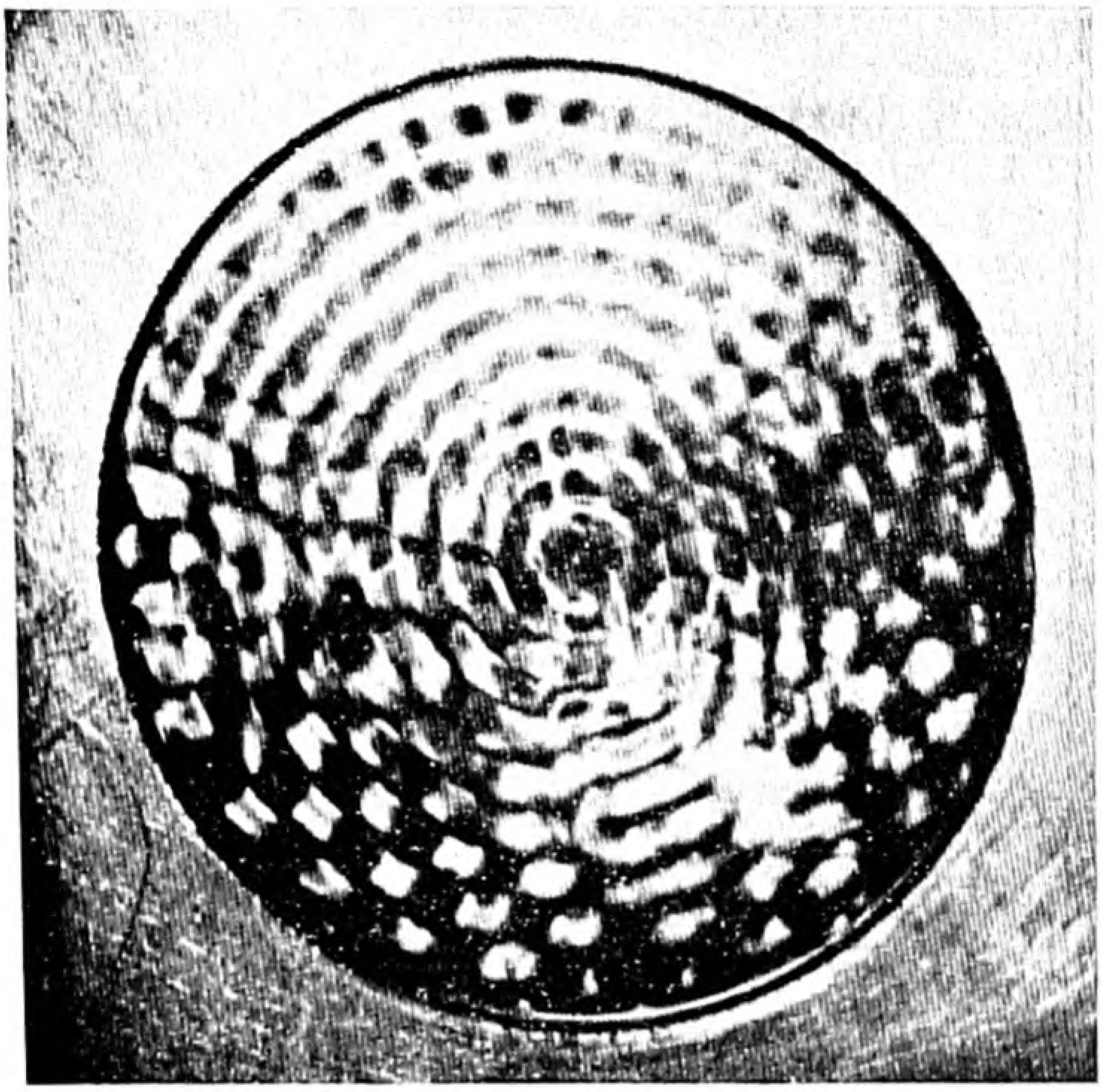


Fig. 5.35.

$$m\,.\,\delta x\,\delta y\,.\frac{d^2z}{dt^2}=T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\delta x\,\delta y=T\left(\frac{d^2z}{dx^2}+\frac{d^2z}{dy^2}\right)\delta x\,\delta y,\ \text{approx.}$$

i.e.
$$\frac{d^2z}{dt^2}=\frac{T}{m}\left(\frac{d^2z}{dx^2}+\frac{d^2z}{dy^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This is the standard form of wave equation for motion in two dimensions, and the velocity of propagation of waves along the



(a)

[Robinson and Stephens, Phil. Mag.

(b)

Fig. 5.34.

membrane is therefore given by $C_M=\sqrt{\frac{T}{m}}$. Although the motion of a *bounded* membrane may be very complicated, yet it can be analysed into a large number of wave trains travelling to and fro over the surface with this speed C_M. The frequencies of the permitted vibrations are strictly determined from the boundary conditions and are sometimes called "characteristic" values, but they are probably better known by the German designation of *Eigenwerte*. This latter term is a familiar one in advanced atomic theory, being first employed by Schroedinger in his formulation of wave-mechanics. The circular membrane with its edge clamped provides a greater interest for the physicist than the square or rectangular membrane, but to derive its various modes of vibration equation (24) is more conveniently expressed in polar coordinates with the centre of the membrane as origin. The mathematical analysis is too difficult to attempt here, but the general pattern of the nodal lines may be stated to consist of radial lines from the origin, together with a system of nodal circles based upon the origin as centre. The clamped outer edge of the membrane must obviously coincide with one of these circles as is to be noted from Fig. 5.34.

Vibrations of diaphragms and plates

When a membrane possesses appreciable stiffness it approximates to a thin plate; elastic control now becomes important and with thicker plates the tension factor is negligible by comparison. The mathematical analysis of vibrating plates is very difficult, although their modes of vibration can easily be demonstrated by experiment. The simplest case is the diaphragm or circular disc rigidly clamped at its edge, and Lord Rayleigh derived the following expression for its fundamental frequency of vibration:

$$N=\frac{2{\cdot}96}{2\pi}\cdot\frac{h}{r^2}\sqrt{\frac{E}{\rho(1-\sigma^2)}} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where r is the radius and h the thickness of the diaphragm, E is Young's modulus, σ is Poisson's ratio, and ρ is the density of the material of the diaphragm. The above expression may also be written as

$$N=\frac{2{\cdot}96}{2\pi}\cdot\frac{h}{r^2}C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where C is the velocity of propagation in a thin plate of the same material and thickness but infinite in extent.

Chladni (1787) made the first experimental investigation of the vibrations of plates by clamping the plate at a centre of symmetry and simultaneously bowing and touching selected points. The simplest mode of vibration, *i.e.* when the plate is vibrating at its fundamental frequency, is that in which there are four segments and four corresponding nodes. The patterns are usually exhibited by means of fine sand dusted on the surface of the plates, the sand arranging itself along the nodal lines. Savart discovered that very fine powders such as lycopodium collected at the anti-nodes and not the nodes, and this effect has been further investigated by Andrade.

Chladni's laws of vibrating plates follows immediately from the above formula (25) of Lord Rayleigh. If two plates show the same nodal patterns then (i) if they are of similar shape, their frequencies vary directly as their thicknesses, and (ii) if they possess the same thickness then their frequencies will vary *inversely* as the *square* of their diameters.

A general idea of the principles underlying the development of Chladni figures may be obtained by a method due to Wheatstone, in which the plate is regarded as being built up from a series of parallel thin bars as indicated by equidistant lines in Fig. 5.36 *a* and *b*. C is the clamp at the centre of the plate, so that on the above supposition if the plate is bowed at B_1 or B_2 and damped at D_1 or D_2, it would appear in its fundamental mode, to perform lateral vibrations in the manner of a bar clamped at its centre. The lower sketch of Fig. 5.36*a* indicates the displacement at the edge of the plate (O being the nodal point) when looking in the direction D_2CD_1. If the bowing is too strong there is a danger of the higher frequencies being elicited, although these tend to die out more quickly than the fundamentals. On interchanging the bowing and damping points (Fig. 5.36*b*), a similar type of vibration to that observed above (Fig. 5.36*a*) is again noted when viewed in the direction D_2CD_1. The resultant form of pattern obtained by superposing the figures of Fig. 5.36 *a* and *b* is shown in *c*, which is self-explanatory, and it may be experimentally realised by bowing at either corner B_1 or B_2 and touching an *adjacent* corner (D_1 or D_2). It is to be noted with any form of pattern that adjacent segments move in opposite directions, and consequently there are always an *even* number of segments.

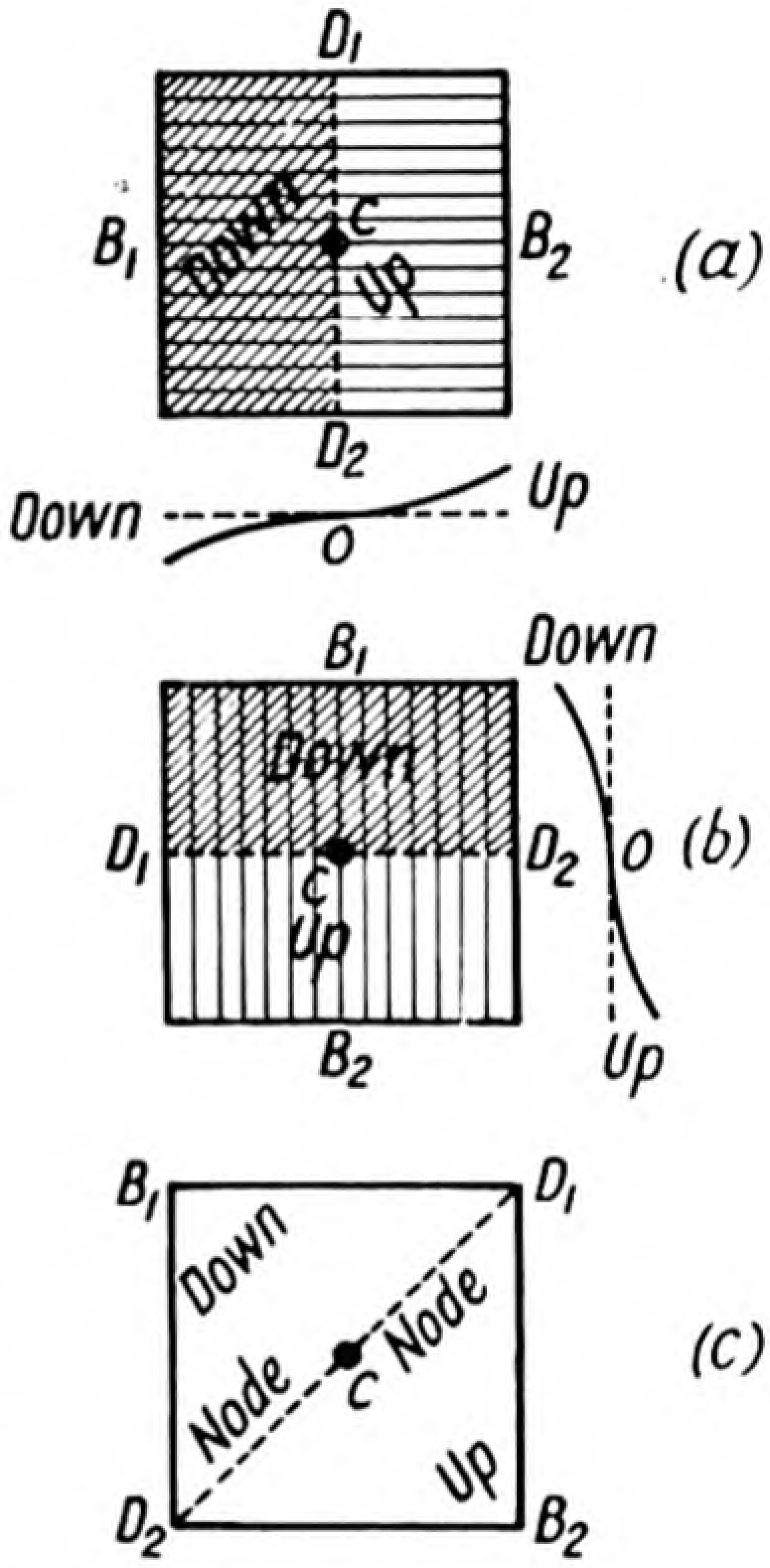


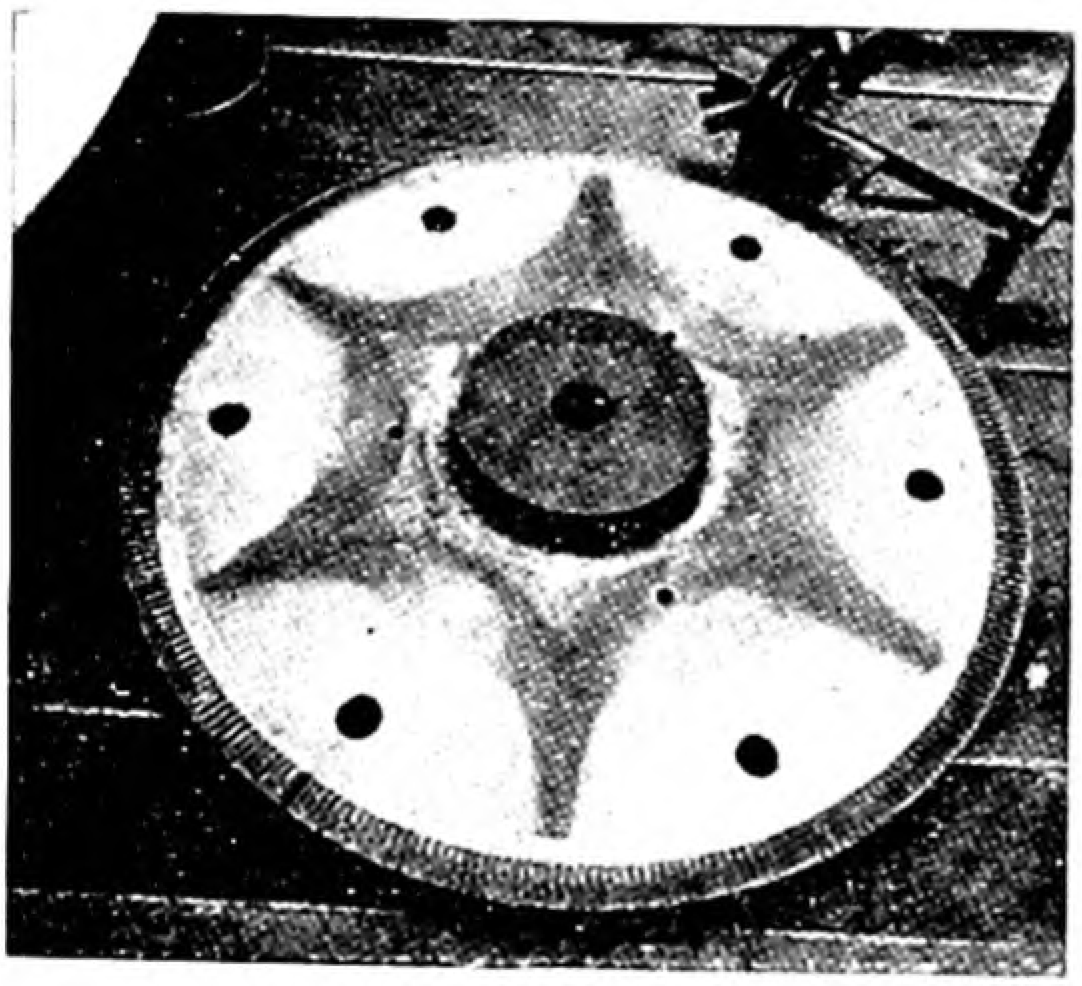
Fig. 5.36.

Besides the familiar method of bowing there are two other methods of exciting the vibrations of a plate. If the latter is composed of iron, or has a suitably small insert of this metal, then electromagnetic excitation may be used. The exciter will be a small electromagnet carrying an alternating current of a frequency appropriate to the plate, and hence in practice the frequency of the source should be variable. This method is shown in Fig. 5.37 applied to the determination of the natural frequencies of a disc which carries the blades of a steam turbine. The resonance is elicited by means of the A.C. magnet shown in the right-hand top corner, the pattern being rendered visible by means of a fine powder.

The other method is one due to Mary Waller and consists in bringing solid carbon dioxide, of sufficiently high density, into contact with

a metallic plate. The solid CO_2 sublimes at $-80°$ C., so that when brought into contact with the warmer metal considerable local gas pressures are developed, due to the rapid formation of gas. An impulse is therefore imparted to the plate, and it is set into and maintained at its natural frequency by receiving additional impulses at each approach to the solid CO_2. The method is very simple and effective and requires no adjustment of frequency as the excitation agent possesses no natural period of its own. Dr. Waller's attention was first directed to the phenomenon by an ice-cream vendor, who showed how he could cause his bicycle-bell to chatter by bringing a piece of "Drikold" into contact with it.

All good thermal conductors may be excited in this manner, and so the method provides a means for the geologist to differentiate rapidly between semi-conductors and insulators. The detection of the lack of uniformity in metal plates by the corresponding distortion of the Chladni figures is another interesting application of the technique. It is important to note that with the solid CO_2 method of excitation the object under examination need not be rigidly supported, and so is particularly appropriate for the investigation of the modes of vibration of an irregularly shaped body like a spanner which cannot be easily, if at all, deduced mathematically. The range of frequencies most easily excited by this method is between 1000 and 4000 c.p.s. Figs. 5.38 *a* and *b* were obtained by Dr. Waller using the solid CO_2 method, the nodal lines being rendered visible by lycopodium powder.

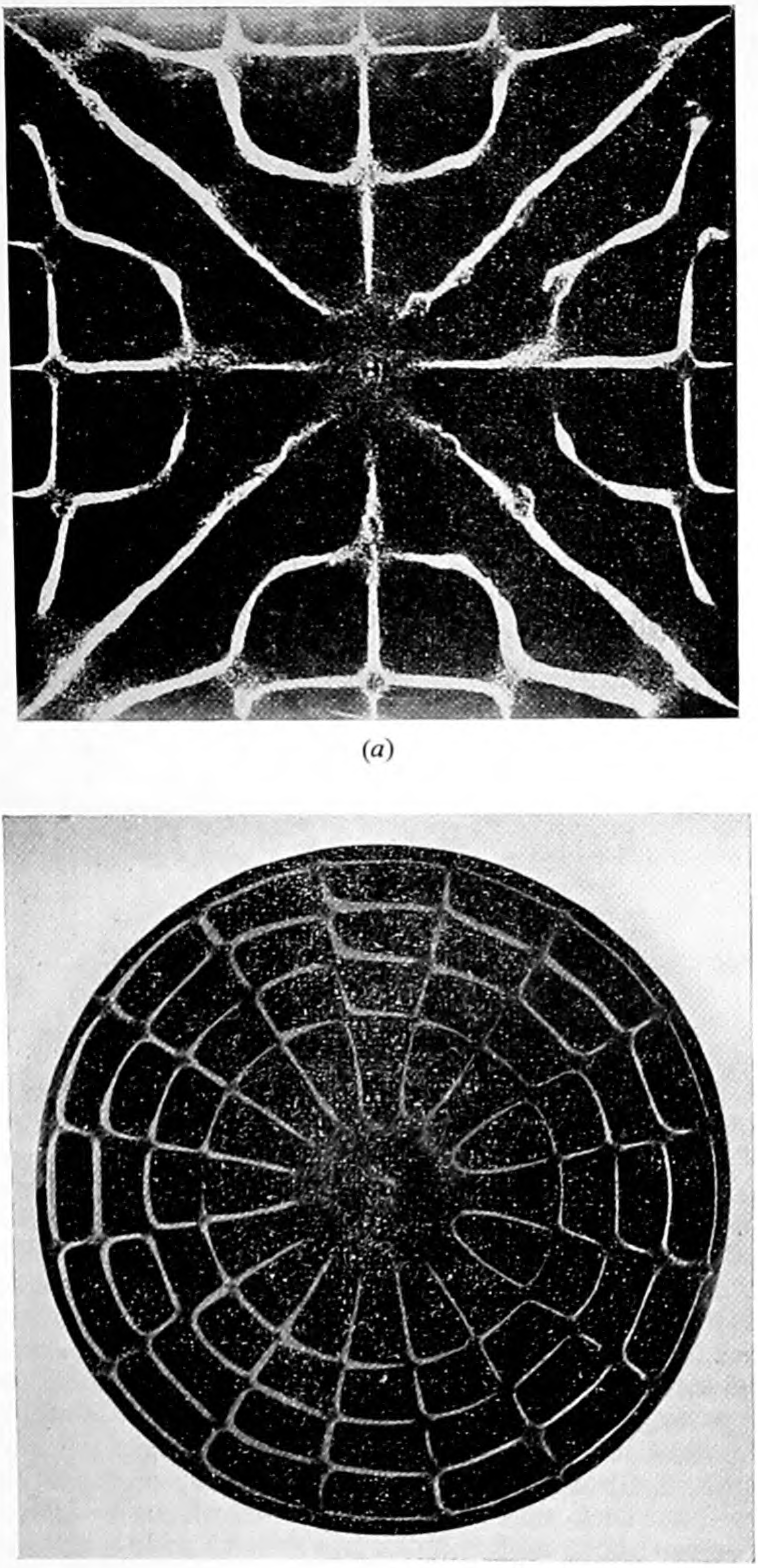


[*J. Appl. Phys.*

Fig. 5.37.

Although not directly employed in the form of a musical instrument, the diaphragm is widely used for the recording and reproduction of music and speech, as in various types of microphone (see p. 271). In these applications care has to be taken that its inherent fundamental frequency is greater than the highest frequency to be recorded, or, alternatively, the diaphragm is suitably damped at its natural period. The ordinary telephone microphone has a fundamental frequency of about 800 c.p.s. so that acoustical distortion would be very noticeable, in the absence of appreciable damping, if the incident sound wave has a prominent component of this frequency.

Bells bear a similar relation to plates as tuning-forks to rods since they may be regarded as *curved* discs loaded in the centre.



(a)

[Waller. Proc. Phy. Soc.

(b)

Fig. 5.38.

CHAPTER 6

REFLECTION, REFRACTION AND ABSORPTION

Reflection

The simple laws of reflection in optics also apply to acoustics provided that the limitations imposed by diffraction are carefully recognised; viz. that the dimensions of obstacles and apertures must be large compared with the wave-length of the sound. This condition implies that for the more common lower audio-frequencies, reflecting surfaces, assumed to be smooth and regular, must be at least of the order of 10 ft. in their linear dimensions for the laws of geometrical optics to be directly applicable.

These laws, to remind the reader, are as follows:—

(1) The incident ray, the reflected ray and the normal at the point of incidence all lie in the same plane, and

(2) The angle of reflection is equal to the angle of incidence.

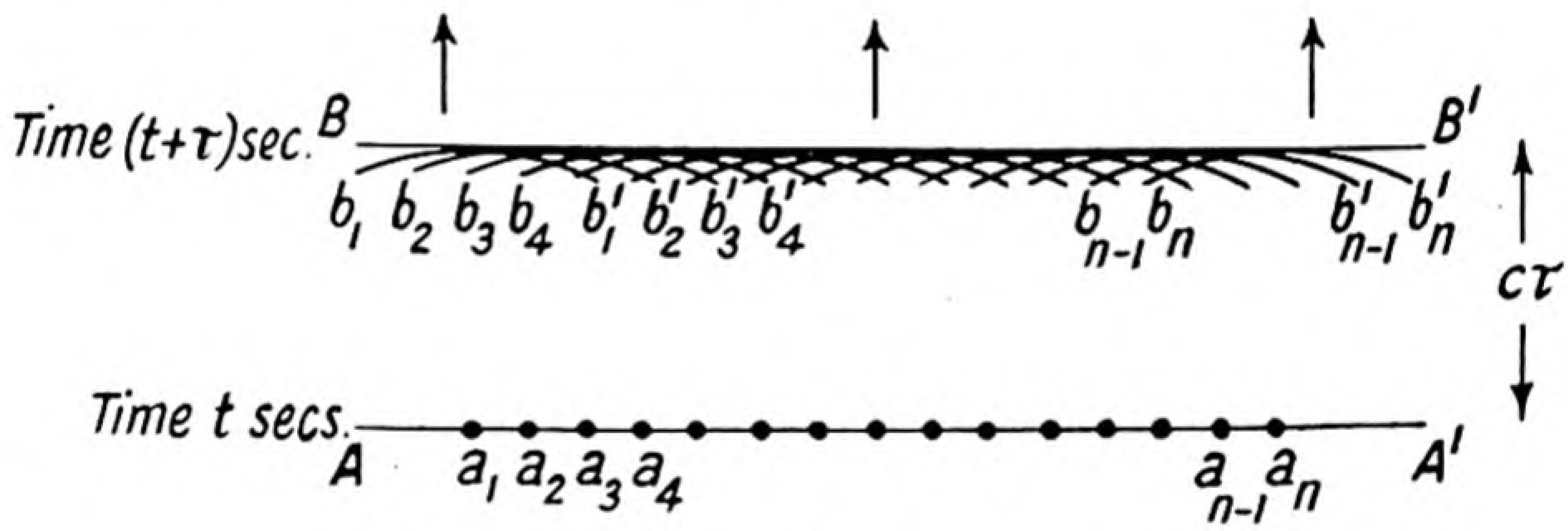


Fig. 6.1.

Construction of wave-fronts. The graphical construction of the form and position of an advancing wave-front at a future instant, is deducible from a principle enunciated by Huyghens which states, "every point on a wave-front is to be regarded as the centre of a spherical disturbance spreading outwards from it, so that the resultant disturbance at any point in front of the advancing waves will be produced by the summation effect of all the elementary centres of disturbance." For example, suppose a plane wave has reached a position given by AA' (Fig. 6.1) which therefore represents the wave-front at a particular instant t sec., say, *i.e.* it is the *locus* of all the points which the wave motion has reached at a given moment. Then the wave-front at a subsequent time $(t+\tau)$ sec. will be obtained, following Huyghens' construction, by drawing a series of circles (since only a plane section of the complete motion is shown) with centres at points a_1, a_2, etc., in AA' and common radius $c\tau$, where c is the velocity of propagation of the longitudinal waves in the medium. It will be seen that all these wavelets touch BB', which therefore forms an *envelope*

to these wavelets. A closer consideration of the problem would suggest that the centres of these secondary waves should also give rise to a plane wave in a backwards direction, *i.e.* towards the source of the waves, but Kirchhoff has shown theoretically that the effect of the secondary waves in this direction is nullified by mutual interference.

Similar reasoning to the above is applicable to the construction of the wave-front of a spherically diverging wave, partly shown in section as AA' in Fig. 6.2, where S represents the centre of the wave system. It is easily seen that the wave-front at a subsequent time $(t+\tau)$ sec. will be a portion BB' of the spherical surface of radius $c(t+\tau)$ with centre S.

Reflection at a plane surface—acoustical images

Consider a plane reflecting surface AB (Fig. 6.3) of *infinite* extent, and let a point source be situated at O. Then waves diverge from O and undergo reflection from all parts of the reflecting surface, and the effect at any point, L say, is the resultant of the aggregate effect of all the reflected waves, together with the direct sound from O.

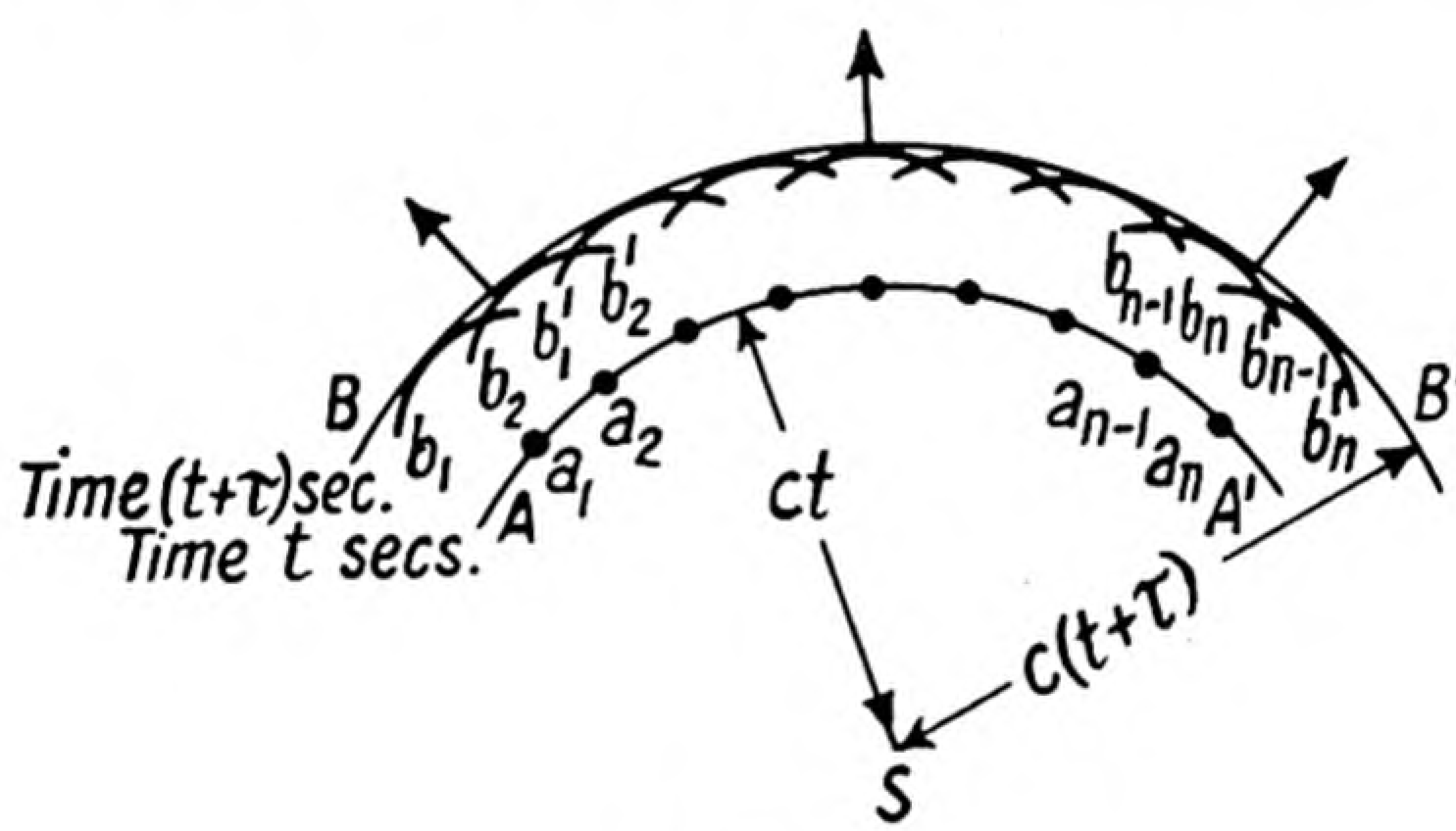


Fig. 6.2.

The problem of these reflected waves becomes considerably simplified if a fictitious source of sound is placed at I on the normal to and on the other side of the surface, such that $NI=NO$. This image source is assumed to be exactly similar in intensity, phase, and frequency to the actual source at O.

Adopting the argument followed by Lindsay and Stewart: Suppose the reflecting surface to be removed and consider the combined effect of the sources at O and I at a point P in the plane AB. Then the spherical disturbances (indicated by the dotted lines) which are generated by O and I, and which pass through P, will subject the particles at that point to equal forces in the directions PL and PM. The resultant displacement will evidently lie in the plane AB so that the latter may now be replaced by a hypothetical reflecting surface of *zero* thickness without affecting the resulting wave motion. Hence on the side of AB nearer the source, the effect of the presence of the reflecting surface on the sound waves emanating from O is the same as if this surface is removed and an image I of the source is placed as indicated.

The reflected sound is termed an *echo*, and if a brief hand clap is made at O then an observer at L will hear the sound by the direct path OL and by means of the echo along the path OPL. The hearer will perceive the echo as separate and distinct from the direct sound if the difference in length of the two paths is sufficiently large, and this fact is of prime importance when considering the acoustical properties of a lecture or concert hall. It is found that if the difference in *times* of arrival of the direct and reflected sounds of a spoken word is about $\frac{1}{20}$ sec., then the word is usefully reinforced, but if delayed longer the echo interferes with the direct sound. Assuming the velocity of sound in air at room temperatures to be 1120 f.p.s., then to avoid "blurred" speech the above condition implies that a speaker must not be further away from a back-wall reflector than $\frac{1}{2}(\frac{1}{20}\times 1120)=\frac{56}{2}=28$ ft. At a distance of about 33 or 34 ft. the echo will be heard quite distinctly from the sound received directly from the source. Use is made of this phenomenon for obtaining an approximate estimate of the depth of wells, etc., by timing the arrival of an echo. The comparatively recent development of ultrasonic generators, which enable *beams* of sound to be easily produced, has resulted in the employment of the echo-sounding method to the charting of ocean depths, to the detection of icebergs, and, in wartime, of submarines, etc. In the modern technique, the sending of the sound signal and the reception of the echo are accurately timed by electrical means, thus eliminating the errors of the human observer. Alternatively the echo-method may be applied to the measurement of the velocity of sound if the path-length is known, and it has been suggested that Newton made such a measurement in the long walk, or passage, at Trinity College, Cambridge, which bears his name. This passage possesses a flat roof and end walls, so that if a brief sound is created at a position near one end it will travel up and down between the end walls. Hence, if an observer situated near the sound source times the hearing of, say, 6 echoes in t sec. and l is the length (in centimetres) of the passage, then the velocity of sound in air will be given by $\dfrac{2l}{\frac{t}{6}}=\dfrac{12l}{t}$ cm. per sec.

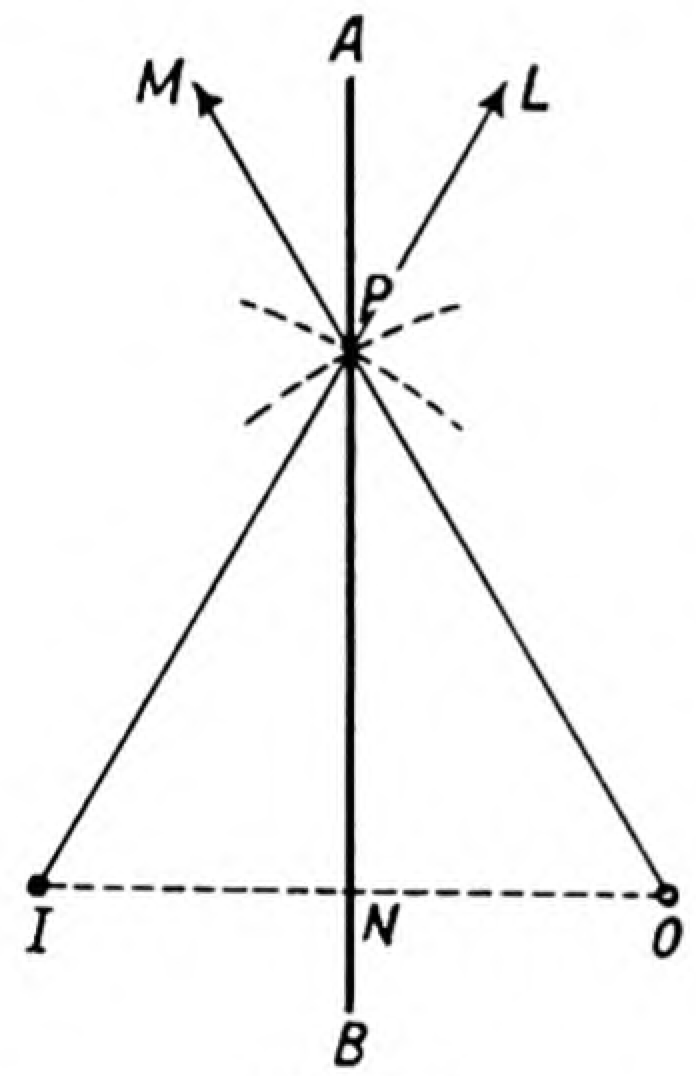


Fig. 6.3.

In a closed room the repeated reflection of sound at the containing surfaces produces an enhancement of the sound intensity at any point in the enclosure as compared with that due to the direct sound alone, as would occur if the walls were absent. This building-up of the sound intensity within an enclosure is governed by the volume of the room and the nature of the surfaces of the walls, ceiling, etc., and the phenomenon is known as *reverberation* (see Chap. 14).

Whispering galleries. In the majority of these galleries whispered sounds produced at any point near the circumference, give rise to diverging sound waves which become focused at another point where

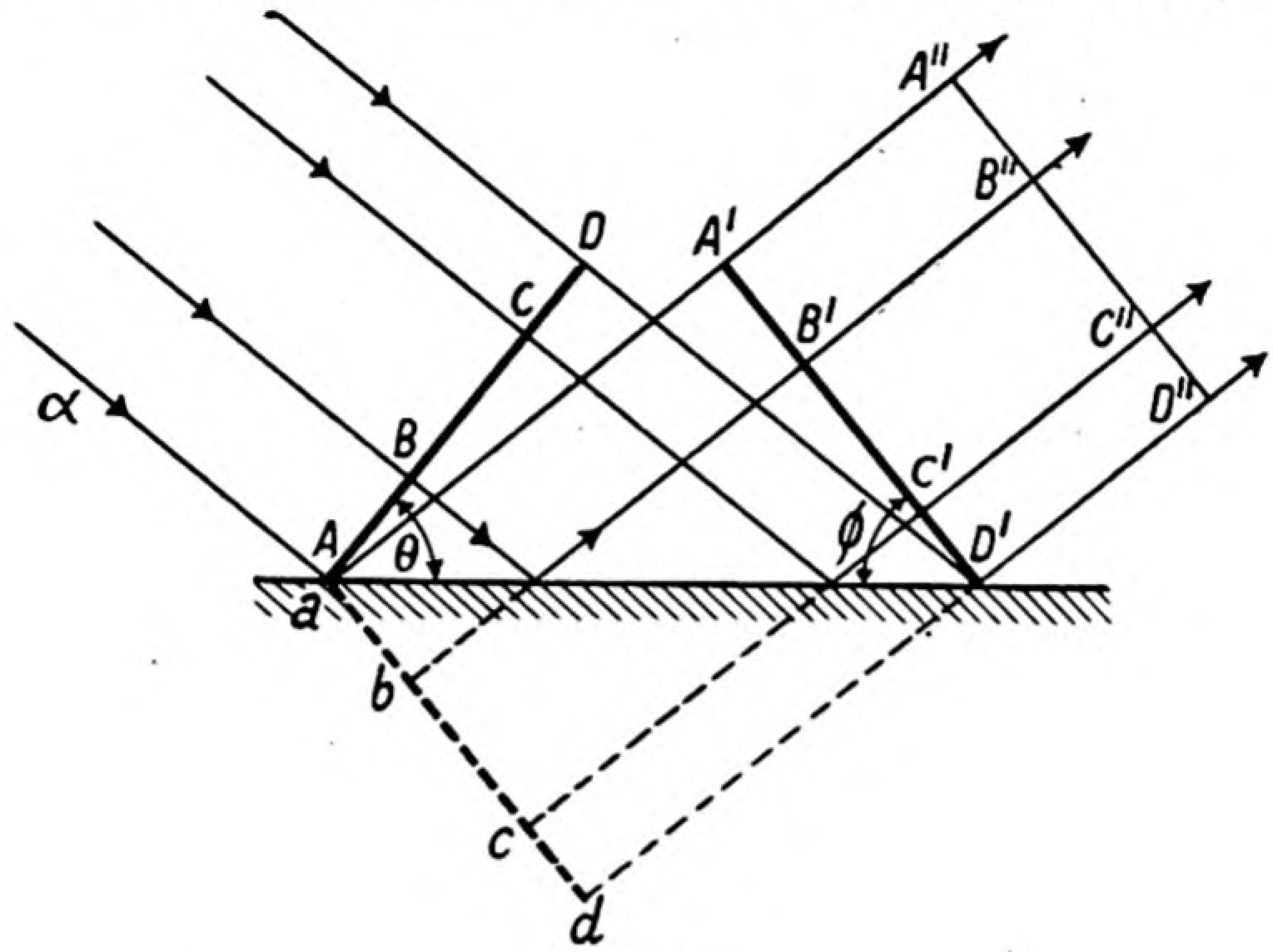


Fig. 6.4.

an observer may clearly hear the whispered conversation. This phenomenon has been explained in terms of the reflection of waves at a concave surface, but it is not sufficient to explain the effect observed in some galleries such as that of St. Paul's Cathedral. In these cases a whisper uttered near the wall can be heard by a listener in a like position *anywhere* around the circumference. The reason put forward to account for this phenomenon is that the walls are slightly inclined downwards and, together with the floor of the gallery, prevent a great deal of the spreading of the sound waves so that they become largely restricted to a circular channel near the wall of the gallery.

Reflection of a plane wave at a plane surface

The problem of the reflection of a plane sound wave at a plane surface is a natural extension of the similar problem for the single source. Let A, B, C, D, etc., and AD' represent respectively sections of the planes, normal to the plane of the paper (Fig. 6.4), of the incident wave-front and of the reflecting surface. The *direction* of the incident wave will be given by the normal to the wave-front, and if this latter makes an angle θ with the reflecting surface, then the direction of the waves will make an angle θ with the *normal* NA (Fig. 6.5) to the reflecting surface. It should be noted that if the conditions of rectilineal propagation are satisfied then the normals to the wave-front may be described as *rays* of sound.

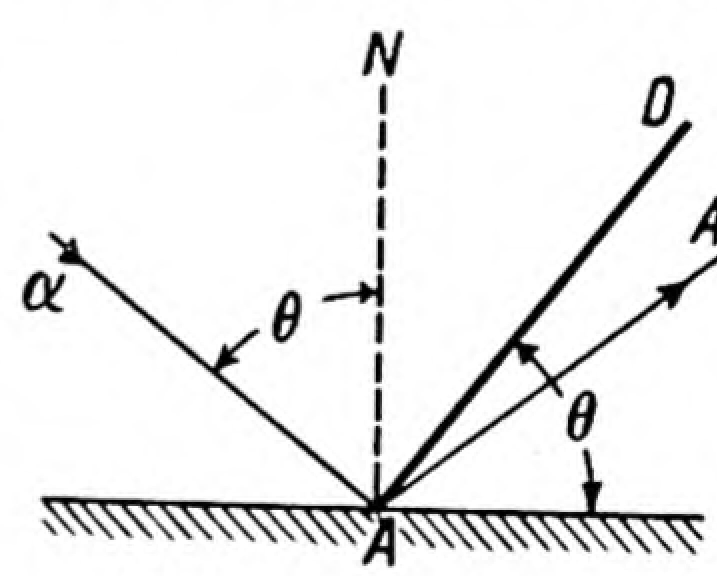


Fig. 6.5.

Now using the concept of acoustical images, all points A, B, C, D, etc.,

of the incident wave-front will respectively give rise to such images at a, b, c, d, e, etc., which will behave as virtual centres of spherical disturbances. Since all the particles A, B, C, etc., are vibrating in phase, as follows from the definition of a wave-front, then a, b, c, etc., will also be in phase with each other. If $aA'=bB'=cC'$, etc., then it is evident that the envelope of these spherical wavelets will form the reflected wave-front $A'B'C'D'$ at some given time, t sec. say, after the wave-front meets the surface. If v is the velocity of sound in the upper medium, then in the particular case shown in the diagram $t=\frac{DD'}{v}=\frac{AA'}{v}$, and at a later time t' the reflected

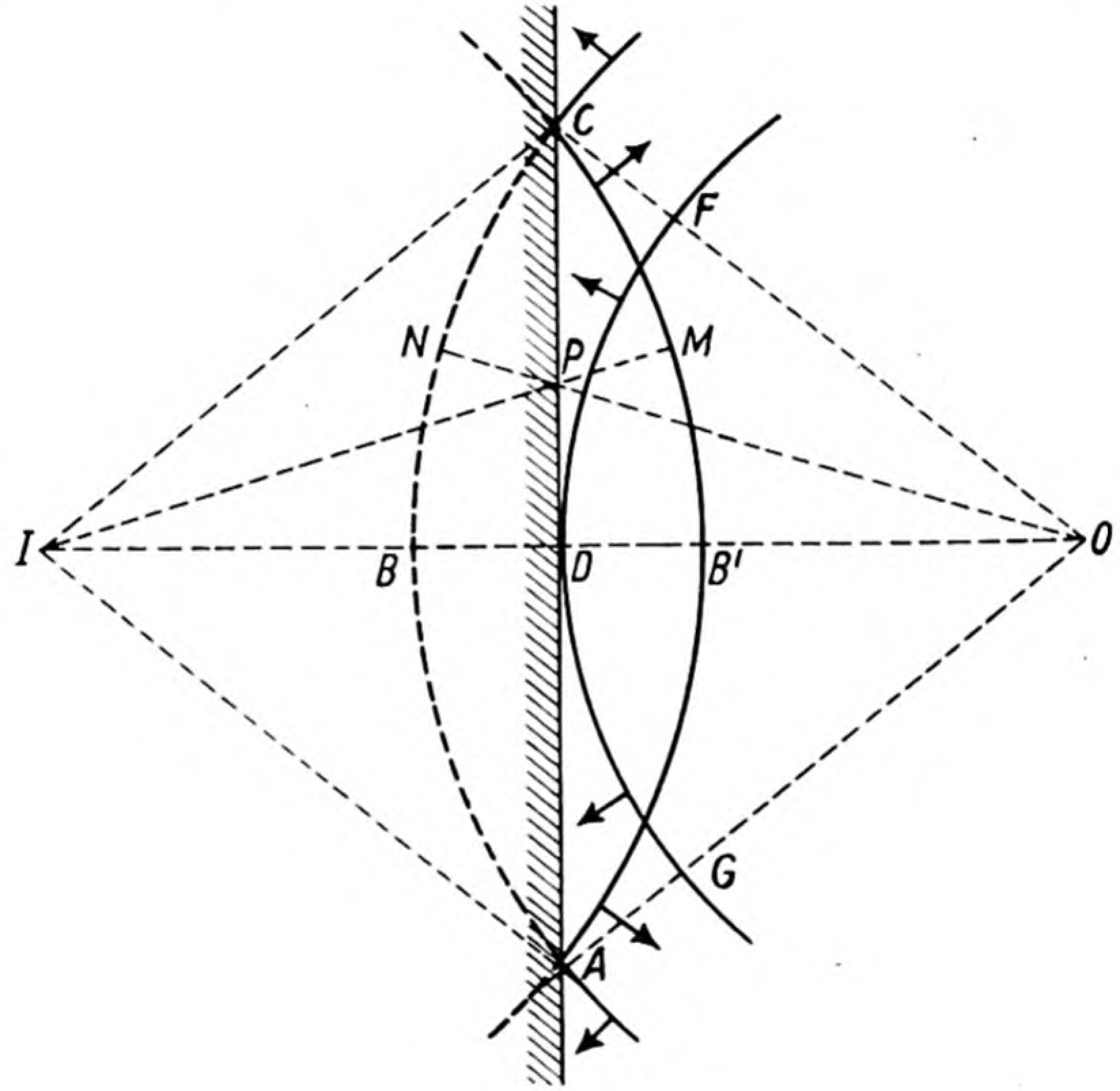


Fig. 6.6.

wave-front will have reached the position $A''B''C''D''$ where $\frac{AA''}{v}=t'$. It follows from the geometry of the figure that since $A'B'C'D'$ is parallel to $abcd$ and bB is normal to the surface AD', then angle $A'D'A$=angle $D'Ad$=angle DAD', *i.e.* $\phi=\theta$ or the angle which the reflected wave makes with the surface is equal to the angle between the incident wave-front and the surface. Since the sound rays are normal to the wave-fronts the above deduction immediately implies that the incident and reflected rays make equal angles with the normal to the surface at the point of incidence. Furthermore, since AD, AD' and $A'D'$ are sections in the plane of the paper of planes perpendicular to that of the paper, then it follows that the incident and reflected rays, together with the normal to the surface at the point of incidence, all lie in the same plane.

Reflection of a spherical wave at a plane surface

Let O be the source of spherical waves (Fig. 6.6) which are to undergo reflection at the plane surface represented by ADC, then the acoustical image of O will be situated at I, where $ID=OD$, and ODI is perpendicular to ADC. Consider the wavelet GDF which has just touched the surface at D, such that $OD=vt$ where t is the time taken for a disturbance at O to reach D, v being the velocity of sound in the medium. Now in the time taken for this wavelet to progress, such that the wave-front between A and C has come into contact with the reflecting surface, a reflected wave such as $CB'A$ will have been formed. In the absence of the reflecting surface the incident wave would have reached the position ABC, where $DB=CF=DB'$. Consequently the reflected wave "emanating" from the virtual image I will form part of a spherical surface of radius $IB'=OB=OD+BD=ID+DB'$. Consider any point P on the reflecting surface, then the wave reflected from this point will have reached M where $IM=IB'$, and $PM=PN$ where N would be the corresponding position to M on the "unreflected" wave-front ABC.

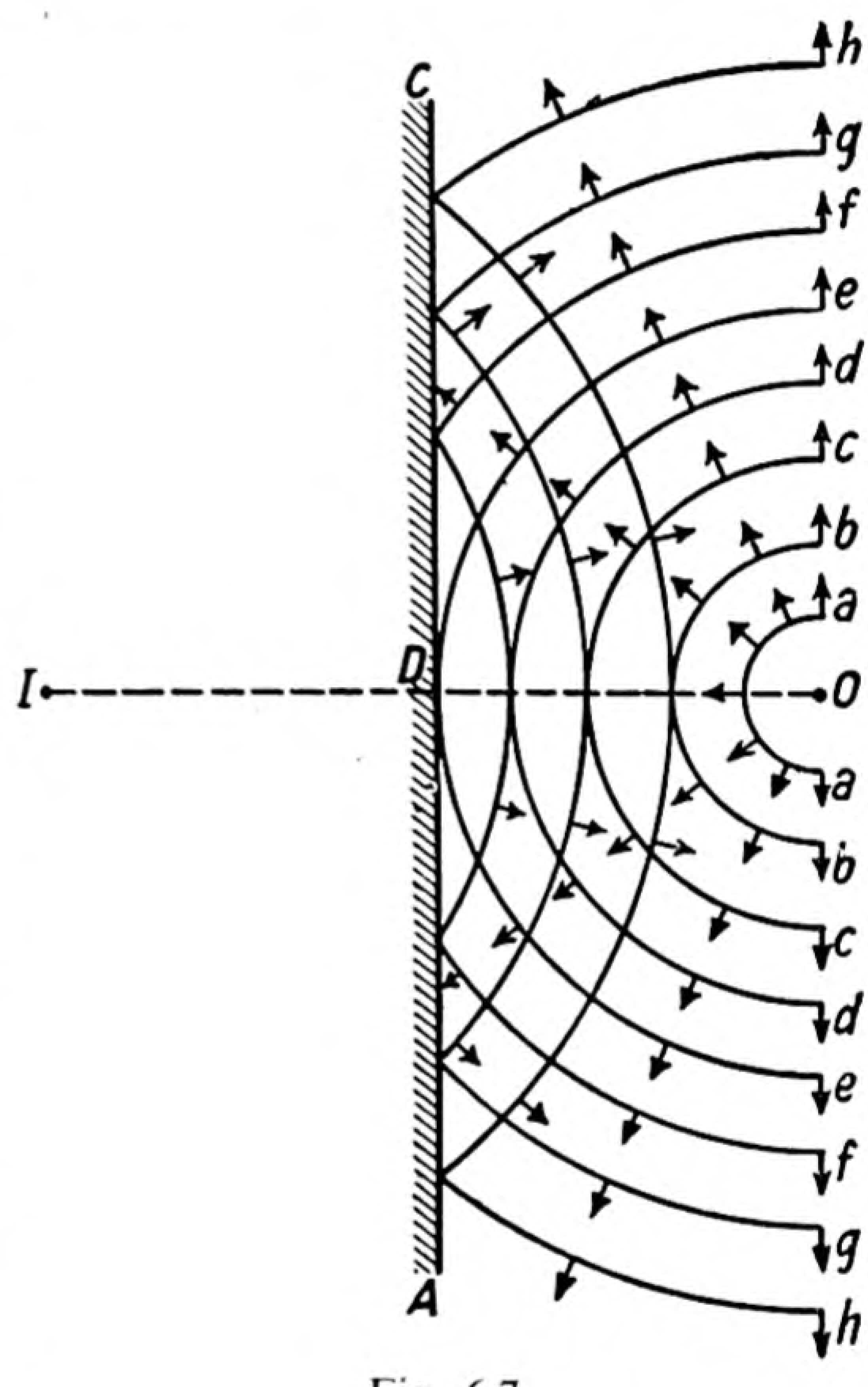


Fig. 6.7.

It is evident therefore that an alternative method of constructing the reflected wave-front is to draw the *envelope* $CB'A$ to the spherical wavelets from many points between A and C which have all become centres of disturbance as a result of the impact of the incident wave. The radius of any one of these wavelets will be the difference between OP and OB, where OP represents the straight line drawn from O to any point between A and C. Summarising, it may be said that a spherical wave diverging from the source O is converted, after reflection, into a wave of equal radius diverging from the image position I, the effect of the surface therefore being merely to reverse the curvature.

Now suppose the source O to be a continuous generator of spherical waves, the surfaces of maximum compression at any instant being represented by the circular lines aa, bb, etc., in Fig. 6.7. Certain wavelets f, g, h are shown to have reached the reflecting surface ADC and given rise to reflected wavelets after the manner previously mentioned. The wavelet e has just reached the reflecting surface.

The pattern of the sound field produced by the combined effect of the incident and reflected waves will be given by the right-hand side of YY' in Fig. 6.8, where S_2 will correspond to the source O and S_1 to its image I. It is evident that if the source is placed very close to the reflecting surface its image will also be correspondingly near. Consequently since source and image vibrate in phase with the same frequency, the displacement in the sound field will be approximately *double* that due to the source alone, provided the separation of source and image is small compared with the wave-length of the sound waves.

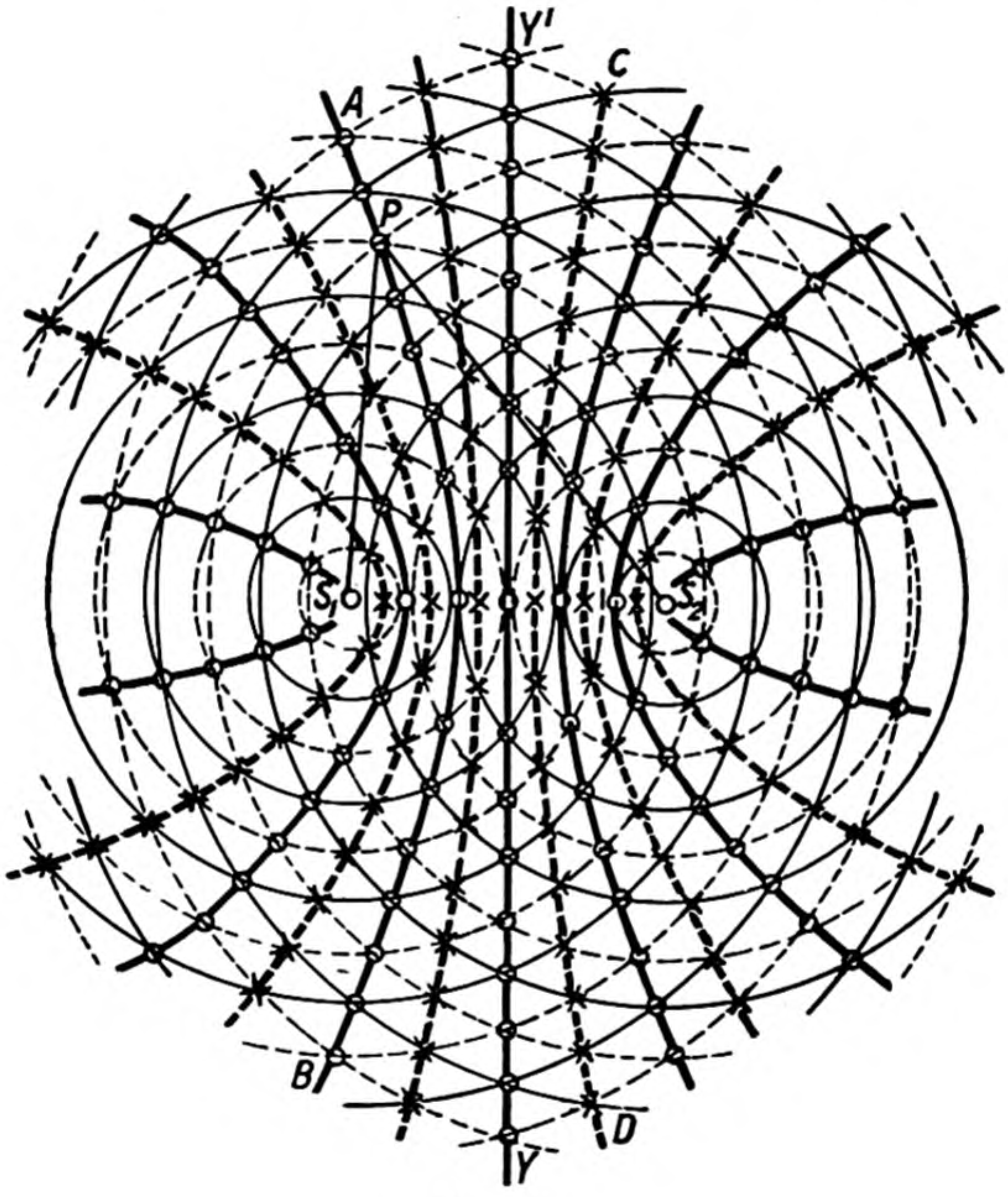


Fig. 6.8.

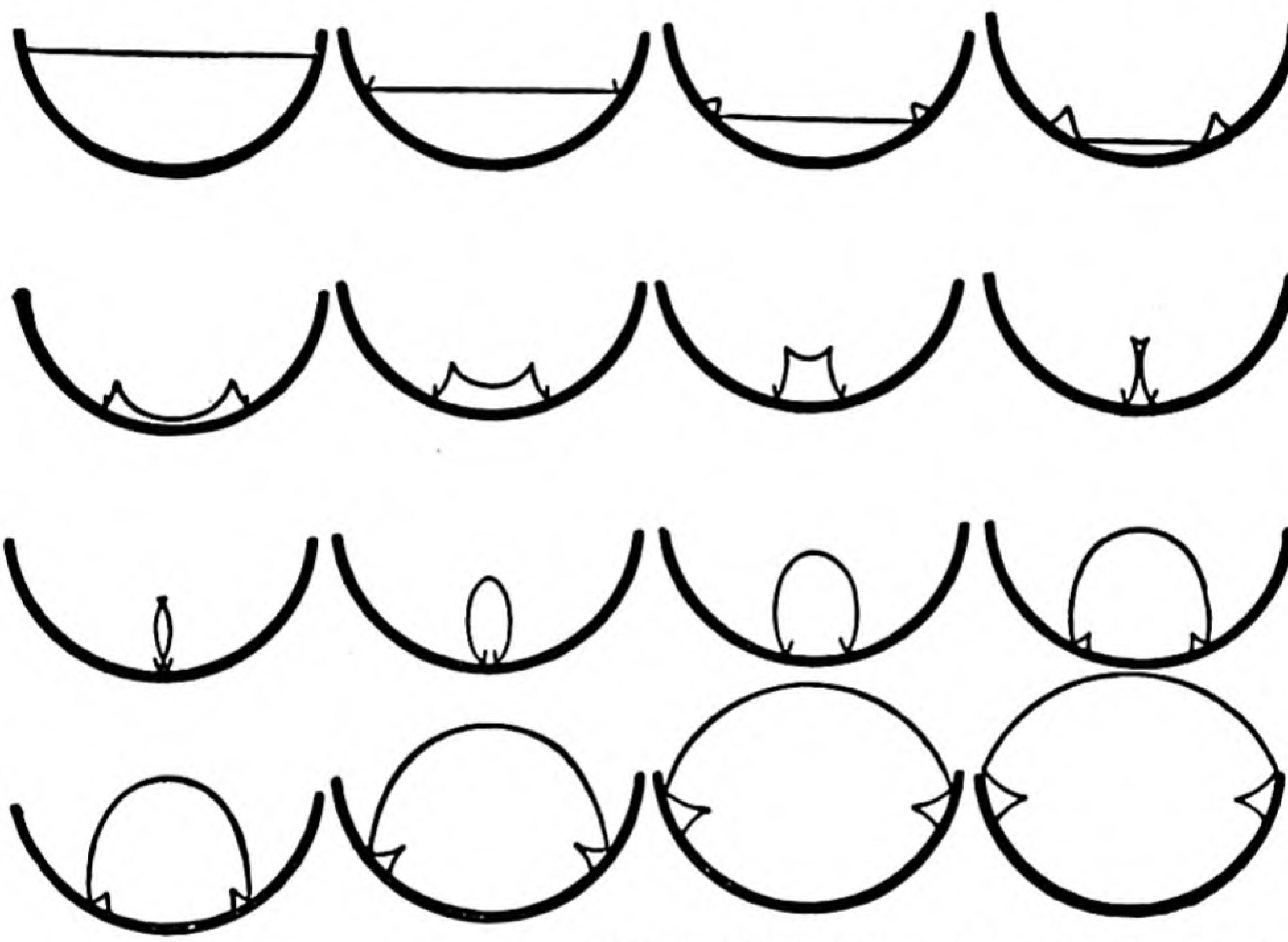
An important application of this effect is the location of a loud-speaker unit at the centre of a baffle board.

Reflection at curved surfaces

The reflection of plane and spherical waves at curved surfaces is considered below, but only briefly, since these cases are fully worked out in books on optics, the method of solution following that employed previously when dealing with plane surfaces. It is assumed that the aperture of the *spherical* reflecting surface is small compared with the radius of the surface, as otherwise the effects become complicated. In this connection R. W. Wood has investigated theoretically the case

of the reflection of a plane wave at a *hemispherical* concave surface and has constructed a series of diagrams giving a number of successive positions of the reflected wave-front (Fig. 6.9). He experimentally verified his predictions by using a modified form of Toepler's method to photograph successive positions of a sound wave during its reflection at a cylindrical surface.

(1) Plane wave at a concave surface. Let P_2BP_1 (Fig. 6.10) represent the trace of the spherical surface and N_2BN_1 the trace of the plane wave-front of the incident wave, which would have reached the position shown but for the presence of the reflecting surface. The effect of the latter is to give rise to the reflected wave M_1BM_2 where in the limit $P_1M_1=P_1N_1$ (also $P_2M_2=P_2N_2$), and P_1M_1 is the distance travelled



[*Wood: "Physical Optics." Macmillan Co., New York.*

Fig. 6.9.

by the reflected disturbance at P_1, while the centre B of the incident wave travels over the distance P_1N_1.

Actually the reflected disturbance at P_1 will reach a point A, say, where the normal P_1F cuts the reflected wave-front. The aperture as shown in the diagram is, however, greatly exaggerated, and in practice the angle M_1P_1A will be sufficiently small that its cosine may be taken as unity and so $P_1M_1=P_1A$ to required degree of accuracy.

If R is the radius of the reflecting surface and ρ that of the reflected wave, then from a well-known property of the circle

$$(P_1N_1)(2R-P_1N_1)=(N_1B)^2=(M_1N_1)(2\rho-M_1N_1),$$

or

$$\frac{R}{\rho}=\frac{M_1N_1}{P_1N_1}=\frac{2P_1N_1}{P_1N_1}=2 \text{ approx.}$$

Hence $\frac{\frac{1}{\rho}}{\frac{1}{R}}=2$, or the curvature of the reflected wave is twice that of the reflecting surface. The point F on the axis to which the reflected wave converges is known as the principal focus of the mirror, and it is evident that with the assumed degree of approximation $FB=\rho=\frac{R}{2}$. This distance from the focal point to the pole B of the mirror is known as its *focal length* as in optics. The waves on passing through the focus will commence to diverge.

In the case of a convex surface the centre of curvature C and the focal point F will lie on the side of the mirror remote from the reflecting surface, but the above relation between focal length and radius of curvature will still be satisfied.

(2) Spherical wave at a spherical surface. As typical of this class of reflection, that taking place at a convex surface N_1BN_2 is chosen in Fig. 6.11. The incident wave-front $M_1N_1PN_2M_2$ is shown diverging from V and to have reached a position where the portion of the wave-front between N_1 and N_2 has been reflected and appears to diverge from a point Q. From the geometry of the figure it follows that $BP=BT$, $SP+ST=PT=2BT$, and $SB=BT-SP$. Hence as before, $2SB=ST-SP$ or $\frac{2}{R}=\frac{1}{\rho_2}-\frac{1}{\rho_1}$

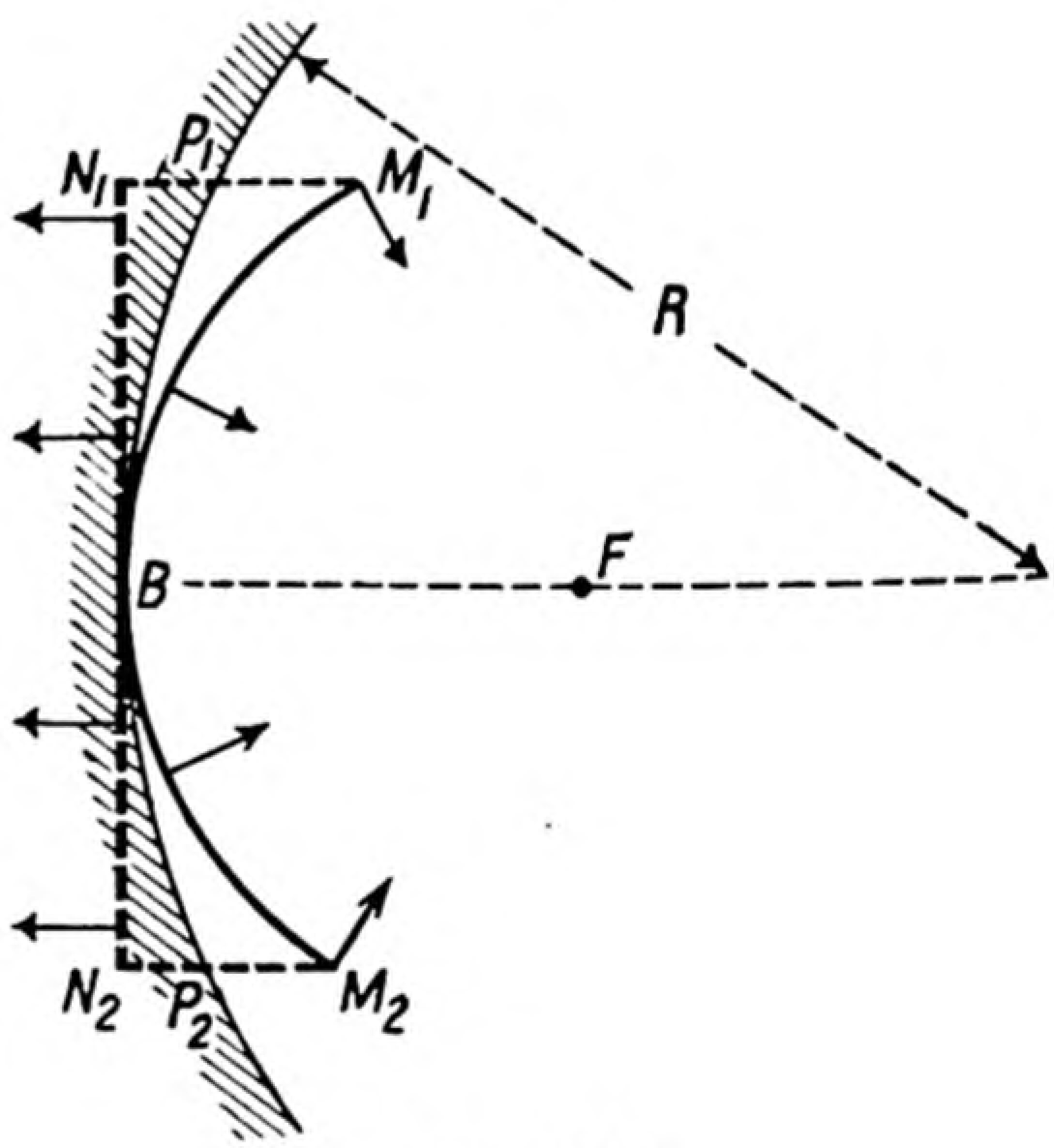


Fig. 6.10.

where ρ_1, ρ_2 and R are respectively the radii of curvature of the incident wave, the reflected wave and the reflecting surface. If note is taken of the opposite signs of ρ_1 and ρ_2, then the above relation may be written as $\frac{2}{R}=\frac{1}{\rho_2}+\frac{1}{\rho_1}$, which will then represent the reflection of any spherical wave at any spherical surface.

Summarising, therefore, it may be said that the curvature of the reflecting surface forms the arithmetical mean between the curvatures of the incident and reflected waves. The points V and Q are referred to as conjugate foci with respect to the mirror, *i.e.* a wave diverging from either will after reflection diverge from (or converge to) the other.

Reflection from a parabolic mirror

If a small source of sound is placed at the focus of this form of reflector the spherical waves diverging from that point are transformed into plane waves on reflection. This effect is seen by reference to

Fig. 6.12, where F is the focus of the parabolic mirror which is shown in section. A number of spherical wave-fronts are shown diverging

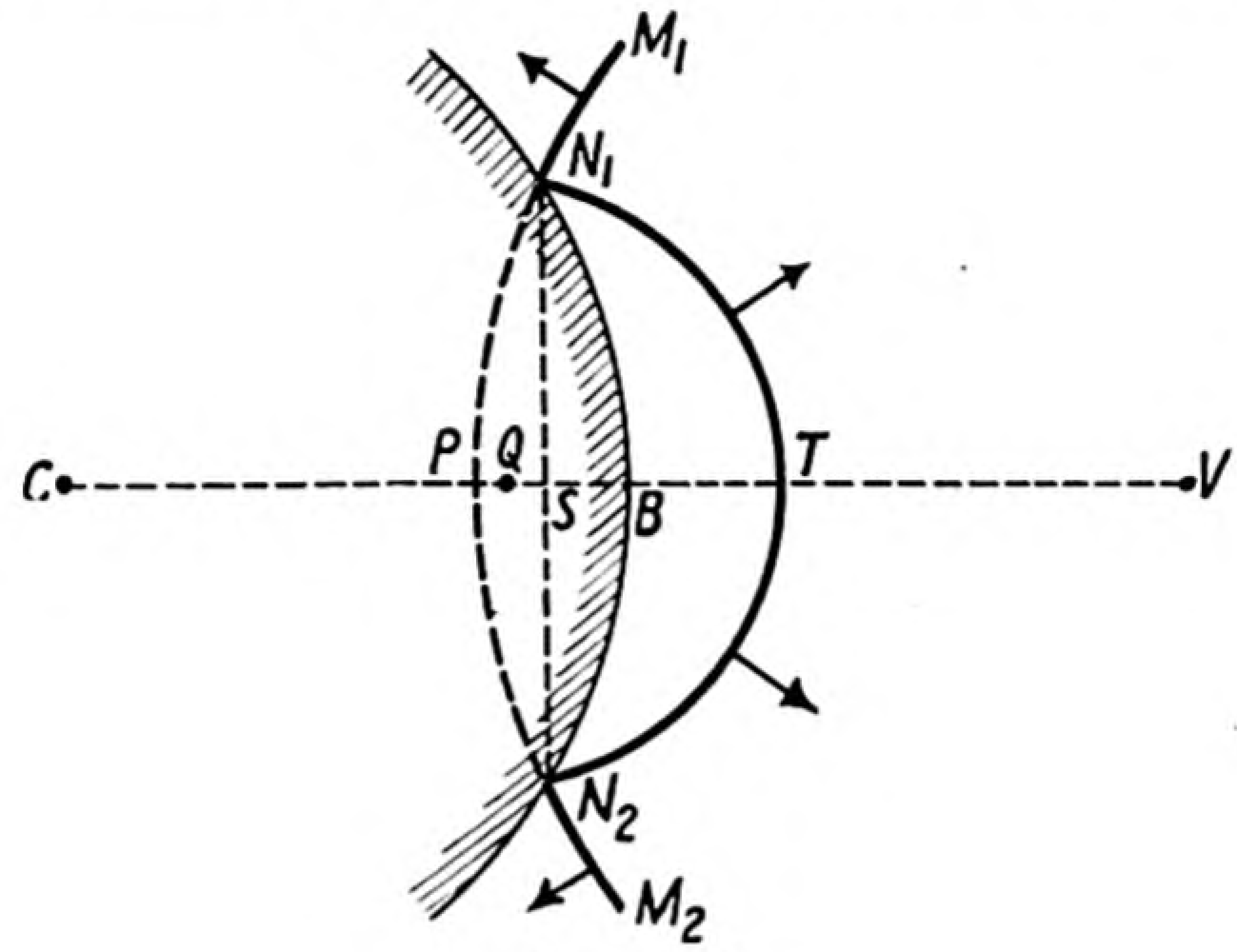


Fig. 6.11.

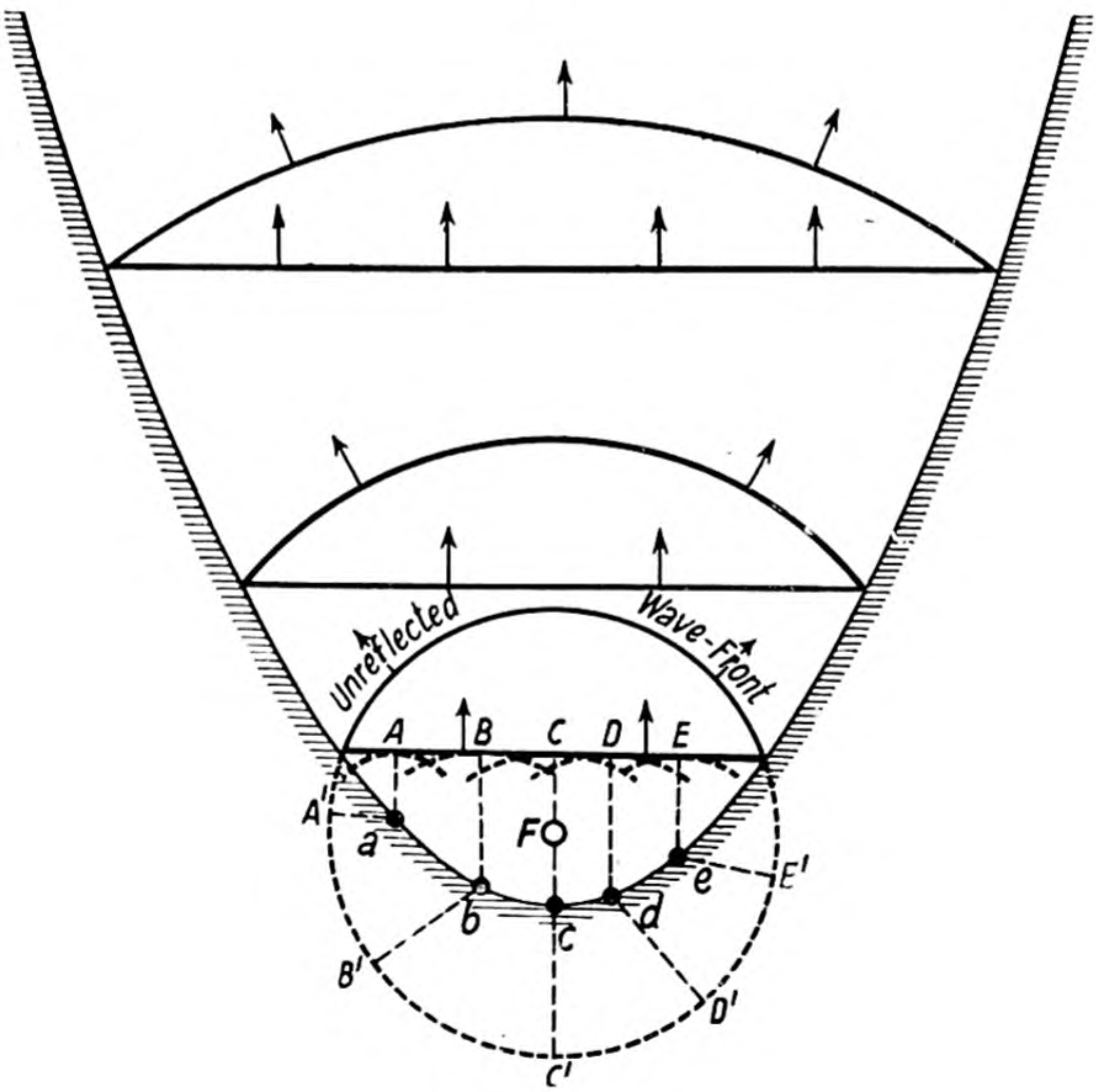


Fig. 6.12.

from F, and portions of these wave-fronts have already undergone reflection at the surface of the mirror. In particular, a consideration of the smallest wavelet of radius $FA' = FB'$, etc., reveals that the

reflected portion would have reached the dotted circular position $A'B'C'D'E'$ in the absence of the mirror, and hence the various points a, b, c, d, e, etc., will have become centres of disturbances. With radii equal to the distances of these points from the imaginary dotted wave-front $A'B'C'D'E'$, measured along its radii, secondary wavelets are constructed with the respective points as centres. From a property of the parabola, viz. that the paths FbB, FaA, etc., are equal, it follows that the centres of disturbance A, B, C, D, E, etc., are all in phase, *i.e.* the envelope of the wavelets from a, b, c, etc., is the straight line $ABCDE$ representing the section of a plane wave-front.

Parabolic mirrors are employed in film studios to pick up the voices of leading actors in crowd scenes, and another application is their use in the location of aircraft. A typical apparatus designed for this purpose is shown schematically in Fig. 6.13. The reflector R is composed of plaster of Paris, which has been suitably coated to give it a hard and weatherproof surface, and mounted in a manner which permits it to be rotated and elevated at any desired angle. A piezo-electric microphone M is mounted at the focal point of, and faces, the reflector which is adjusted in position and elevation for maximum sound intensity in the telephones. The axis of the mirror will then give the direction of the sound source.

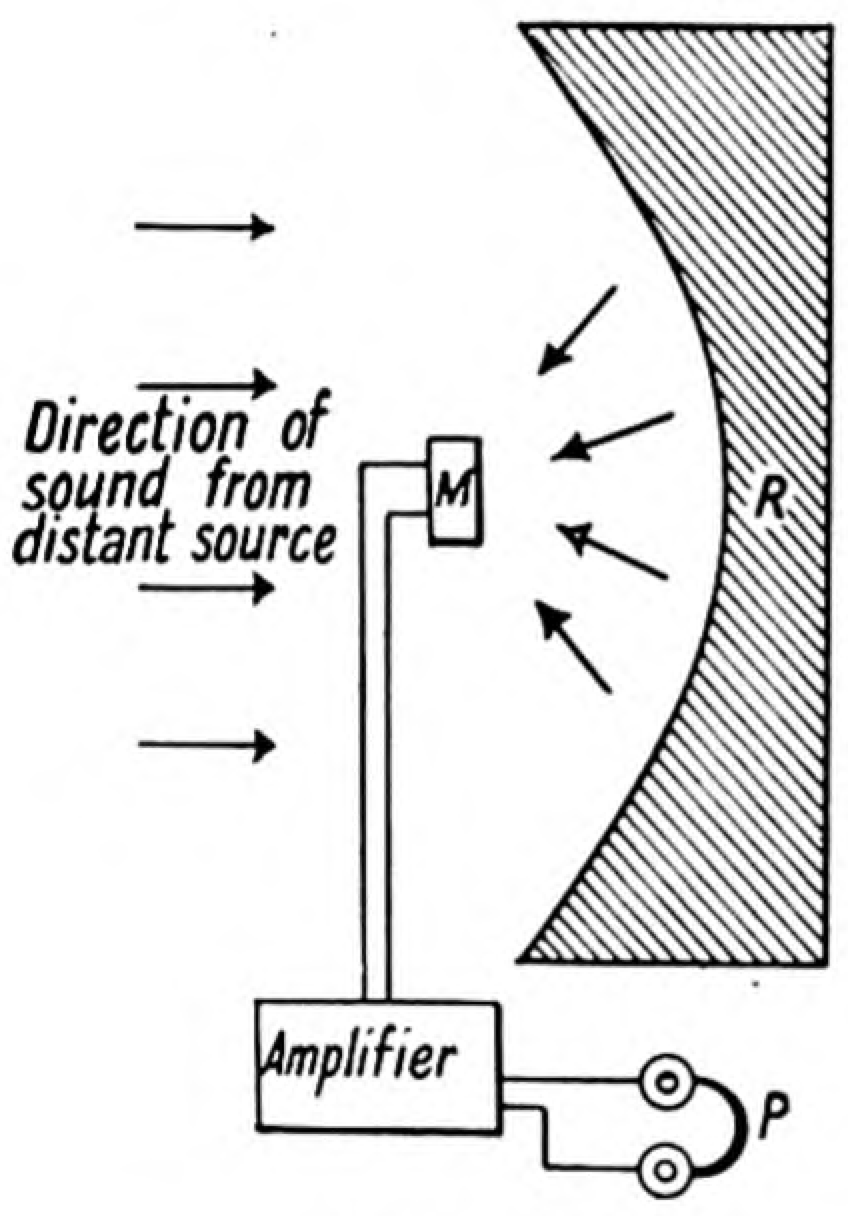


Fig. 6.13.

A method in which the focusing properties of the parabolic mirror are utilised to determine the velocity of sound in air was suggested by A. Michelson, the American physicist, celebrated for his optical interferometer experiments, although the actual experimental work was performed by T. C. Hebb. The experiment was designed essentially to overcome the uncertainties of sound velocity measurements over *long* distances in the open air, where wind, temperature and humidity are liable to show appreciable variation over a baseline of several miles. Incidentally this method of Hebb's also has an advantage over another stationary wave-method, that of Kundt, in which there arises the uncertainty of tube corrections.

The apparatus is shown schematically in Fig. 6.14, where M_1 and M_2 are two *large* parabolic reflectors (5 ft. aperture) made of plaster of Paris coated with varnish, and a telephone transmitter or microphone is placed at the respective focus, F_1 and F_2, of each mirror. T is a telephone transformer supplied with two primaries P_1 and P_2, which are respectively connected in series with F_1 and F_2, and a battery. The secondary winding of the transformer, which is linked with each primary, is connected in series with a telephone receiver R. The source of sound S was also located at the focus of M_1, and in Hebb's

experiment it was a whistle blown at constant pressure and giving a high-pitched frequency of approximately 2400 c.p.s. Now the intensity of the sound heard at the receiver R will be the vector sum of the signals received by F_1 and F_2. Their relative phase may be

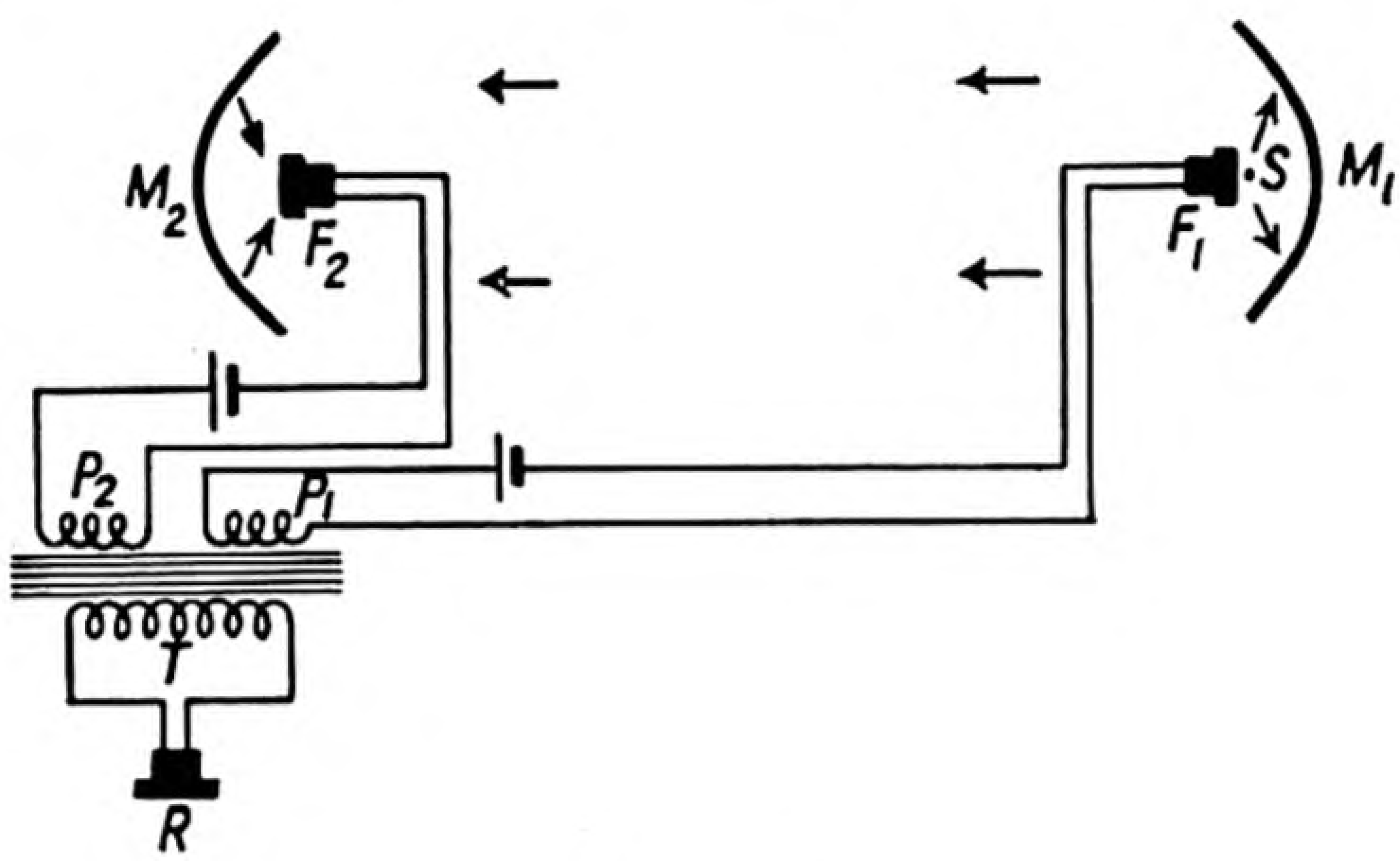


Fig. 6.14.

adjusted by a relative displacement of M_1 and M_2, so that if the position of M_2 (together with F_2) is gradually varied, at certain places a wave-length apart, the sound intensity at the receiver is a minimum. If it is possible to move M_2 a hundred or more wave-lengths, *i.e.* a distance of the order of 20 to 30 metres, the value of the wave-length may be accurately estimated, and if the frequency is measured to a corresponding degree of accuracy the velocity of sound in air is known to within 0·1 per cent. or less. Hebb obtained a value of 331·44 metres per sec. for the velocity of sound in air at 0° C.

Refraction

Sound waves crossing obliquely the boundary separating two media, in which their velocities of propagation are different, undergo an abrupt change in direction. The laws governing this phenomenon of sound refraction are the same as those applicable to light waves, viz. (i) that the incident ray, the normal to the refracting surface at the point of incidence, and the refracted ray are in the same plane; and (ii) that the sine of the angle (i) between the incident ray and the normal bears a constant ratio to the sine of the angle (r) between the refracted ray and the normal (Fig. 6.15). The incident and refracted rays are normal to their respective wave-fronts. Expressed symbolically the second law may be written as $\frac{\sin i}{\sin r}=\text{constant}=\frac{C_1}{C_2}$, where C_1 and C_2 are the velocities of sound in the two media as indicated in Fig. 6.15. In the case of sound waves

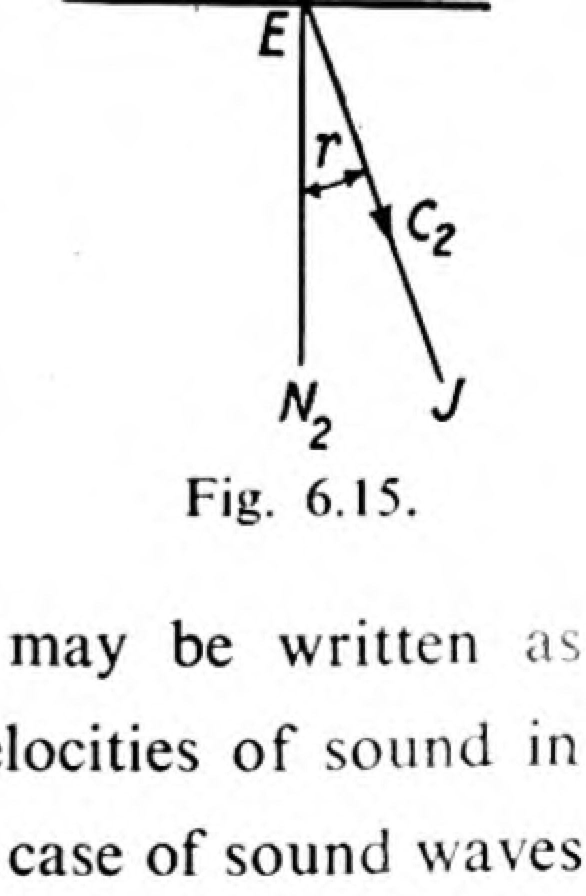


Fig. 6.15.

at ordinary frequencies it is preferable, from the physical point of view, to regard i and r as respectively the angles between the incident and refracted wave-fronts and the plane surface (Fig. 6.16).

Let I_1I_2 represent a section of the advancing wave-front (Fig. 6.16) when I_1 has reached the boundary surface EF, and A_1I_1 and A_2R_2 are the rays at the confines of the incident wave train. As each point P_1, P_2, P_3, etc., of the incident wave-front reaches the boundary surface it becomes the origin of spherical wavelets spreading out into the second medium. Hence, following Huyghens' construction, R_1R_2, the envelope of these secondary wavelets, will represent the wave-crest of the refracted beam in medium 2. The distances I_2R_2 and I_1R_1 will therefore have been described in the same interval of time,

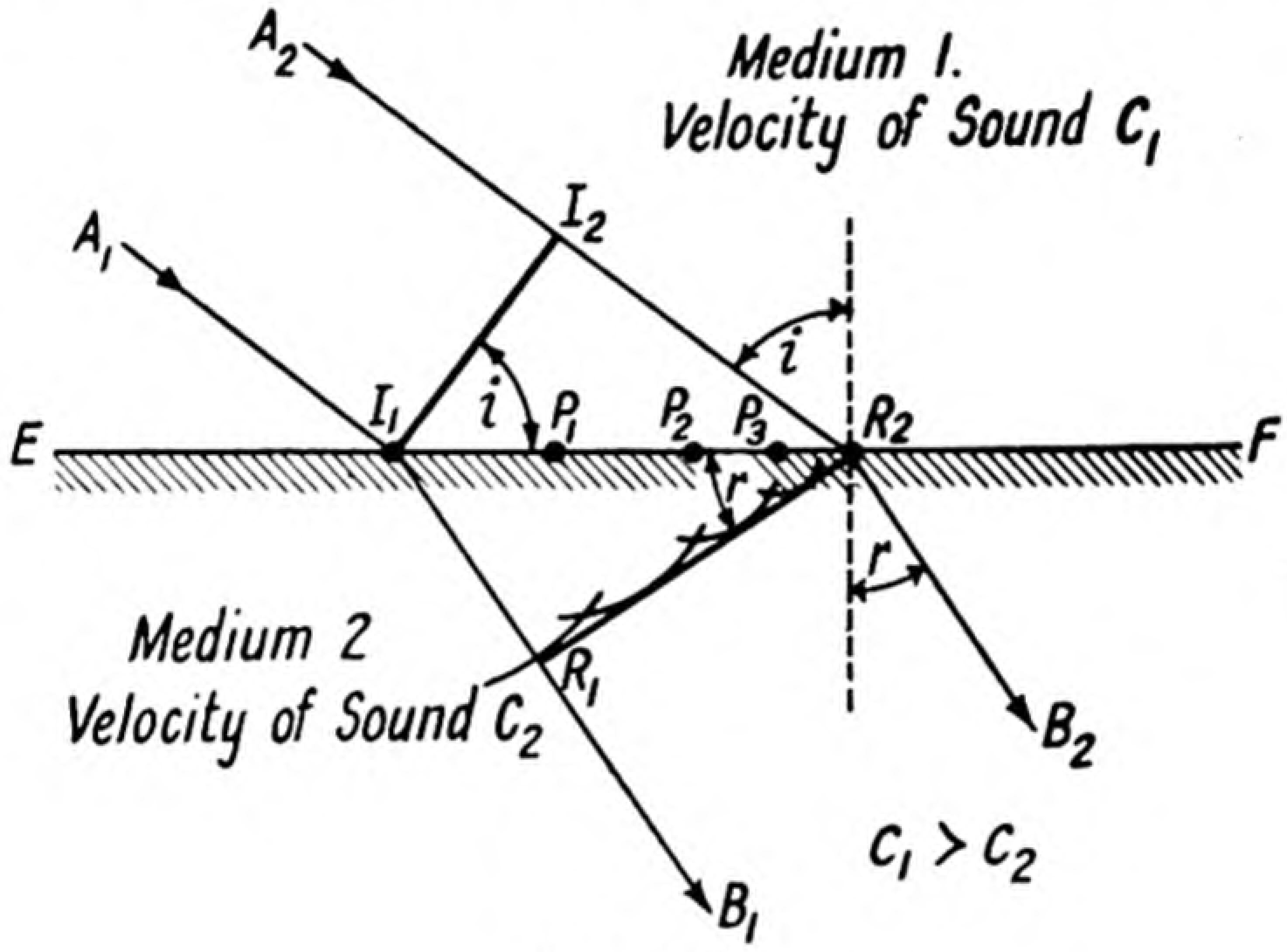


Fig. 6.16.

i.e. $\frac{I_2R_2}{C_1}=\frac{I_1R_1}{C_2}$. But $\frac{I_2R_2}{I_1R_1}=\frac{\sin i}{\sin r}$ and hence by combining these relations Snell's law is obtained.

Refraction of spherical sound waves at plane and at spherical surfaces lead to the analogous formulae obtained in optics, and since in acoustics these cases are not particularly important only a brief reference to them will be made. In Fig. 6.17 a spherical wave ABD is shown to be diverging from the source at O, and to be incident upon the plane surface NBL separating the two media. If the medium existed also to the left of the boundary the wave-front at a later instant would, by Huyghens' construction, assume the spherical front NML, but since the velocity of sound C_2 in medium 2 is less than the velocity C_1 in medium 1, the radius of the secondary wavelet from B will be $BM'=\left(\frac{C_2}{C_1}\right)BM$. Hence $LM'N$ becomes the new wave-front in medium 2, having its centre of curvature at O'. Denoting OM and $O'M'$ by R and R' respectively it is easily shown that to a first approximation $\frac{R'}{R}=\frac{BM}{BM'}=\frac{C_1}{C_2}$, provided BM and BM' are small compared with R and R' respectively.

Fig. 6.18 showing a spherical wave undergoing refraction at a concave surface should be self-explanatory, since the reasoning and

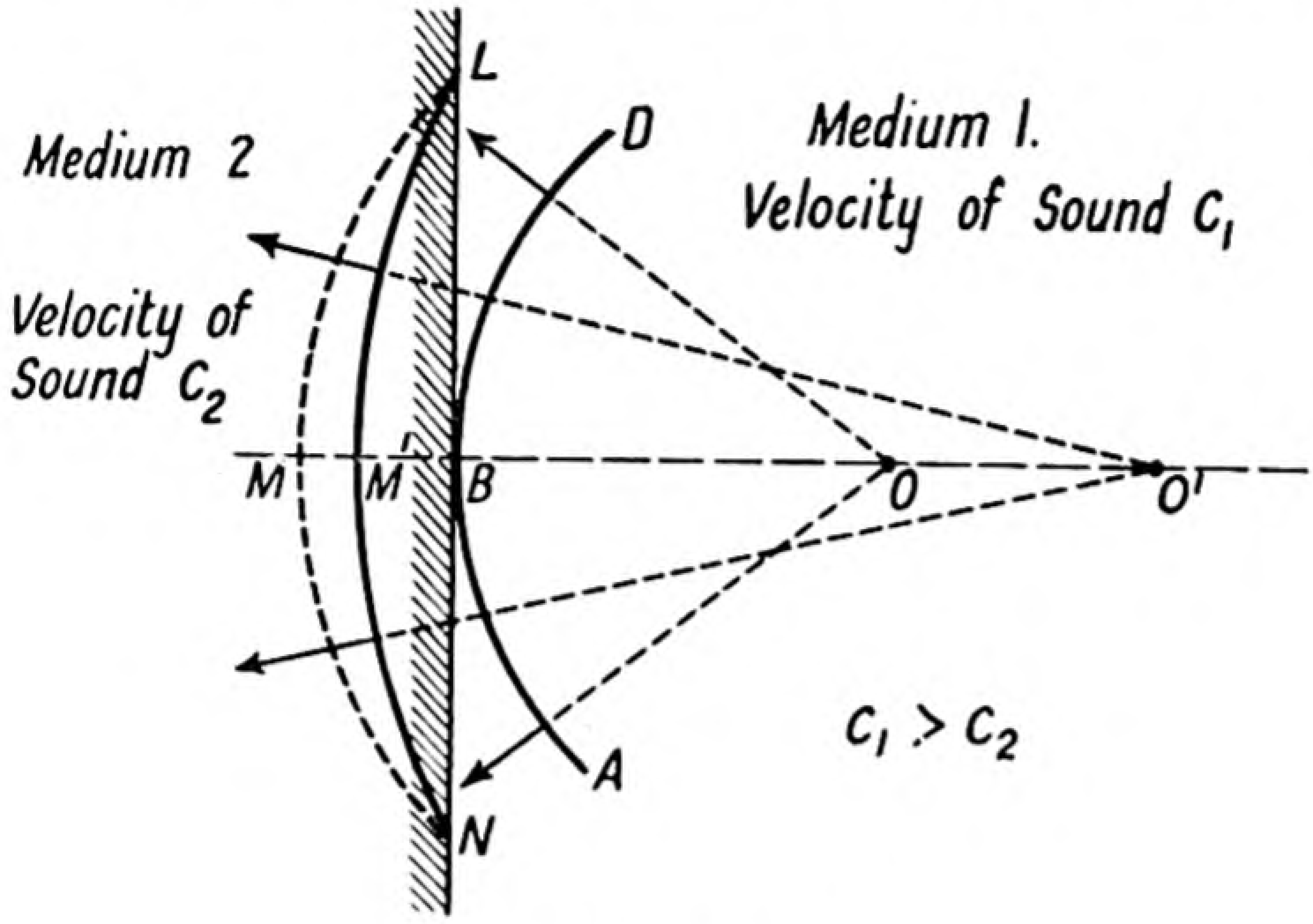


Fig. 6.17.

deductions follow the theme of the previous case. C is the centre of curvature of the reflecting surface.

So far only an abrupt change in the properties of a medium have been considered, but many examples occur in nature in which the

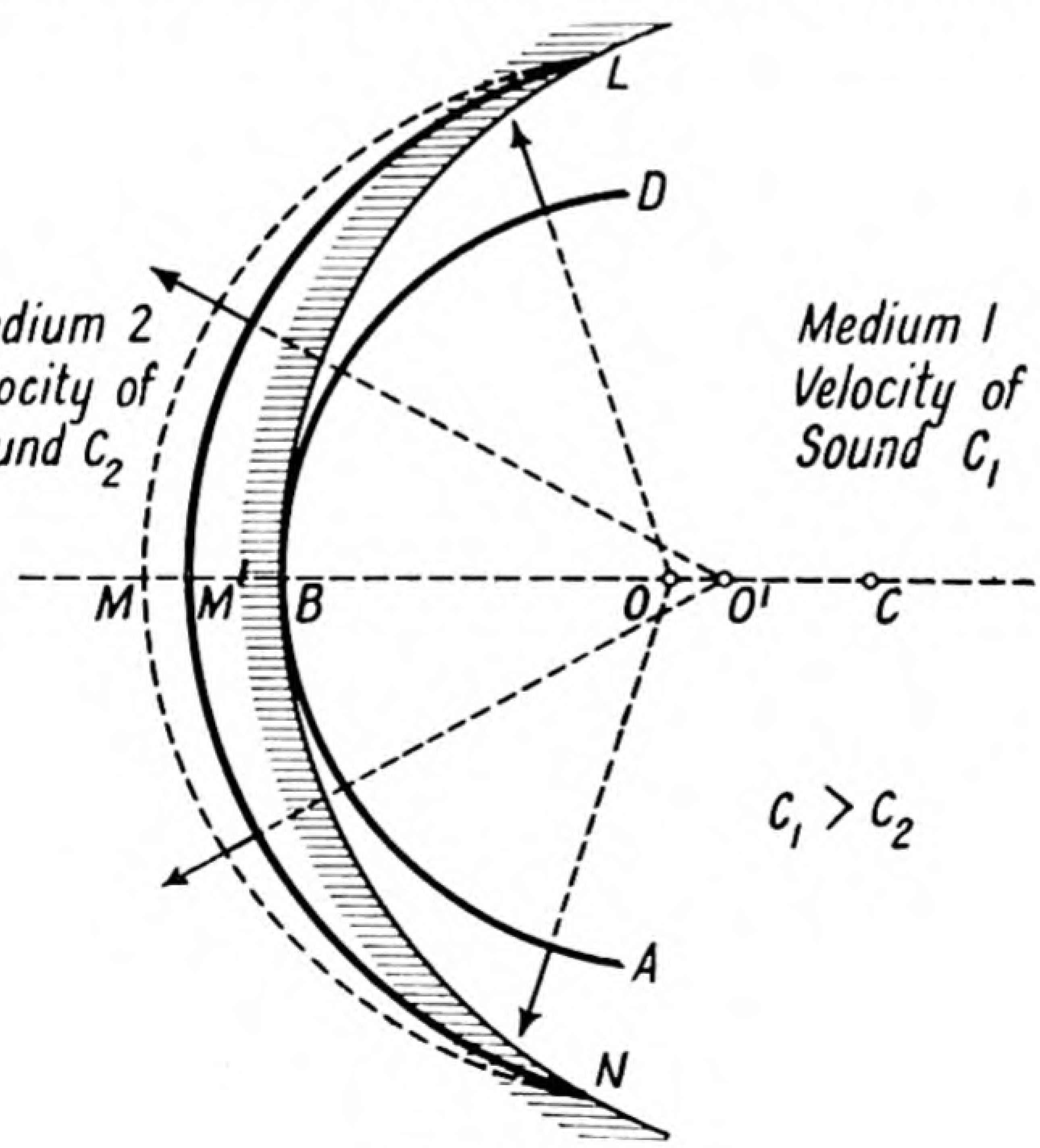


Fig. 6.18.

change takes place gradually, and in these cases the change in direction of the refracted ray will also be correspondingly gradual as indicated in Fig. 6.24.

Partial reflection and measurement of sound absorption

Plane sound waves are propagated with constant velocity and without change of type, in any direction within a homogeneous medium, but the presence of any discontinuity, *i.e.* a change of mechanical properties at any portion of the medium, will immediately disturb the uniformity of propagation. So far consideration has only been given to the case of sound waves reflected at a perfectly rigid and smooth surface so that the amplitude of the reflected wave is equal to that of the incident, *i.e.* the reflection may be said to be one hundred per cent. In practice this degree of completeness of reflection is never attained, although it is interesting to note that it is more complete for a sound wave incident upon a hard polished surface than for a light wave upon a correspondingly good flat optical surface. This imperfectness of reflection will imply that the strength (defined in Appendix 21) of the image I (Fig. 6.7) is no longer to be taken as equal to S, the strength of the source O, but to some value fS where f is fractional and for most materials will vary with the angle of incidence of the impinging wave upon the reflecting surface. The square of the factor f, i.e. f^2, is known as the coefficient of reflection (of sound energy) of a surface.

As an introduction to the general problem of partial reflection, consider the case of plane waves travelling down a tube closed at one end by a plug of the material under investigation. The reflection is thus restricted to one of normal incidence, but the problem is a more general case of the resonance tube dealt with on p. .

Let $\eta_1 = a \sin(\omega t - kx)$ represent the incident wave where η_1, the displacement of a particle at any time t, varies with its position x along the tube and has a maximum value a. ω is the pulsatance defined by $\omega = 2\pi N = \frac{2\pi}{T}$, where N is the frequency and T is the period of the waves, while $k = \frac{2\pi}{\lambda}$ and λ is the wave-length. The displacement η_2 due to the reflected wave will be given by $\eta_2 = fa \sin(\omega t + kx)$. Hence the resultant displacement at any point and time will be given by

$$\begin{aligned}\eta = \eta_1 + \eta_2 &= a \sin(\omega t - kx) + fa \sin(\omega t + kx)\\ &= a(1+f)\cos kx \sin \omega t - a(1-f)\sin kx \cos \omega t, \quad \text{(1)}\end{aligned}$$

which represents two standing wave systems both of period $T = \frac{2\pi}{n}$ but $\frac{T}{4}$ out of phase with one another. An inspection of the above equation shows that the amplitudes of the two sets of waves vary from position to position and are respectively given by $a(1+f)\cos kx$ and $a(1-f)\sin kx$. The maximum values of the former occur when $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$, etc., and for the latter when $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$, etc.; these values are respectively $a(1+f)$ and $a(1-f)$.

Since the maximum amplitude of one standing wave system occurs at places where the amplitude of the other is zero, it follows that these maximum amplitudes may be measured separately and their ratio is given by $\frac{a(1+f)}{a(1-f)} = \frac{A_1}{A_2}$, say.

Since the energy of a wave is proportional to the square of the amplitude of a wave (see p. 43), then the coefficient of reflection $\alpha = f^2 = \left(\frac{A_1 - A_2}{A_1 + A_2}\right)^2$.

The fraction of the incident sound energy which is not reflected passes

into the reflecting medium itself and becomes absorbed to an extent dependent upon the nature of the medium and usually also upon the frequency of the waves. On reaching the second boundary face of the specimen the *attenuated* plane wave will emerge as a plane wave with a constant but reduced amplitude. This latter wave is usually referred to as the transmitted wave, while the wave within the material is known as the refracted wave. It is usually recognised that the absorption of the material is measured by the fraction of the incident energy which is not reflected, *i.e.* by

$$\beta=1-\alpha=1-f^2=\frac{4A_1A_2}{(A_1+A_2)^2}.$$

This expression for β may alternatively be written as

$$\beta=\frac{4}{2+\frac{A_1}{A_2}+\frac{A_2}{A_1}}$$

which is more convenient for computing β from the observed ratio of A_1 to A_2.

It is instructive to rewrite equation (1) in the form

$$\eta=[a-fa]\sin(\omega t-kx)+fa[\sin(\omega t-kx)+\sin(\omega t+kx)] \quad . \quad . \quad (2)$$

whence $$\eta=a[1-f]\sin(\omega t-kx)+2fa\sin\omega t\cos kx \quad . \quad . \quad . \quad . \quad (3)$$

The latter equation comprises two terms, the first term represents a *progressive* wave and the second a standing wave. The first is only zero if

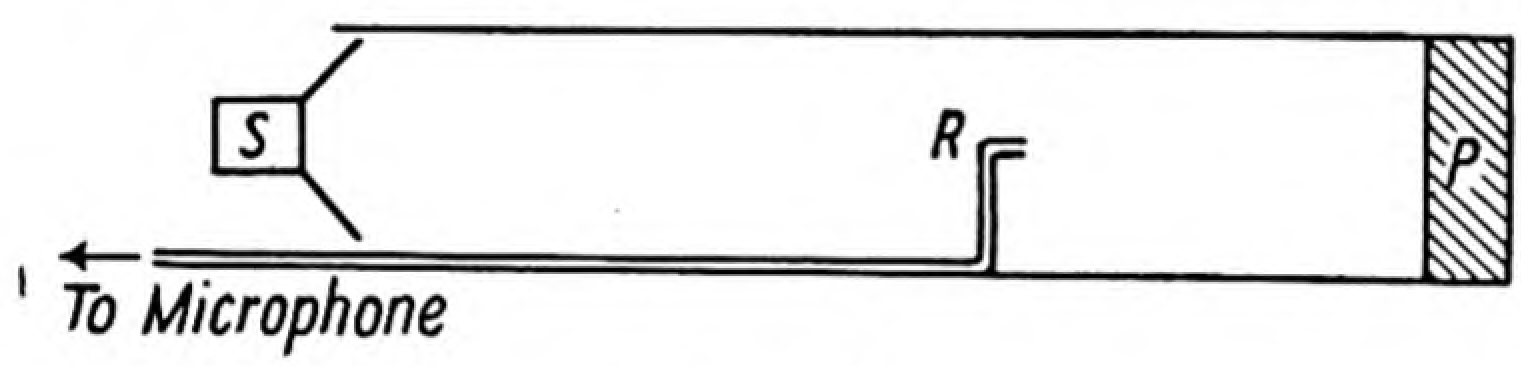


Fig. 6.19.

$f=1$, *i.e.* if complete reflection occurs, and this is the condition to be expected when no energy is carried forward into the medium closing the end of the tube.

The general procedure underlying the method adopted by Taylor, Paris, and others to measure the sound absorption coefficients of material is shown by Fig. 6.19 where the specimen under test forms a plug P at one end of the resonating tube, which is energised by the loud-speaker S. The investigation of the pressure variation along the axis is carried out by means of the small exploring tube R which is connected to a microphone with its amplifier and indicating meter. The formula derived above may be applied and the tube should be sufficiently long for a number of maxima to be observed. This tube method of determining sound absorption coefficients is limited in its application to small samples of material, which must therefore be fairly homogeneous to be representative of the whole. Furthermore, the method only measures the absorption coefficient for normal incidence, whereas in practice its value for all angles of incidence is what is chiefly required.

It should be emphasised again here that when sound waves undergo reflection at the boundary of two media, changes of phase occur in the reflected and refracted portions of the dispersed incident beams. In the case where the "reflecting medium" is a perfectly rigid material there is no change of phase in the refracted wave but the reflected wave is π radians out of phase with the incident wave, whereas reflection at the "open" end of a resonance tube occurs without reversal of displacement phase for the

reflected wave. These two cases correspond to $f=-1$ and $f=+1$ respectively, and it follows that for a porous and flexible solid material the value of the phase change will assume an intermediate value dependent upon the density and rigidity of the substance. The effect of the phase change is merely to shift the stationary wave system, so that the above analysis does not become invalid when estimating α (see Appendix 13).

Partial reflection of plane sound waves for any angle of incidence upon the boundary surface of two extended media

The reader is referred to Rayleigh's "Sound" (Vol. II, § 270) for a more rigorous analysis of the problem, but the following treatment at this stage has the merit of emphasising the physical principles involved.

It is assumed that the linear dimensions of the boundary surface are large compared with the length of the incident sound waves, so that the geometrical laws of reflection and refraction are applicable as for the corresponding problem in optics. If this condition is not satisfied then the problem becomes essentially one of diffraction.

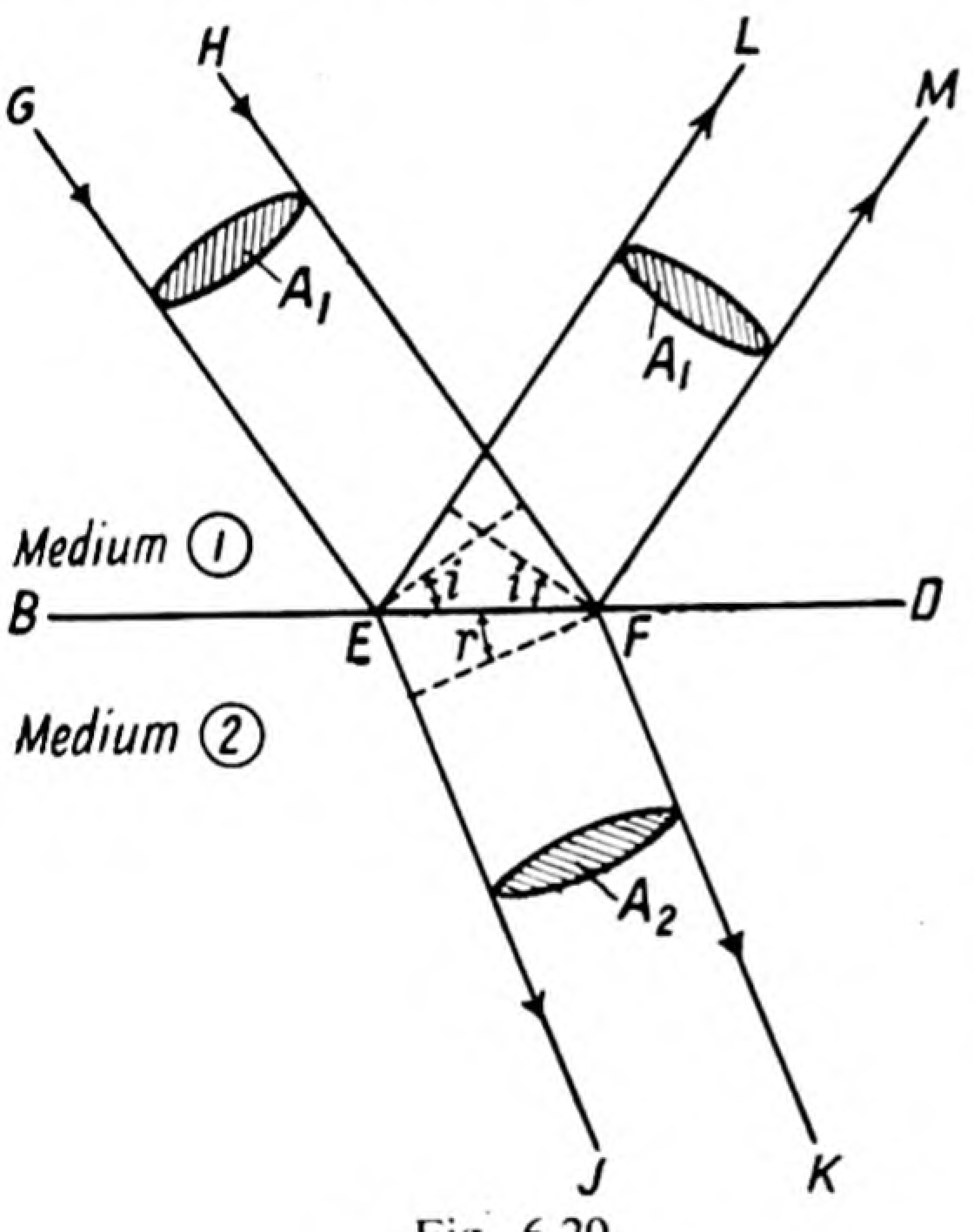


Fig. 6.20.

Let $BEFD$ (Fig. 6.20) represent a section of the boundary surface and $GEFH$, $EFML$ and $EFKJ$ sections of the incident, reflected and refracted beams respectively. The cross-sections of the appropriate beams are indicated by A_1 and A_2 in the figure, the suffix 1 referring to the medium in which the sound wave is initially travelling and the suffix 2 to the second medium. The velocities of sound in the two media are therefore c_1 and c_2 respectively as indicated in the ray diagram (Fig. 6.15), the corresponding densities of the media being ρ_1 and ρ_2.

The geometrical law of reflection states that the angle of reflection is equal to the angle of incidence (i), and Snell's law of refraction gives $\dfrac{\sin i}{\sin r}=\dfrac{c_1}{c_2}$, where r is the angle of refraction. The angles i and r are measured between their respective wave-fronts and the boundary surface (Fig. 6.20) or the equivalent angles between the normals N_1E and N_2E and the normals to the wave-fronts (Fig. 6.15).

Two conditions require to be satisfied at the boundary surface as follows:—

(i) **Continuity of displacement.** This condition is a consequence of the assumption that the interface between the two media remains undisturbed during the passage of the sound wave, *i.e.* that there is no separation or relative motion of the media perpendicular to their common surface, or slipping of one medium upon the other parallel to this surface.

If a_i, a_r, a_t be the amplitudes, *i.e.* the maximum displacements, of a particle due to the passage of the incident, reflected and refracted (alternatively known as transmitted) waves respectively, then it follows from the above

condition that the normal components of these on each side of the boundary must be equal,

i.e.
$$a_i \cos i - a_r \cos i = a_t \cos r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The negative sign accrues from the reversed direction of propagation of the reflected wave from that of the incident and refracted waves.

(ii) Conservation of energy. This condition assumes that the energy in the incident sound wave reaching the boundary surface per second must equal the sum of that carried away per second in the reflected and refracted waves. This assumption will not be valid if the second medium absorbs any part of the refracted beam, such absorbed energy appearing as a heating of the medium.

Now the energy per unit volume of a medium due to the passage of plane sound waves is given by equation 23 on page 43 as $\frac{1}{2}\rho a^2 \omega^2$, where ρ is the density of the medium, a is the amplitude of motion and $\frac{\omega}{2\pi}$ is the frequency of the waves. The energy reaching the surface BD per second is contained within a volume $A_1 \times c_1$, and that carried away by reflected and refracted waves within volumes $A_1 \times c_1$ and $A_2 \times c_2$ respectively. Hence it follows from the above condition that

$$\rho_1 A_1 c_1 (a_i)^2 \omega^2 = \rho_1 A_1 c_1 (a_r)^2 \omega^2 + \rho_2 A_2 c_2 (a_t)^2 \omega^2 \quad . \quad . \quad . \quad (5)$$

But $\frac{A_1}{\cos i} = \frac{A_2}{\cos r}$, therefore (5) may be written as

$$\rho_1 c_1 (a_i^2 - a_r^2) = \rho_2 c_2 a_t^2 \frac{\cos r}{\cos i}$$

or
$$\rho_1 c_1 a_i^2 (1 - f^2) = \rho_2 c_2 a_t^2 . \frac{\cos r}{\cos i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where
$$f = \frac{a_r}{a_i}.$$

(6) may now be rewritten, after squaring each side, as

$$a_i^2 (1 - f)^2 \cos^2 i = a_t^2 \cos^2 r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From (6) and (7) it follows that

$$\left(\frac{1+f}{1-f}\right) = \frac{\rho_2 c_2}{\rho_1 c_1} . \frac{\cos i}{\cos r} = \frac{\rho_2}{\rho_1} . \frac{\sin r}{\sin i} . \frac{\cos i}{\cos r} = \frac{\rho_2}{\rho_1} \frac{\cot i}{\cot r}$$

as by Snell's law
$$\frac{\sin i}{\sin r} = \frac{c_1}{c_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The ratio of the reflected to incident amplitude of vibration is therefore given by

$$f = \frac{\left(\frac{\rho_2}{\rho_1} - \frac{\cot r}{\cot i}\right)}{\left(\frac{\rho_2}{\rho_1} + \frac{\cot r}{\cot i}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The condition for *complete transmission*, *i.e.* no reflected wave, is easily deduced from equation (9) as

$$\frac{\rho_2}{\rho_1} - \frac{\cot r}{\cot i} = 0 \quad \text{or} \quad \frac{\rho_2}{\rho_1} = \frac{\cot r}{\cot i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence from (8) and (10) it may be shown that for complete transmission to occur, the angle of incidence must satisfy the relation

$$\cot^2 i = \frac{\frac{c_1^2}{c_2^2} - 1}{\frac{\rho_2^2}{\rho_1^2} - \frac{c_1^2}{c_2^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

i.e. $\frac{c_1}{c_2}$ must be intermediate in value between unity and $\frac{\rho_2}{\rho_1}$, for $\cot^2 i$ to be positive for real values of ρ_1, ρ_2, c_1 and c_2. This condition holds in the case where both media are gases. On page 38 it is shown that $c=\sqrt{\frac{\gamma p}{\rho}}$ where γ, the ratio of the specific heats of a gas, does not vary appreciably for the simple gases. Hence it follows that $\frac{c_1}{c_2}=\sqrt{\frac{\rho_2}{\rho_1}}$ approximately, and

therefore $\cot^2 i=\dfrac{\left(\frac{\rho_2}{\rho_1}-1\right)}{\left(\frac{\rho_2{}^2}{\rho_1{}^2}-\frac{\rho_2}{\rho_1}\right)}=\frac{\rho_1}{\rho_2}$, which is essentially positive.

The general amplitude relation may now be expressed in a simplified form for the case of two gases by substituting $\frac{\sin i}{\sin r}=\frac{c_1}{c_2}=\sqrt{\frac{\rho_2}{\rho_1}}$ in equation (9). This leads to $f=\frac{\tan(i-r)}{\tan(i+r)}$, which, it may be noted, is also the relation given by Fresnel for the reflection of light polarised perpendicularly to the plane of incidence.

Total reflection

If c_1 is less than c_2 (Fig. 6·20) then from Snell's law it follows that there will be a critical angle of incidence i_c for which the angle of refraction r is 90°,

i.e. $$\frac{\sin i_c}{\sin 90°}=\frac{c_1}{c_2} \quad \text{or} \quad i_c=\sin^{-1}\left(\frac{c_1}{c_2}\right).$$

The velocity of sound in water is greater than that in air, hence total reflection can occur for waves travelling from air to water but not in the reverse direction,

e.g. $$c_{\text{water}}=4{\cdot}6c_{\text{air}}; \qquad \therefore\ i_c=\sin^{-1}\frac{1}{4{\cdot}6}=12° \text{ approx.}$$

Hence 12° is the greatest angle of incidence for which sound is refracted into water from air, and in a succeeding paragraph it will be shown that very little sound energy enters the water even at normal incidence.

Normal incidence. In this case i and r both tend to zero, and it follows that

$$\frac{c_1}{c_2}=\left(\frac{\sin i}{\sin r}\right)_{\substack{i\to 0\\ r\to 0}}=\left(\frac{\tan i}{\tan r}\right)_{\substack{i\to 0\\ r\to 0}}=\left(\frac{\cot r}{\cot i}\right)_{\substack{i\to 0\\ r\to 0}}$$

Hence equation (9) now becomes

$$f=\frac{\left(\frac{\rho_2}{\rho_1}-\frac{c_1}{c_2}\right)}{\left(\frac{\rho_2}{\rho_1}+\frac{c_1}{c_2}\right)}=\frac{\rho_2 c_2-\rho_1 c_1}{\rho_2 c_2+\rho_1 c_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The condition for complete transmission at normal incidence is therefore that $\rho_1 c_1=\rho_2 c_2$. The product of velocity and density is variously known as the characteristic acoustical impedance for unit area of the medium or its radiation resistance. Denoting $\rho_1 c_1$ and $\rho_2 c_2$ by R_1 and R_2 respectively, equation (12) may now be written simply as

$$f=\frac{R_2-R_1}{R_2+R_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The above analysis indicates therefore that in order to obtain the maximum transfer of acoustical energy from one medium to another, the characteristic acoustical impedance of each must be as nearly equal to the other as possible. This problem of *impedance matching*, as it is termed, occurs in many branches of physics, as for instance in order to obtain the maximum energy output from a battery the external load resistance should be chosen equal to that of the battery. Again, in optics light is reflected at the surface of discontinuity of two media, *e.g.* air and glass, but if the surfaces of the media are separated by a film which is a quarter of a wave-length thick and has a suitably chosen refractive index, then reflection for light of this particular wave-length is eliminated. The value of this refractive index μ must be intermediate between the refractive indices μ_1 and μ_2 of the two media, or more exactly $\mu=\sqrt{\mu_1\mu_2}$. In this example the optical film plays the part of an impedance-matching device with respect to the media which it separates, and its behaviour is thus analogous to the electrical transformer in alternating current circuits. A similar prodecure is employed in acoustical technique (see also p. 347) as a means of matching two media of widely different acoustical impedances R_1 and R_2 respectively. The matching medium is sandwiched between the other two and should be the appropriate thickness relative to the wave-length of the sound to be transmitted, and its acoustical impedance R should be as nearly equal to $\sqrt{R_1R_2}$ as possible.

It is appropriate to point out here that only *sudden* discontinuities of the properties of a medium give rise to reflection and these changes should occur within distances which are *short* compared with the length of the waves employed. A transitional change of properties in small steps would seem therefore to be the obvious means of reducing reflection to a minimum; in other words, the objective should be to simulate the characteristics of a transmission line. Such a device in sound is provided by the acoustical horn, whereby the acoustical impedance of the pipe feeding the horn is matched to the impedance of the free air at the mouth. This result is brought about by a *gradual* change in the cross-section of the horn which results in radiation of acoustical energy from its mouth, in contrast to being reflected to and fro with the formation of useless standing waves as is the case with the ordinary resonance tube.

Reflection and transmission coefficients at normal incidence

The reflection coefficient $\alpha=\dfrac{\text{reflected sound energy}}{\text{incident sound energy}}$

$$=\frac{\rho_1c_1a_r^2}{\rho_1c_1a_i^2}$$

$$=\frac{a_r^2}{a_i^2}$$

$$=\left(\frac{R_2-R_1}{R_2+R_1}\right)^2 \qquad (14)$$

The transmission coefficient $\beta=\dfrac{\text{transmitted (or refracted) sound energy}}{\text{incident sound energy}}$

$$=\frac{\rho_2c_2a_t^2}{\rho_1c_1a_i^2}$$

$$=\frac{R_2}{R_1}\cdot\frac{a_t^2}{a_i^2} \qquad (15)$$

But from (7)

$$\frac{a_t^2}{a_i^2}=(1-f)^2\frac{\cos^2 i}{\cos^2 r}=(1-f)^2 \text{ at normal incidence.}$$

Hence (15) now becomes

$$\beta=\frac{R_2}{R_1}.(1-f)^2=\frac{R_2}{R_1}\left[1-\left(\frac{R_2-R_1}{R_2+R_1}\right)\right]^2$$

$$=\frac{4R_2R_1}{(R_2+R_1)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

An inspection of the expressions for the reflection and transmission coefficients shows that they are both independent of the wave-length of the sound, and consequently any incident wave which is complex in type, *i.e.* comprises a mixture of frequencies, will be reflected or refracted with its wave-form unchanged. Furthermore, α and β may be written respectively as $\dfrac{\left(\dfrac{R_2}{R_1}-1\right)^2}{\left(\dfrac{R_2}{R_1}+1\right)^2}$ and $\dfrac{\dfrac{4R_2}{R_1}}{\left(\dfrac{R_2}{R_1}+1\right)^2}$, *i.e.* both coefficients are symmetrical with respect to R_1 and R_2; hence the ratios of the reflected and transmitted energies will be identical on whichever side of the boundary surface the wave is incident.

Transmission of sound from a medium of small to one of a large acoustical impedance

In this case R_1 is small compared with R_2 and almost complete reflection is obtained even at normal incidence. The coefficients α and β can now assume the approximate expressions

$$\alpha=\frac{\left(1-\dfrac{R_1}{R_2}\right)^2}{\left(1+\dfrac{R_1}{R_2}\right)^2}=\frac{\left(1-\dfrac{2R_1}{R_2}\right)}{\left(1+\dfrac{2R_1}{R_2}\right)}=1-\frac{4R_1}{R_2}$$

and $\beta=\dfrac{4R_1}{R_2}$.

R_1 for air $=41$, R_2 for water $=1{\cdot}43\times 10^5$ c.g.s.

Hence percentage of acoustical energy entering water from the air at normal incidence $=\dfrac{400R_1}{R_2}=0{\cdot}12$.

Transmission of sound from a liquid to a solid

The acoustical impedances of the two media are now more nearly comparable. For example, consider the transmission from water to steel. R_1 for water $=1{\cdot}43\times 10^5$ c.g.s.; R_2 for steel $=3{\cdot}9\times 10^6$ c.g.s.;

$$\therefore \alpha=\left(\frac{3{\cdot}9-{\cdot}14}{3{\cdot}9+{\cdot}14}\right)^2=\left(\frac{3{\cdot}76}{4{\cdot}04}\right)^2=0{\cdot}866.$$

Hence 86·6 per cent. of the incident sound energy is reflected and 13·4 per cent. is transmitted.

Detection of under-water objects. The foregoing theory has an important application in the possibility of locating under-water objects, such as icebergs or wrecked ships, by echo-sounding methods. The success or otherwise of these methods is dependent upon the intensity of the sound beam

reflected from the object being sufficiently large for detection. Now the acoustical impedance for ice $=1{\cdot}9\times10^5$ c.g.s., hence $\alpha=\left(\frac{1{\cdot}9-1{\cdot}43}{1{\cdot}9+1{\cdot}43}\right)^2=0{\cdot}022$, or only just over 2 per cent. of the incident sound energy is reflected at normal incidence. Such circumstances would not be favourable for using echo-sounding methods but for the fortunate fact that icebergs contain appreciable inclusions of rock and air cavities, and their presence leads to a considerable enhancement of the sound energy reflected.

Reflection of sound at the interface of two gaseous media

On page 110 it is shown that $\frac{c_1}{c_2}=\sqrt{\frac{\rho_2}{\rho_1}}$ where c and ρ refer to the velocity of sound and density of a gas respectively. Hence from equation (12),

$$f=\frac{\left(\frac{\rho_2}{\rho_1}-\frac{c_1}{c_2}\right)}{\left(\frac{\rho_2}{\rho_1}+\frac{c_1}{c_2}\right)}=\frac{\left(\frac{\rho_2}{\rho_1}-\sqrt{\frac{\rho_2}{\rho_1}}\right)}{\left(\frac{\rho_2}{\rho_1}+\sqrt{\frac{\rho_2}{\rho_1}}\right)}$$

i.e.
$$f=\frac{\sqrt{\rho_2}-\sqrt{\rho_1}}{\sqrt{\rho_2}+\sqrt{\rho_1}}\quad\text{or}\quad\frac{c_1-c_2}{c_1+c_2}\quad\ldots\ldots\quad(17)$$

If the difference in velocities is very small then

$c_2=c_1+\Delta c_1$ say, and therefore $f=\frac{-\Delta c_1}{2c_1}$ approx. . . . . (18)

Suppose that a sound wave is travelling from nitrogen (density $=0{\cdot}001251$ gm. per cc.) to oxygen (density $=0{\cdot}001429$ gm. per cc.) at normal incidence, then substituting in (17) gives the ratio of the reflected to incident amplitude as

$$f=\frac{\sqrt{\cdot001429}-\sqrt{\cdot001251}}{\sqrt{\cdot001429}+\sqrt{\cdot001251}}$$
$$=\frac{\cdot0024}{\cdot0732}$$
$$=\cdot033.$$

Hence the reflected energy is $(\cdot033)^2$, *i.e.* 0·0011, of the incident energy.

If the sound wave travels initially in hydrogen (density $=0{\cdot}00008837$ gm. per cc.) instead of in nitrogen, then 0·36 of the incident energy is reflected.

A sound wave passing through the air may suffer reflection from strata differing in temperature and humidity although the magnitude of the effect under conditions usually prevailing in the atmosphere is quite small. The appropriate formula for calculating the effect due to differences in temperature between different regions of a gas is obtained by combining (18) with the relation $c\propto\sqrt{T}$ which leads to the numerical result $f=\frac{1}{4}\frac{\delta T}{T}$ where T is the absolute temperature.

The foregoing theory does not reveal completely how the acoustical energy of a sound wave is actually distributed after its impact upon a solid medium. It is found that besides the reflected and refracted compressional waves, two other types may be produced, known respectively as flexural waves and surface, or Rayleigh (p. 114), waves. Furthermore, a portion of the energy of the transmitted, otherwise refracted, wave may become dissipated as heat within the medium.

In the case of a porous body this dissipation will result from the viscous resistance opposing the vibratory motion of the air within the narrow cavities. The energy lost in this manner represents what is usually understood as the true acoustic *absorption* of the medium. The energy which emerges in the form of sound waves from the surface of the solid remote from the source will consist of the transmitted compressional wave, the motion in this case being handed on from particle to particle of the medium in the manner of the transfer of heat by conduction, and in addition the disturbance due to the flexural vibrations of the solid. The latter result from the vibration of the solid as a whole, under the forcing action of the incident waves. In the case of the partition walls commonly used in houses, the larger percentage of the sound energy passing from one room to another is conveyed through the air-vents and by the flexural vibrations of the walls.

Wave motion and geophysics

If the waves in a solid elastic medium of unlimited extent are the result of a local disturbance within its interior, then these waves diverge from the source or focus, as it is often termed, with two distinct

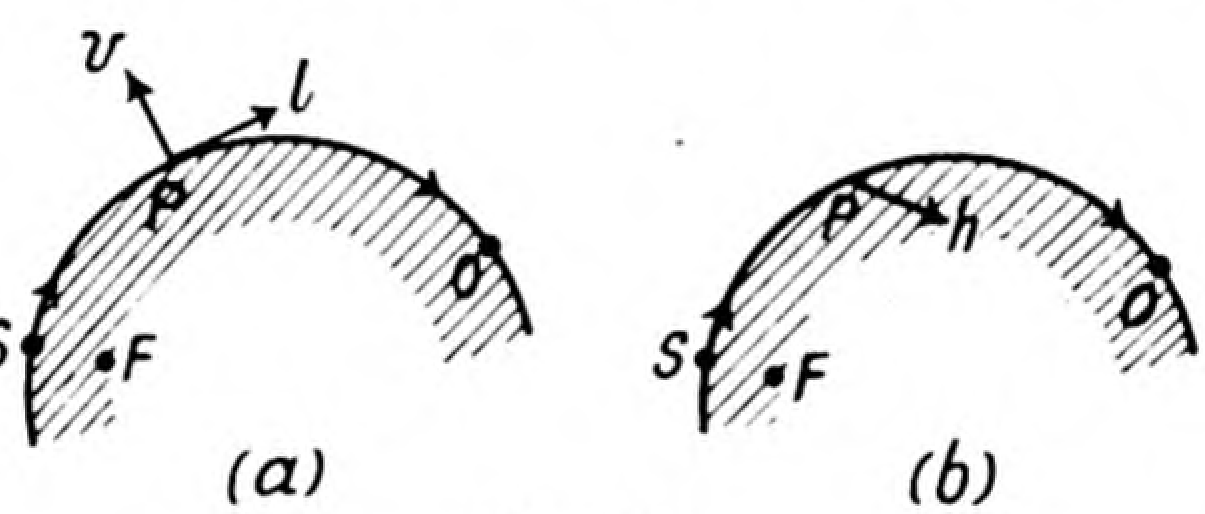


Fig. 6.21.

velocities C_l and C_t (p. 61), corresponding respectively to the longitudinal and transverse modes of propagation. An example of this type of wave propagation is provided by the so-called seismic waves generated by an earthquake. The instruments used to record these waves, known as seismographs, give indications of four types of waves, the two mentioned above which are propagated directly to the observer through the material of the earth, and two types of surface waves named after their discoverers, Rayleigh and Love respectively. These surface waves are confined to a surface layer and their common line of propagation to the listening point P (Fig. 6.21*a*) follows that of a great circle, starting from a point S on the earth's surface immediately above the focus F, and hence their times of arrival at O are delayed behind those of the compressional and distortional waves by amounts greater than those due to the difference in the velocities of propagation. In actual practice the seismic records show a very wavy trace, which is to be expected when it is considered that the earth does not constitute a homogeneous medium and so the original waves leaving the source become split up by refraction and reflection at the various strata or discontinuities in the earth's crust.

Although the scope of this book does not permit the development of the mathematical analysis of the Rayleigh and Love waves, it is

possible in a general way to see how the waves arise by considering again the propagation of longitudinal waves along a solid cylinder or prism. If the linear dimensions of the cross-section of the solid body are small compared with the length of the waves, then the velocity of propagation in the solid is given (p. 38) by $C_s=\sqrt{\frac{E}{\rho}}$, where E is Young's modulus and ρ the density of the medium. The sides of such a body being without restraint are free to move, which accounts for the difference in the velocities C_s and C_l. It is therefore a natural deduction to presume that when the longitudinal (or compressional) and transverse (or distortional) waves travelling in the so-called infinite medium approach a free boundary then the conditions of their propagation undergo modification. In point of fact, the velocity of the Rayleigh waves C_R for an *incompressible* solid can be shown theoretically to be equal to $0{\cdot}955C_t$. In the case of the Rayleigh waves the surface particles at any point P (Fig. 6.21) in their path will possess component vibrations vertically and along Pl, both being in the vertical plane containing the direction of propagation. On the other hand, for Love waves the particle displacements are restricted

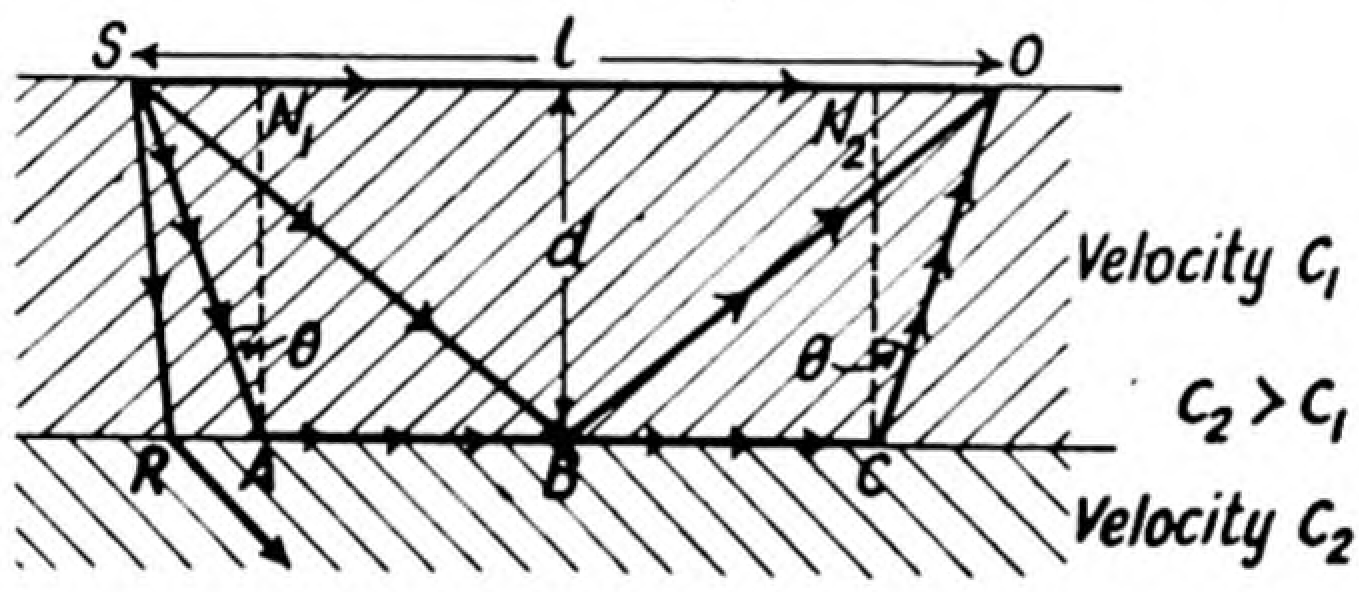


Fig. 6.22.

to the horizontal direction Ph (Fig. 6.21b), *perpendicular* to the direction of propagation. Both types of waves appear intermingled in the seismic record.

Since the surface waves diverge only in two directions they are less attenuated than the C_l and C_t waves, and hence become of increasing importance at large distances from the source.

Seismographic methods are often employed in the location of oil deposits, and the method to be described depends for its operation on the marked difference in the velocity of propagation of earthquake or sound waves in oil- and non-oil-bearing strata, *e.g.* in Persia the velocity of propagation in the limestone strata which bear the oil is $4{\cdot}7\times10^4$ cm. per sec., as compared with a velocity of $3{\cdot}7\times10^4$ cm. per sec. in the overlying soil.

If an explosion of suitable magnitude is created at or just below the earth's surface at S, say (Fig. 6.22), then the earthquake wave produced will travel out in all directions, and three possible ways in which the effect of the disturbance can reach O are shown in the diagram. The direct wave SO and the wave SBO reflected from the interface $RABC$ both travel entirely in the upper strata, while waves

such as SR pass completely into the lower strata. The wave in the direction SA, striking the interface at the critical angle $\theta = \sin^{-1}\frac{C_1}{C_2}$, travels in the boundary layer and the disturbed particles give rise to Huyghens wavelets in the upper media, and one such diffracted wave CO is shown. If its time of arrival at O coincides with that of the direct wave then obviously

$$\frac{l}{C_1} = \frac{SA+CO}{C_1} + \frac{AC}{C_2}$$

$$= \frac{2d}{C_1 \cos\theta} + \frac{l-2d\tan\theta}{C_2} \quad . \quad . \quad . \quad . \quad (19)$$

Fig. 6.23*a* shows the positions of a number of seismographic recorders O'', O', O, O_1 and O_2 situated at different distances from the

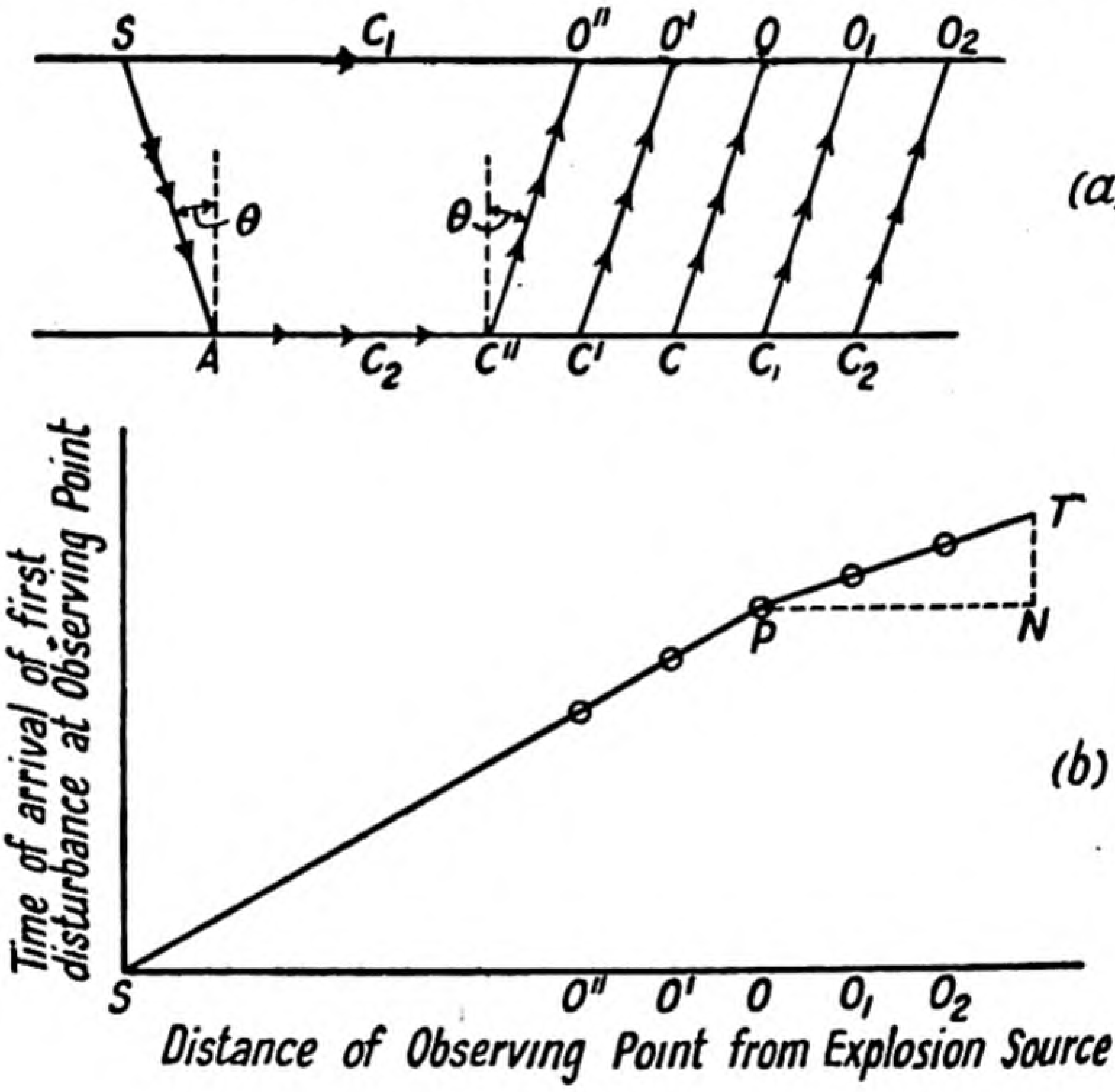


Fig. 6.23.

shot point S, and $C''O''$, $C'O'$, etc., are parallel diffracted waves directed towards the various observation points. In the diagram the direct wave is assumed to arrive at O'' and O' before the diffracted wave, while the reverse is the case for O_1 and O_2; at O, however, both waves arrive simultaneously. The graph drawn in Fig. 6.23*b* is self-explanatory, and it is to be noted that the slope of the line SP as measured by $\frac{SO}{OP}$ gives the velocity C_1 of the waves in the upper medium, while $\frac{PN}{NT}$ is a measure of the velocity C_2 in the lower strata. Since $\sin\theta = \frac{C_1}{C_2}$ and the distance l is given by SO, then sufficient data

are available for the calculation from equation (19) of the required depth d of the interface. The simple conditions chosen above would in practice be complicated by sloping interfaces, etc.

Meteorological acoustics

Since the velocity of sound in a gas varies as the square root of the absolute temperature, and since the earth's atmosphere is never completely quiescent and at a uniform temperature, then it is to be expected that the propagation of sound in the atmosphere will lack uniformity. An interesting case is when the condition known as temperature inversion exists in the atmosphere, which is otherwise quiescent and of uniform composition. Such a state of affairs, in which the temperature increases in moving upwards from the earth's surface, occurs chiefly over stretches of water on summer nights or on frosty nights over the land. Under these conditions sound waves radiated from a source near the ground are refracted in the inversion layer so that they are bent downwards (see Fig. 6.24) towards the earth's surface where

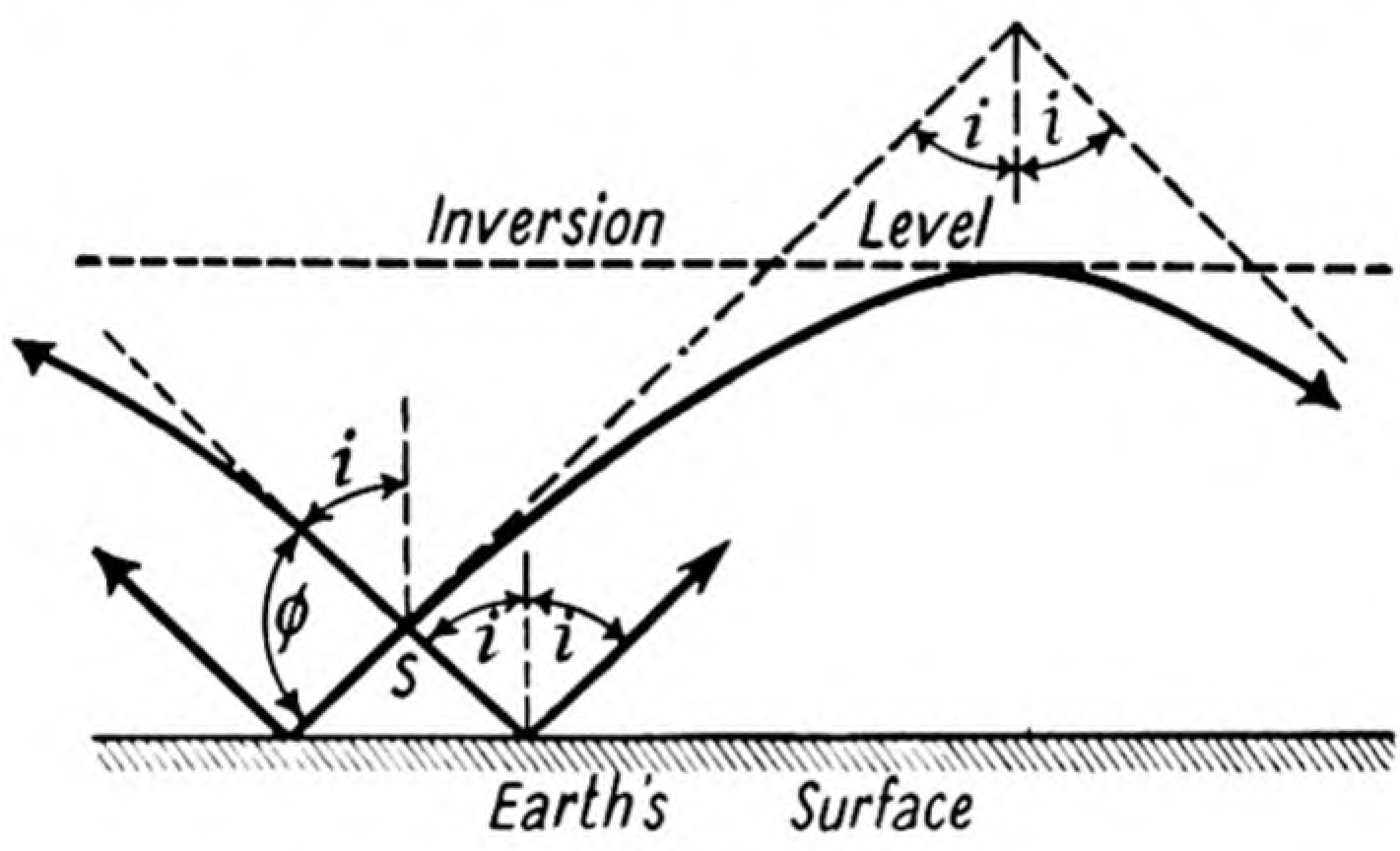


Fig. 6.24.

ordinary reflection occurs. In this manner the propagation of the acoustical energy is largely confined to a "channel" formed by the earth's surface and a horizontal stratum, usually some 15 or 20 metres above ground level, known as the inversion level, which will obviously be the plane showing the highest temperature. The range of sound is thus greatly increased, for the intensity falls off only as the inverse of the distance, and the above phenomenon explains why sounds are often heard at great distances on clear, still nights. The radio enthusiast will discern here a close parallel in the action of the inversion layer and the earth's surface towards sound waves to the similar way that the ionosphere and the earth form a wave-guide for electromagnetic waves.

A simple calculation will lead to the determination of the angle ϕ of the horizontal toroidal wedge, within which the sound is confined, when the temperature of the inversion level is known. Suppose that this temperature is 5° C.; then the smallest angle i for which there is no transmitted wave in the upper atmosphere is given by $i = \sin^{-1} \dfrac{c_1}{c_2}$,

where c_2 and c_1 are the velocity of sound at 5° C. and at 0° C. (the ground temperature) respectively.

Now, $$c_2 = c_1\left(1+\tfrac{5}{273}\right)^{\frac{1}{2}} = c_1\left(1+\tfrac{5}{546}\right) \text{ approx.};$$

$$\therefore\ i = \sin^{-1}\tfrac{546}{551} = 82^\circ \text{ approx.}$$

Hence $$\phi = 180^\circ - 2\times 82^\circ = 16^\circ \text{ approx.}$$

One of the unexpected observations of meteorological acoustics is that thunder cannot usually be heard at distances greater than about 10 km. from its origin, which is small compared with the corresponding distances for big guns. It is suggested that one reason for this is that the sound from a gun emanates from a small volume, whereas thunder is generated over the whole length of the lightning path, but a more important factor is probably the general atmospheric conditions under which the sounds are generated. In a thunderstorm the winds are

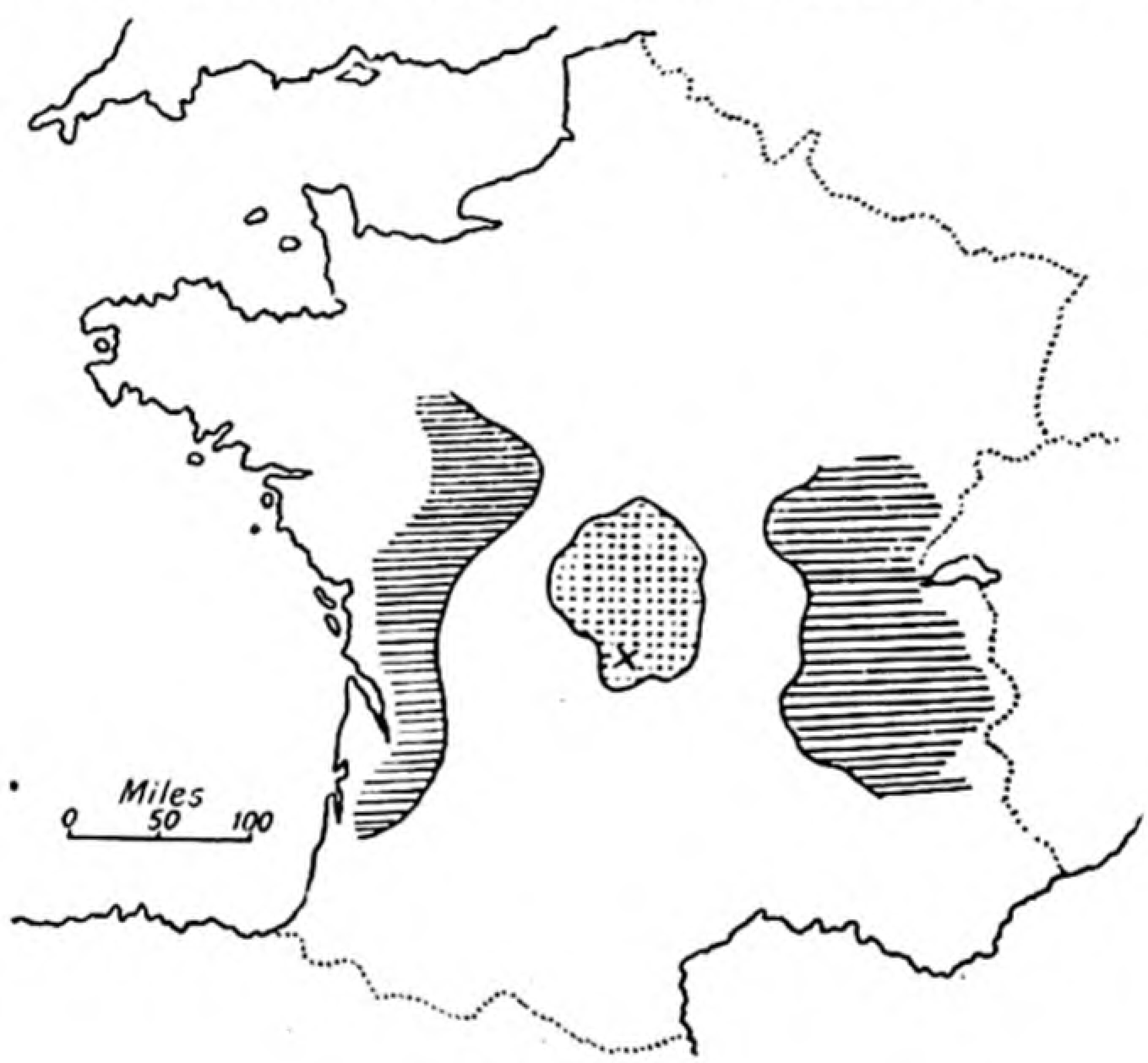


Fig. 6.25.

usually turbulent, and furthermore, the heat generated by the electric discharge will bring about a heterogeneous distribution of temperature in its neighbourhood, thereby setting up irregular convection currents. Consequently the sound waves generated will suffer more or less continuous but haphazard reflection and refraction, and in this way the acoustic energy becomes diffused through a large volume with a reduction in the sound range. It is interesting to note that an analysis of the frequency spectrum of thunder has shown the predominance of long inaudible waves, *i.e.* of the order of 5 c.p.s., whose presence is sometimes indicated by the rattling of windows.

It has been observed that the zone of sound around an explosion centre is followed by one of silence and then a further zone of sound (shaded areas, Fig. 6.25), which illustrates the general form of the sound field observed in the neighbourhood of a pre-arranged explosion at La Courtine (X in diagram) in June, 1924. Now the temperature condition that normally persists to a considerable depth in the lower

atmosphere is that in which the temperature *decreases* uniformly with height, and this atmospheric region is known as the troposphere (Fig. 6.26). Hence if a sound wave is projected upwards from the ground the lower part of the wave-front gains on the upper because the velocity of sound will be greater the nearer the ground, and consequently the wave-front is bent upwards and the sound range at ground level is reduced. In order to account for the existence of a second sound zone the wave-front must at length become bent downwards or, in other words, at very high altitudes the velocity of sound must commence to increase, presumably due to a reversal of the temperature gradient (see stratosphere region in Fig. 6.26). These deductions have been confirmed by actual temperature measurements in the upper atmosphere made by sending up small balloons carrying suitable recording instruments, which are attached to parachutes so that they fall gently to earth when the balloons burst. The latter are often referred to as ballons-sondes, for it was a Frenchman named Teisseronc de Bort who originally used this method of investigation. The distance AB (Fig. 6.26)

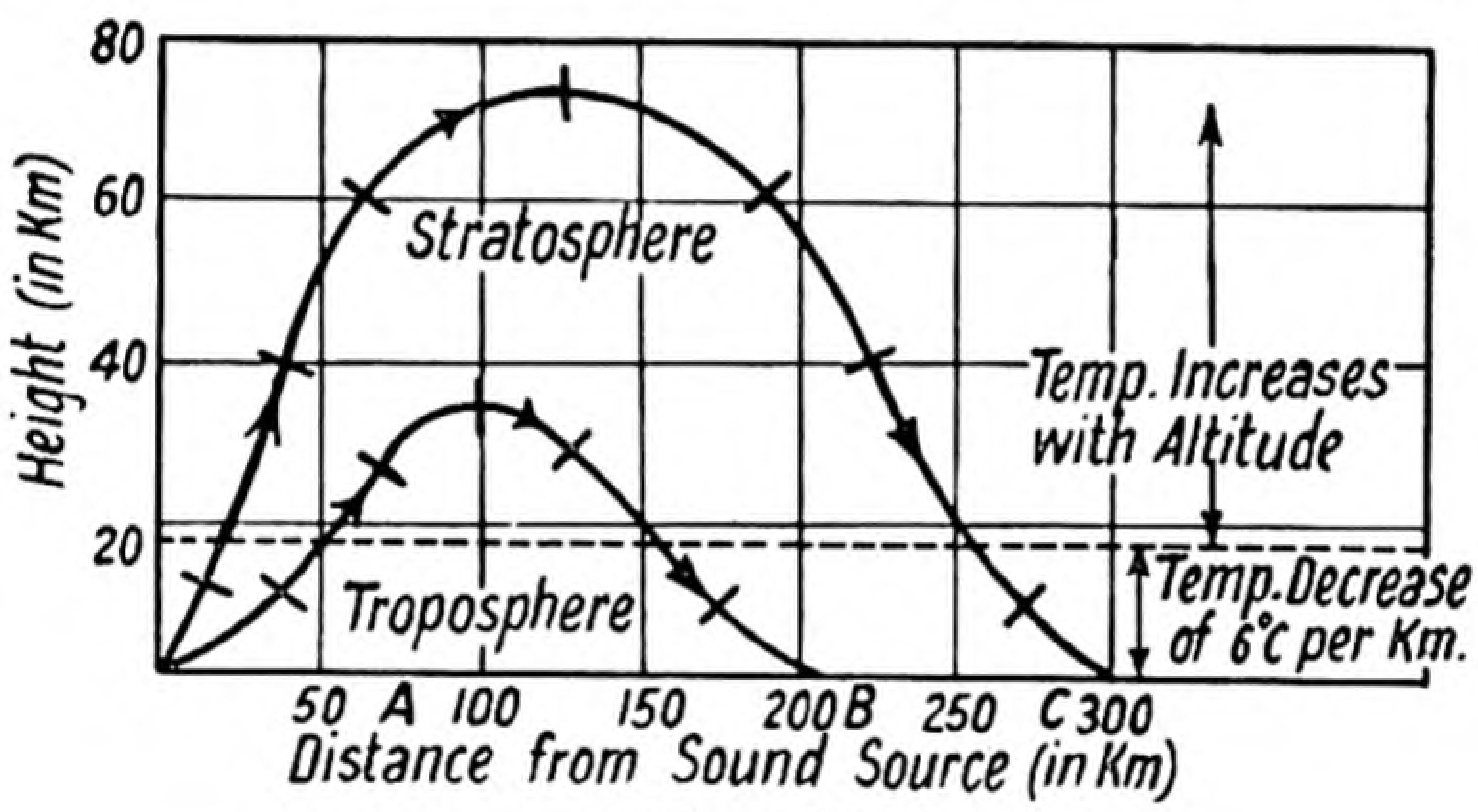


Fig. 6.26.

indicates the zone of silence and is analogous to the "skip" distance experienced in radio-communication.

Another factor, so far unmentioned, which may effect the propagation of sound through the atmosphere is the non-homogeneity of the air due to the presence of rain, dust, fog, etc. Observation has shown that only the higher frequency sounds experience any appreciable absorption, and at ordinary wave-lengths Tyndall has recorded that the sound range is sometimes actually greater than for particle-free air. This anomaly he explains as possibly due to a greater uniformity of *temperature* of the air brought about by the stabilising influence of the particles.

Effect of wind upon sound propagation

The presence of a wind of appreciable velocity makes a marked difference in the distance of transmission of sounds produced near the earth's surface, according as to whether the sound waves travel with or against the wind. If the medium moved as a whole with uniform velocity no such effect would be observed, but it is usual for the wind velocity near the ground to be less than that above,

due to the greater viscous and frictional resistance experienced there. Consequently the relative motion of the sound waves with respect to the ground will vary with altitude, and under these conditions, as a result of refraction, the fronts of the sound waves will be turned down

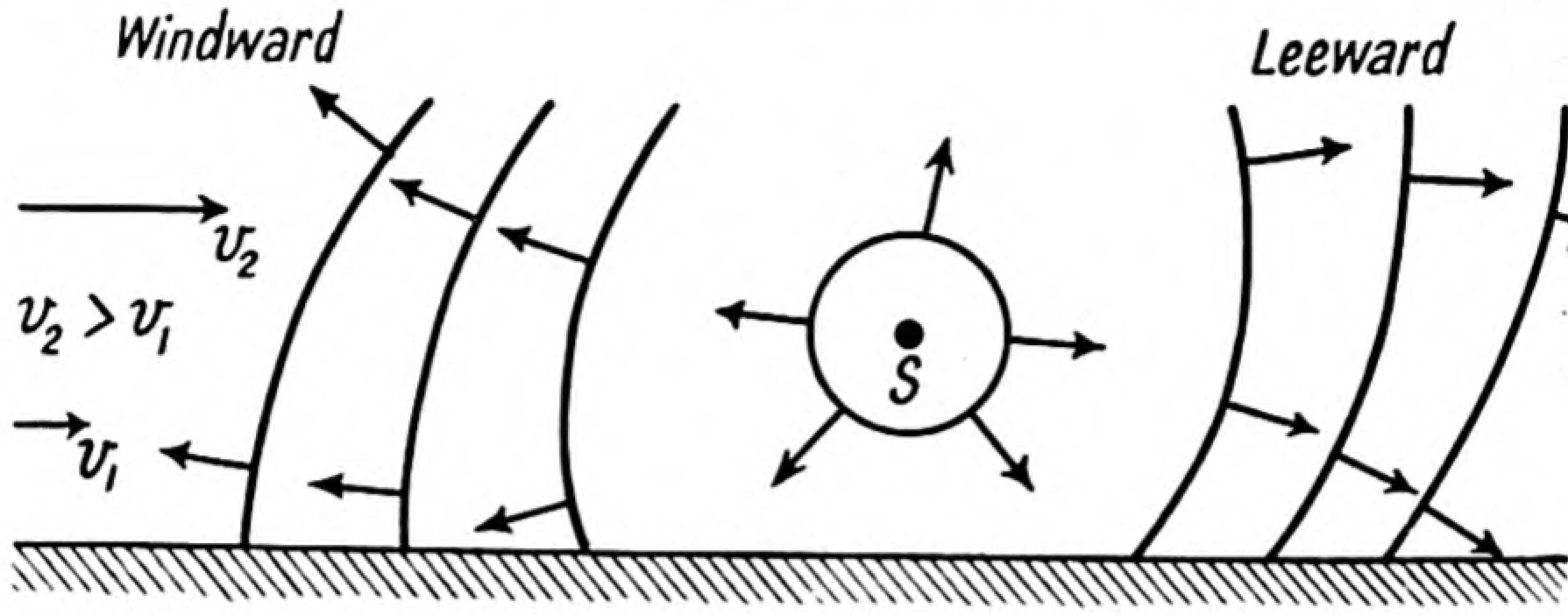


Fig. 6.27.

or upwards away from the ground, according as to whether the direction of the wind is respectively with or against the direction of propagation of sound (Fig. 6.27). It follows therefore that sound will be heard more clearly and at greater distances from the source S when this is situated to windward rather than to leeward of the listener. The completeness of these effects is dependent upon well-defined beams of sound being employed which, of course, is seldom the case in practice. A distinct improvement in the clarity of sound to a listener situated to windward of a source is effected by elevating the latter, *e.g.* church bells in a steeple. Incidentally an elevated source has a further advantage in minimising the "shadow" effects of any obstacles situated at ground level.

The accepted explanation of the above phenomenon was first suggested by Stokes, and will be given by reference to Fig. 6.28.

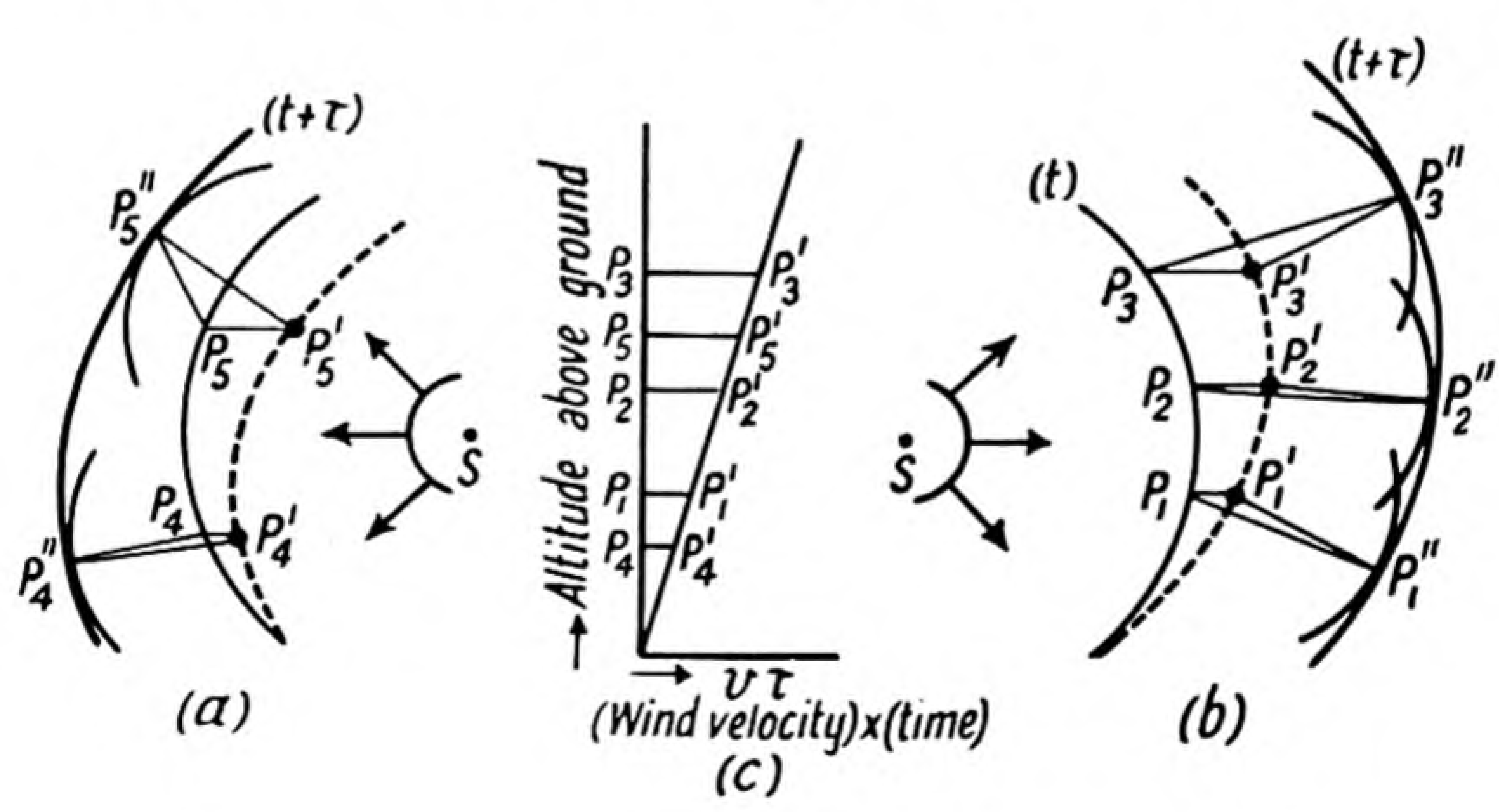


Fig. 6.28.

Consider the case shown in Fig. 6.28*b*, where S is the source of sound and the waves considered are those diverging from S in the same direction as the wind, which is assumed to be horizontal. Let $P_1P_2P_3$ represent the wave-front at time t sec. so that, due to the

motion of the medium itself, the various points P_1, P_2, P_3 on the front have respectively reached positions P_1', P_2', P_3' at time $(t+\tau)$ sec. The magnitudes of these displacements have been calculated on the assumption that the wind velocity v varies linearly with height (see Fig. 6.28*c*). If v is small compared with the velocity of sound c, then P_1', P_2', P_3' can be regarded as the centres of a system of wavelets of radius $c\tau$, and $P_1''P_2''P_3''$ as the new position at time $(t+\tau)$ sec. of the wave-front derived from the envelope of these wavelets. It should be noted that the "sound-ray" velocity at P_1, say, will be given by $\dfrac{P_1P_1''}{\tau}$ and is the resultant of the sound wave velocity and the velocity of the medium. Fig. 6.28*a* shows the modification of wave-front brought about when the sound wave travels in the reverse direction to the wind.

For further reading

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CHAPTER 7

DIFFRACTION AND INTERFERENCE

Two fundamental aspects of wave propagation are interference and diffraction, and the usual approach to the understanding of these phenomena is through the study of optics. Actually, however, the conditions governing their existence are more easily appreciated by studying the corresponding phenomena in acoustics. The advantage of this approach is due to the fact that the longer wave-lengths of ordinary sound permit the use of larger apertures and slits with which to demonstrate interference and diffraction effects. Furthermore, the existence of a longitudinal wave motion in the air can actually be photographed by means of spark photography, and its characteristics, viz. frequency, wave-length, and velocity, may each be *directly* measured without requiring the high degree of experimental skill necessary in optical measurements.

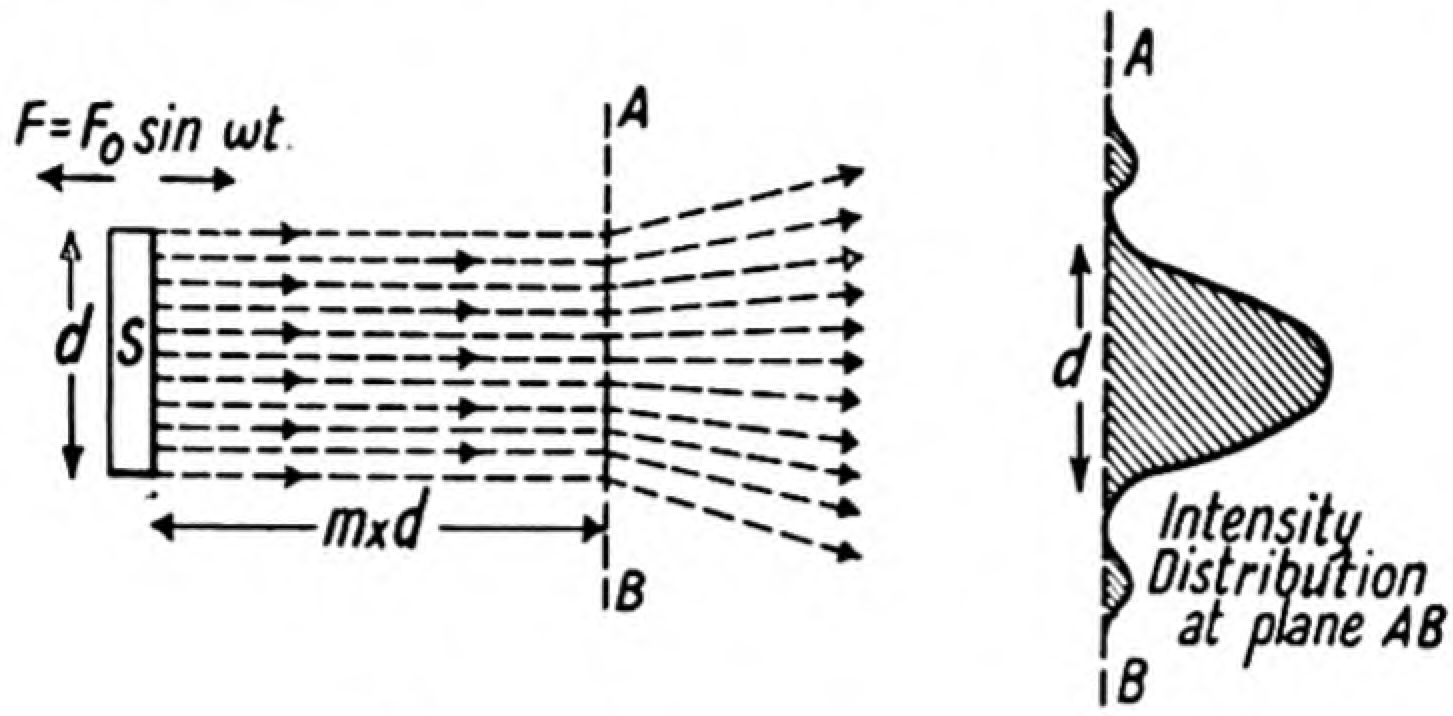
A whole series of these experiments in "phonoptics," as he terms them, are described in some fascinating papers by H. K. Schilling,* and to these the reader is referred for full details, but a brief mention will be made here of some salient features. The necessity for a sound source whose frequency may be varied and which approximates to a point source was satisfied by the use of a Galton whistle operated at a wave-length of 3 to 4 cm. The whistle was mounted inside a "padded" box provided with a small aperture in one side, so that the emerging sound-beam was well defined. In qualitative experiments a sensitive flame was employed as the detector, but for quantitative measurements a receiver system was used which comprised a small crystal-type microphone possessing a useful sensitivity up to 15,000 c.p.s., in conjunction with an audio-frequency amplifier and a suitable electrical filter to cut out frequencies below 6000 c.p.s.

Diffraction

A common occurrence in everyday life is the hearing of near-by sounds when intervening obstacles prevent the listener seeing the actual source. This perception of sound within the "shadow" of an obstacle is a direct consequence of the phenomenon of diffraction. The effect is associated with any progressive wave motion, and to be observable the dimensions of the obstacle (or hole in a screen) must be comparable with the wave-length of the radiation. In the case of light waves (see, for example, Bray's Light) the penetration of light within the geometrical shadow is only seen with very small objects or holes, and the effect is purposely enhanced in the diffraction grating on which a thousand or more parallel and equidistant lines (the obstacles) are drawn per inch of the surface of the reflecting or transmitting medium. The wave-lengths of audible sound are so much

* See "American Physics Teacher," see also Humby, *Proc. Phy. Soc.*, Aug. 1927.

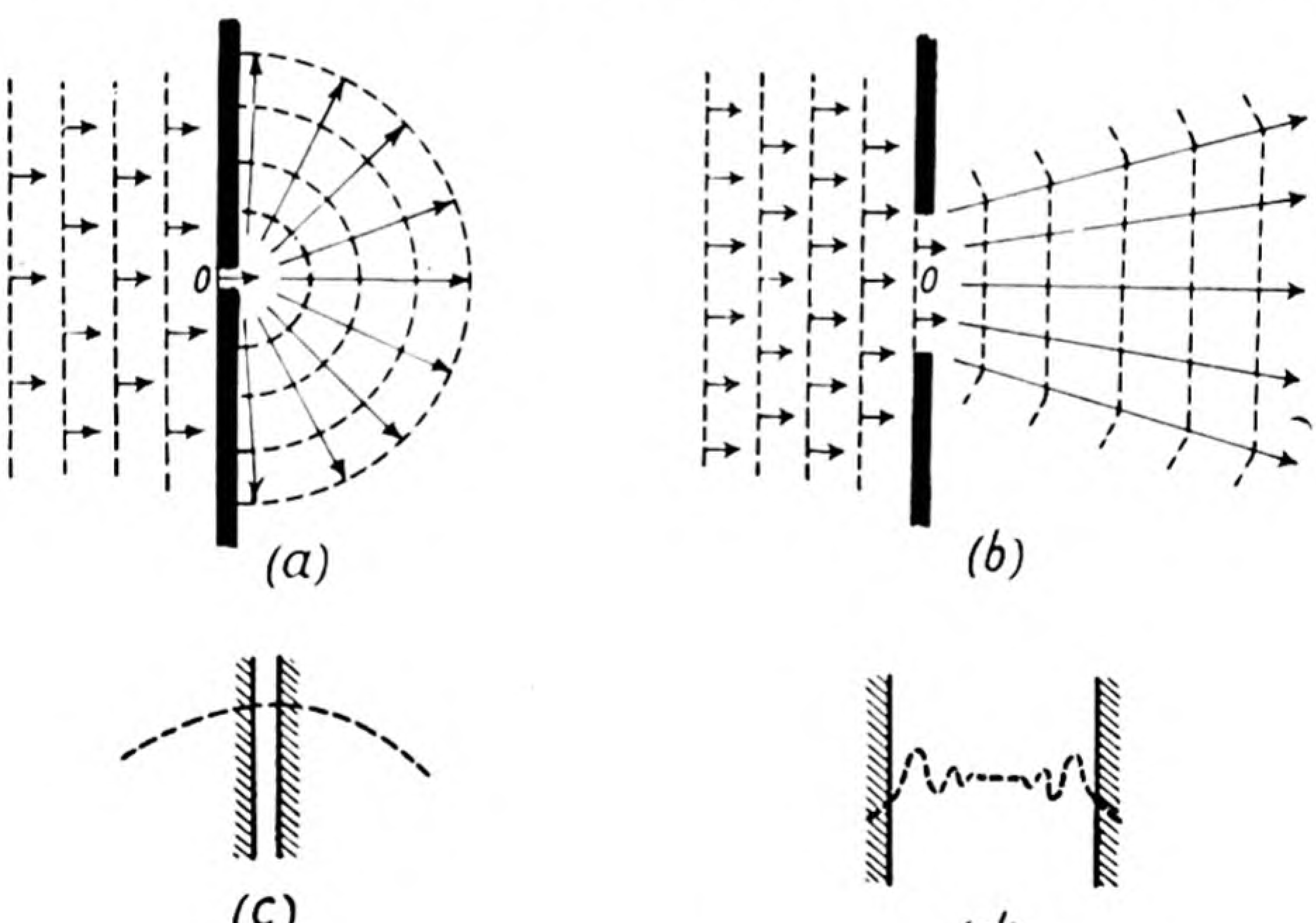
greater than those of light that they are comparable with the linear dimensions of everyday objects, and hence the propagation of sound waves of speech frequencies cannot be regarded as one of linear transmission as is justifiable for light waves under the same conditions.



(a) (b)

Fig. 7.1.

When suitable conditions exist which permit rectilinear propagation, then "optical-like" images and sharp shadows are formed; the reader may here be reminded of the fact that the "spread" of the image of a point-object is of the order of the wave-length of the radiation itself. It follows, therefore, that for sound waves to be much



(a) (b)

(c) (d)

Fig. 7.2.

smaller than ordinary objects and so lead to sharp shadows, the wave-length must be in the ultrasonic region. In the case of ultrasonic *generators*, *e.g.* quartz crystals, where the *linear dimensions of the radiating surface are many times the wave-length*, this will result in the "*beaming*" of the radiation. If the radiating surface S (Fig. 7.1*a*)

is a disc of diameter $d=m\lambda$, where m is greater than unity, the confines of the radiated beam will remain approximately parallel for a distance md, after which appreciable divergence will take place. Between S and the plane AB, however, a fluctuation of intensity occurs as in the diffraction of light through an aperture. Fig. 7.1b shows the distribution of intensity at the plane AB for the case $m=2$, and it will hold either for the case where S represents a radiator, or where it is an opening in a screen through which pass plane waves having the same wave-length as those given by the radiator.

The effects of the passage of plane radiation through an aperture for the cases where it is very small and fairly large compared with the wave-length are shown in Fig. 7.2 a and b respectively, but the corresponding energy distributions actually observed in planes parallel to those of the slits are shown as dotted lines in Fig. 7.2 c and d, and are similar to the optical diffraction patterns. It follows from the considerations of the previous paragraph that the amount of spreading outside the geometrical shadow for a given aperture will diminish, *i.e.* the propagation becomes more linear, as the ratio $\frac{d}{\lambda}$ increases, which, of course, is equivalent to increasing the distance md (Fig. 7.1). Now since d is considered to be constant the above condition implies that the wave-length requires to be small, *i.e.* the frequency large. This effect is readily observed in a public-address system in which a horn loud-speaker is used, for since music or speech comprises a wide range of frequencies, diffraction allows the lower frequencies to spread and to be observed over a wide area, but the higher frequencies do not spread far from the axis. Hence a listener anywhere near the principal axis of the horn will hear all sounds more or less correctly reproduced, but if situated far enough away from the axis may hear only the low frequency components of the source so that the quality of the reproduction will appear to be poor.

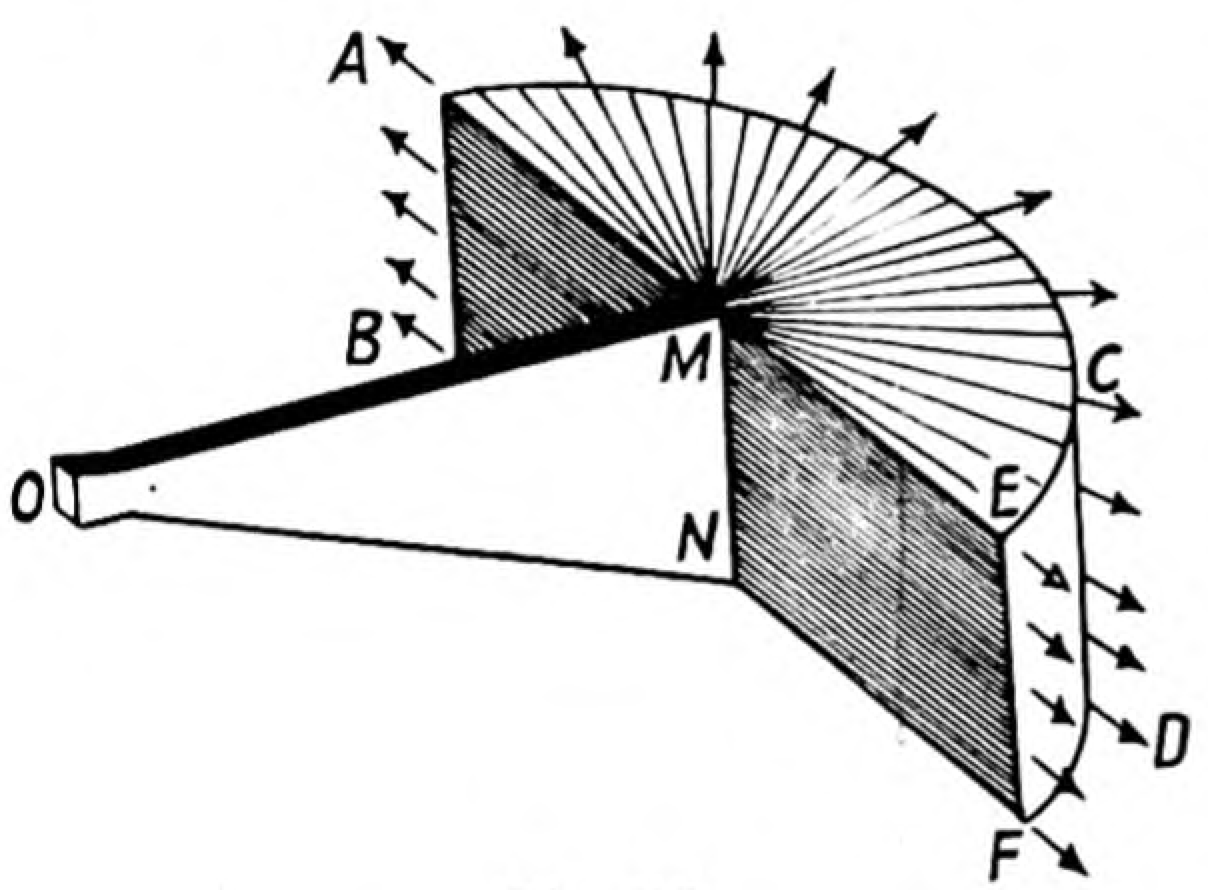


Fig. 7.3.

The diffraction pattern due to a very wide opening resolves itself into the two separate patterns resulting from diffraction at the straight edges of the slit (see a treatise on Light for fuller discussion of problem), and so sound diffraction effects may be expected within the geometrical shadows of the edges of a screen. It is mainly by virtue of this phenomenon that short individuals located behind bigger members of an audience, are able to hear in concert halls, etc.

Applications of diffraction

The *speaking trumpet* or *megaphone* was a familiar device at sports meetings, etc., until the advent of the loud-speaker, and it is represented by the horn OMN shown in Fig. 7.3, having a rectangular cross-section

as originally suggested by Rayleigh. The horizontal dimension at the open end, *i.e.* at MN, is smaller, and the vertical dimension larger, than the mean wave-length of the speech sounds, so that the sound waves issuing from the trumpet spread out horizontally but are limited in a vertical direction. In this way the acoustic energy is more or less confined to the desired directions, but it should be pointed out that a slight divergence in a vertical direction will take place because the dimension MN, in practice, cannot be made large enough compared with the usual wave-lengths of speech sounds.

The *plane acoustical transmission diffraction grating* consists essentially of a number of parallel and similar bars or strips of width b (see B_1, etc., in Fig. 7.4), each separated from its neighbour by a constant

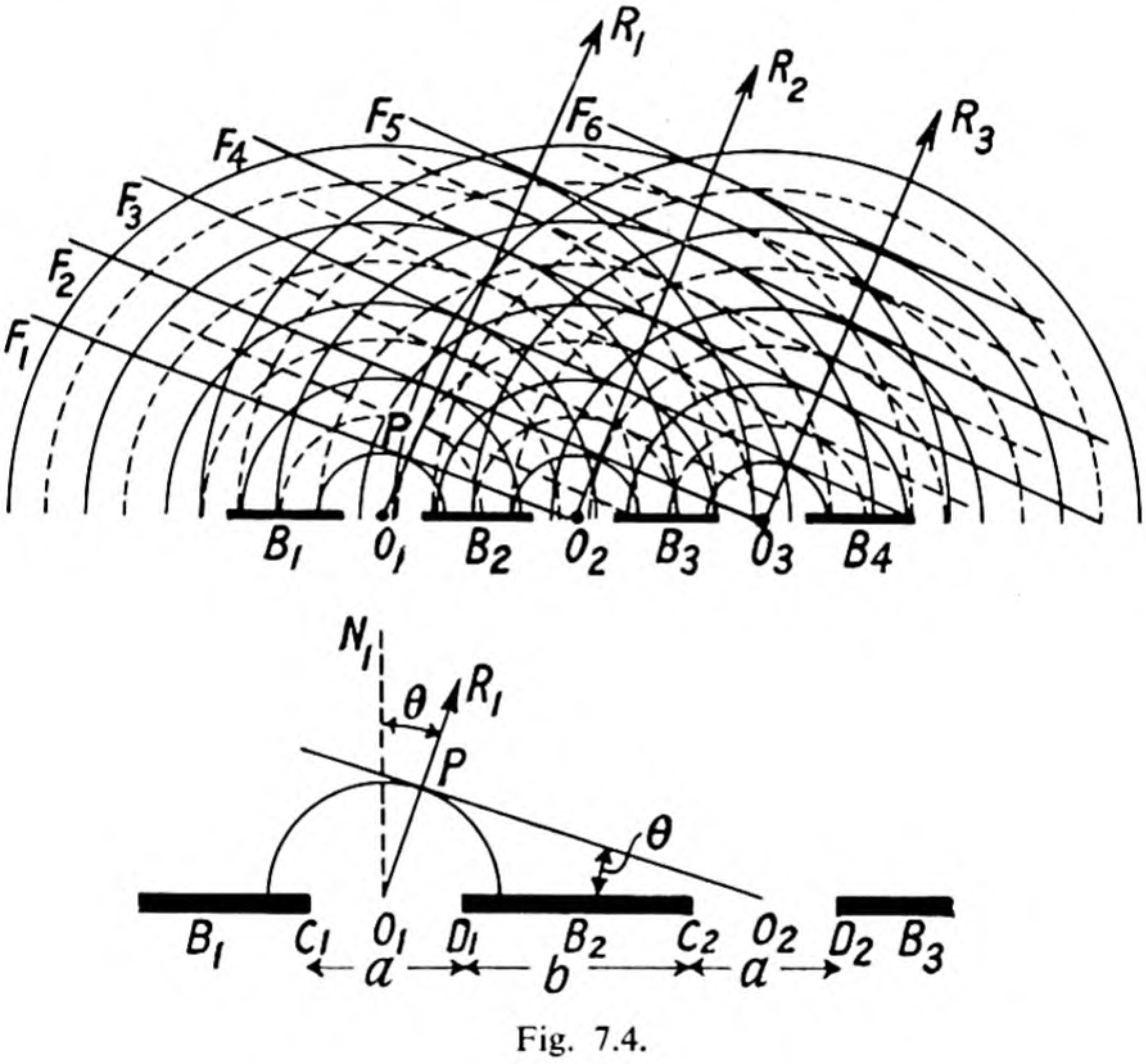


Fig. 7.4.

gap-distance a. According to the Huyghens concept of wave propagation all the particles in the plane of the openings C_1D_1, C_2D_2, etc., become centres of spherical disturbances. Assuming the incident sound wave to be plane, and arguing in the same way as for the corresponding optical problem, the wavelets generated by these particles will be in phase with one another in certain definite directions. One particular direction will obviously be that of the normal (i.e. O_1N_1 in Fig. 7.4) to the plane of the grating. This follows from the fact that the particles in the plane of the openings will be vibrating together in phase, as they constitute a wave-front of the incident plane wave, whose direction of travel is unaltered when considering the normal direction. This direction will be independent of the wave-length of the

radiation, and so will be the same for all frequencies of the source. Now suppose there is another direction in which the wavelets are in phase with one another, and let it be defined by the sense of O_1R_1, O_2R_2, etc., which represent sound "rays" drawn perpendicular to the parallel wave-fronts O_2F_1, O_3F_2, etc., of a diffracted wave system. The particles in these wave-fronts, by definition, must be vibrating in phase, and considering for the moment only the wave disturbances from the particles situated at the centres O_1, O_2, etc., of the openings, it is easily seen (from Fig. 7.4) that the condition which must be satisfied is that $O_1P=n\lambda$, where n is any *integer* and λ is a particular wave-length of the incident radiation. But O_1P will represent the path difference between *all corresponding* points in the grating spaces O_1, O_2, etc. (*e.g.* between particles at C_1 and C_2, at D_1 and D_2, etc.), hence the condition for the *reinforcement* of the diffracted sound is given by

$$n\lambda=O_1P=(a+b)\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the acoustical equivalent of an optical lens is placed on the side of the grating remote from the sound source, then a series of "images" of the latter would be produced in the focal plane of the lens on *each* side of the central undisturbed beam. The location of these images is defined by the order number, as it is termed, which is given to n, viz. 1, 2, 3, etc. Between these "bright" images will be places where interference takes place, the governing condition being that

$$(a+b)\sin\theta=(n+\tfrac{1}{2})\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where, as before, n is any integer. If the sound source is complex in nature, then each component frequency will give rise to its own diffracted image system, and so the grating supplies a convenient means of analysing the frequency spectrum of a given sound source. By making the grating in the form of a concave surface there is no necessity for an additional focusing device; a further reference to a form of acoustic grating is made on p. 342.

Interference

This phenomenon is a direct consequence of the principle of superposition previously considered on pages 5 and 24, and which relates to the fact that each of two (or more) wave trains passing through a given point exerts its effect independently of the other. Hence, when two sound waves act on an air particle the resultant displacement, velocity and pressure produced, will be the sum of the effects due to the separate waves. If the combined effect produces a distribution of energy which is quite different from that due to the separate wave trains, then *interference* is said to have taken place. For example, at a point where the separate displacements are equal but opposite in direction, the resultant displacement (and intensity) is zero, and where these displacements are equal and in the same direction, the resultant will be double that of each component displacement (and hence the intensity will be four times that due to either wave).

The reader should perhaps be reminded here that if the two wave trains possess the same frequency and are moving in the same direction,

they are said to be in the same *phase* at any point if the individual displacements have simultaneous positive (and hence also negative) maximum values at that point, and in opposite phase if their simultaneous maximum displacements are of opposite sign. Moreover, since air pressure is a *scalar* quantity, *i.e.* is non-directive, then even if the two wave trains are moving in directly opposite directions, any air particle acted upon simultaneously by a compression in each wave will be subjected to a resultant excess pressure equal to the sum of the separate excess pressures due to each wave. If two such wave-trains have the same wave-length and amplitude then they give rise to a standing wave train as explained on pages 48 and 106.

Fig. 6.8 shows two sets of circular waves generated by sources S_1 and S_2 of the same frequency, and situated a few wave-lengths apart. The resulting wave system would also, of course, be typical of the cross-section of that produced if the sources were generators of spherical waves. In order to simplify the problem S_1 and S_2 are assumed to be in phase with one another, and around each source is drawn a series of circles in full lines with radii equal to λ, 2λ, 3λ, etc.; these circles indicate the positions of maximum compression at any particular instant. A second set of dotted circles is drawn, with radii $\frac{\lambda}{2}$, $\frac{3\lambda}{2}$, $\frac{5\lambda}{2}$, etc., to show the positions of maximum rarefaction, and they fall half-way between the regions of maximum condensation.

It will be evident from Fig. 6.8 that points of maximum compression and rarefaction will be given by the circles shown, for they indicate respectively where two full-line circles or two dotted circles intersect one another. In other words, they are places where the waves from each source arrive in phase with one another, for taking any point such as P it is easily seen that $PS_2-PS_1=\frac{9\lambda}{2}-\frac{7\lambda}{2}=\lambda$. The points marked by crosses are places where the waves tend to neutralise as they arrive there out of phase with each other. Since the amplitude of a circular wave falls off inversely as the distance [for spherical waves it is inversely as the (distance)^2], then, in general, the neutralisation will not be complete as the distances of the point from S_1 and S_2 will not usually be the same. In the line joining the sources the wave trains are travelling in directly opposite directions and so will give rise to a stationary wave system along S_1S_2.

One further aspect of Fig. 6.8 should be noted, namely that the uniform distribution of sound intensity, or energy, around a single source undergoes a considerable change when the source is placed near to another similar vibrating source. Curves such as BA and DC indicate respectively the directions of maximum and minimum energy flow, but the *total* energy output from the two sources remains unaltered despite the existence of interference.

An interesting example of interference phenomena is provided by the ordinary tuning-fork. As the prongs P_1 and P_2 (Fig. 7.5) move towards one another they create a compression in the gap between them, while a rarefaction is created in their wake, and the reverse effects will take place as the prongs move away from one another.

Consequently two sets of waves will be generated by the motion of the fork, but each will be *out of phase* with the other, so that in the overlapping regions given by *cod* and *aob* they will neutralise each other. The existence of these silent regions may be demonstrated by twirling a vibrating tuning-fork around in front of the ear; furthermore, they can be shown to be solely the result of mutual interference by carefully placing a small cylinder over one of the prongs.

The zone plate

Consider a vibrating plane surface which is circular in shape (Fig. 7.6) and has its centre located at the point O. Suppose, further-

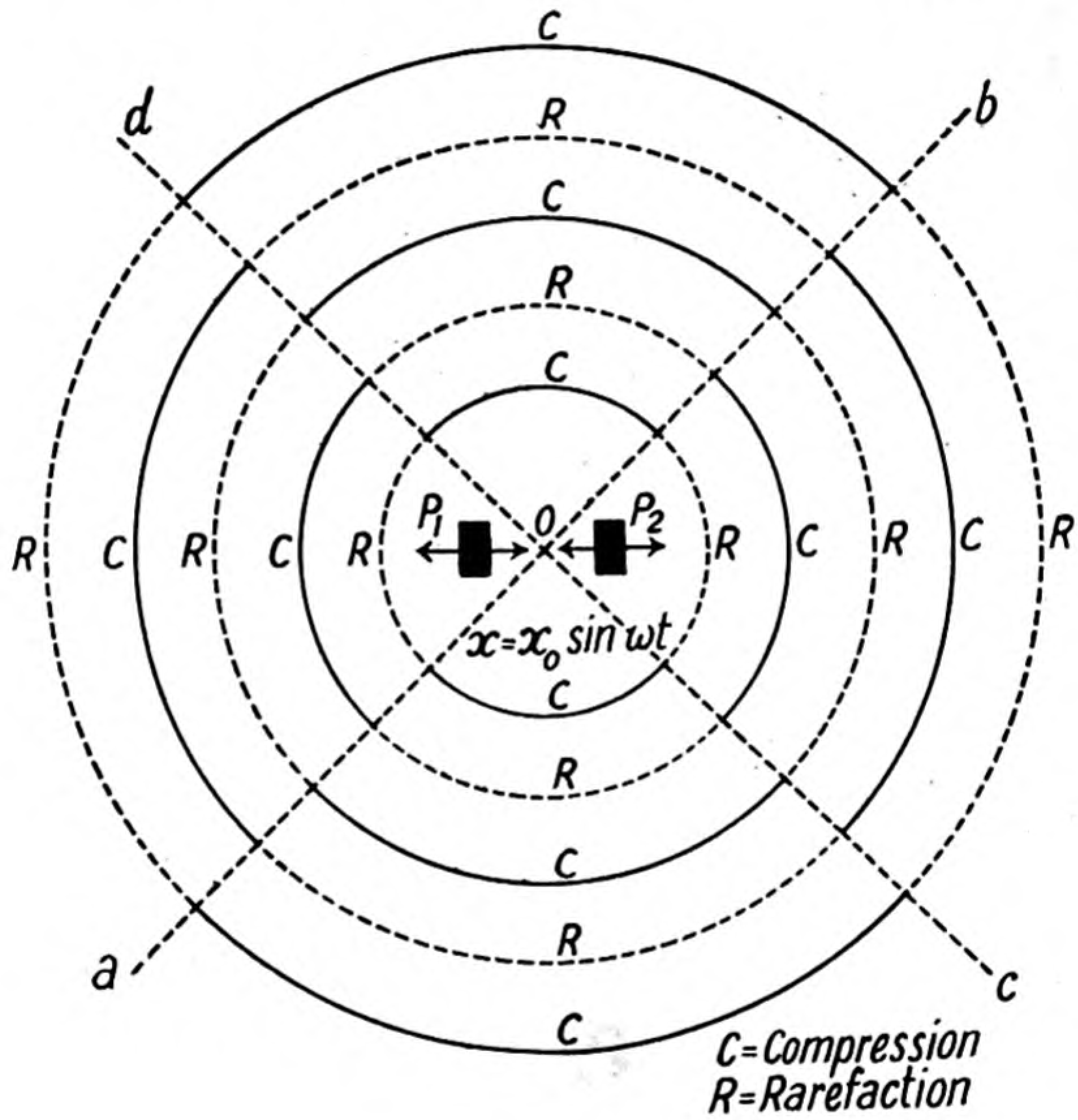


Fig. 7.5.

more, that this surface is divided into a number of rings concentric with O such that the slant distance from the point P of any ring is greater by $\frac{\lambda}{2}$ than that of its neighbour of smaller diameter. This difference of $\lambda/2$, *i.e.* of one half-period, between the vibrations from successive rings or zones is the origin of the term Fresnel half-period zones. It follows that the area of the nth ring will be given by

$$\pi(OD_n{}^2 - OD_{n-1}{}^2) = \pi\left[\left\{\left(x+\frac{n\lambda}{2}\right)^2 - x^2\right\} - \left\{\left(x+\frac{(n-1)\lambda}{2}\right)^2 - x^2\right\}\right]$$

$$= \pi\lambda\left[x + (2n-1)\frac{\lambda}{4}\right] = \pi\lambda x,$$

if λ is small compared with x. It is easily deduced that $\pi\lambda x$ is also the area of the shaded circle and hence each ring or elemental area will possess the same total energy of vibration. The contribution to the resultant amplitude of vibration at P due to any particular ring will be proportional to the ratio $\dfrac{\text{area of ring}}{\text{mean distance of ring from } P}$, neglecting the effect of the obliquity, with respect to OP, of the slant distance.

Now the value of this ratio for the nth ring

$$=\frac{\text{area of ring}}{\text{mean distance of ring from } P}=\frac{\pi\lambda\left[x+\frac{(2n-1)\lambda}{4}\right]}{\left[\left(x+n\frac{\lambda}{2}\right)+\left(x+\frac{(n-1)\lambda}{2}\right)\right]}$$

$$=\frac{\pi\lambda\left[x+\frac{(2n-1)\lambda}{4}\right]}{\left[x+\frac{(2n-1)\lambda}{4}\right]}=\pi\lambda.$$

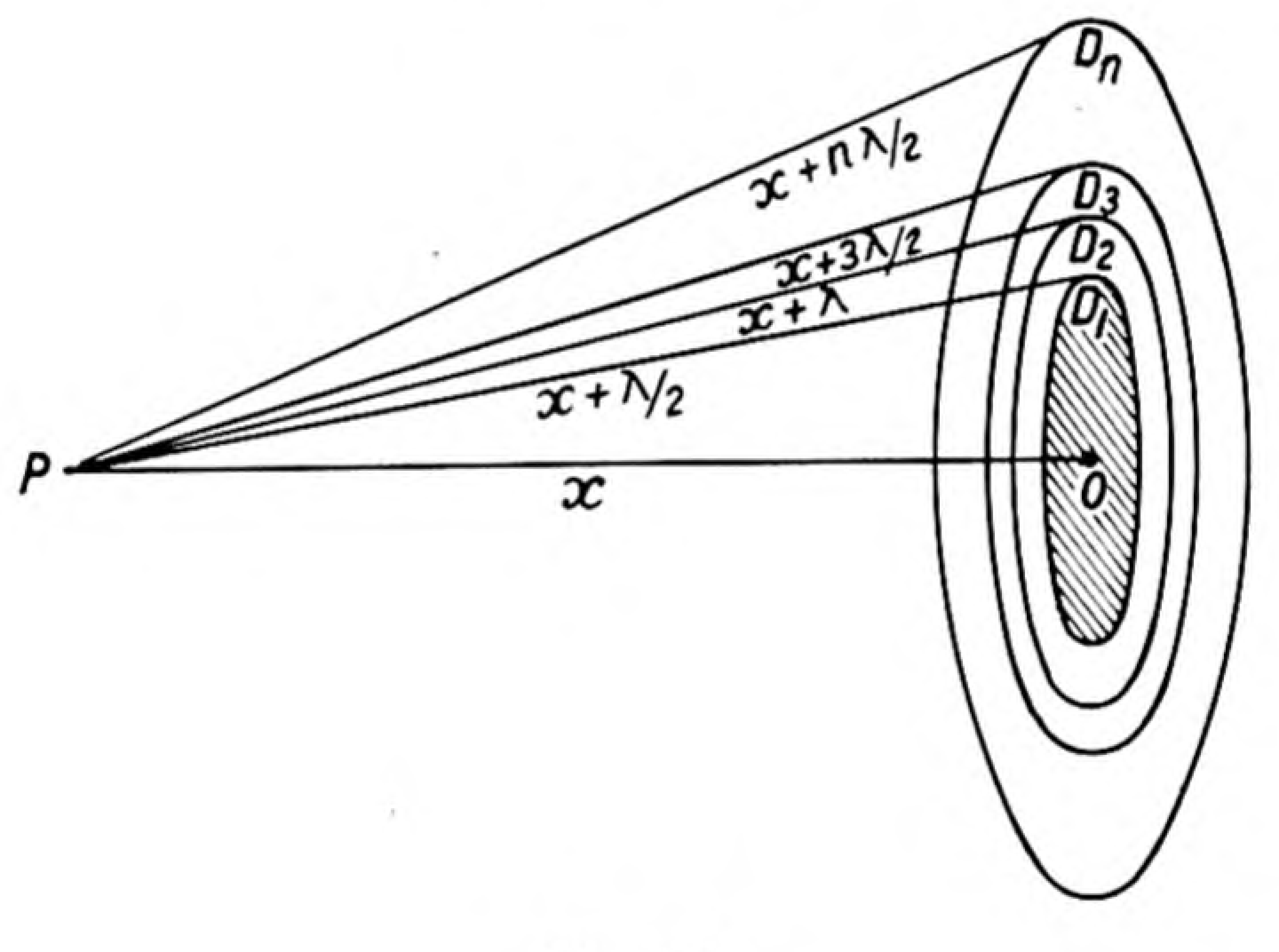


Fig. 7.6.

Hence the magnitude of the contribution of each ring to the resultant amplitude at P will be identical, one with another, except for a slight falling off with increasing diameter of ring due to the increasing obliquity of the slant distance. However, although the magnitude of the effective disturbance from each ring is nearly constant, that due to any particular ring is *out of phase* with the disturbances due to its immediate neighbours, and so it follows that the resultant effect at P will depend upon the number of zones or rings in the plane surface. To obtain an exact solution of the problem requires a fairly extensive mathematical analysis which cannot be attempted here and instead a graphical interpretation will be given.

Now in considering the effect at P of the disturbance due to *each particle* in any elemental area it is evident that only those particles situated in a particular *circle* in a ring will be exactly in phase together

at P, and therefore it is necessary to take into account the gradual change of phase in passing from the inner to the outer circles of particles in a particular ring. The first half-period, *i.e.* the shaded area in Fig. 7.6, is itself, therefore, imagined to be sub-divided into a large number of very narrow rings of equal area and hence, as follows from the previous analysis, are at distances from P (Fig. 7.7) which progressively increase by the same amount in passing from the centre O outwards to D_1. If OP is large compared with OD_1 the effect of obliquity may be ignored and the problem therefore resolves itself into a determination of the resultant of a large number, n, of vibrations of equal amplitude a which progressively differ in phase by a constant amount. The phase difference 2θ between the innermost circular area

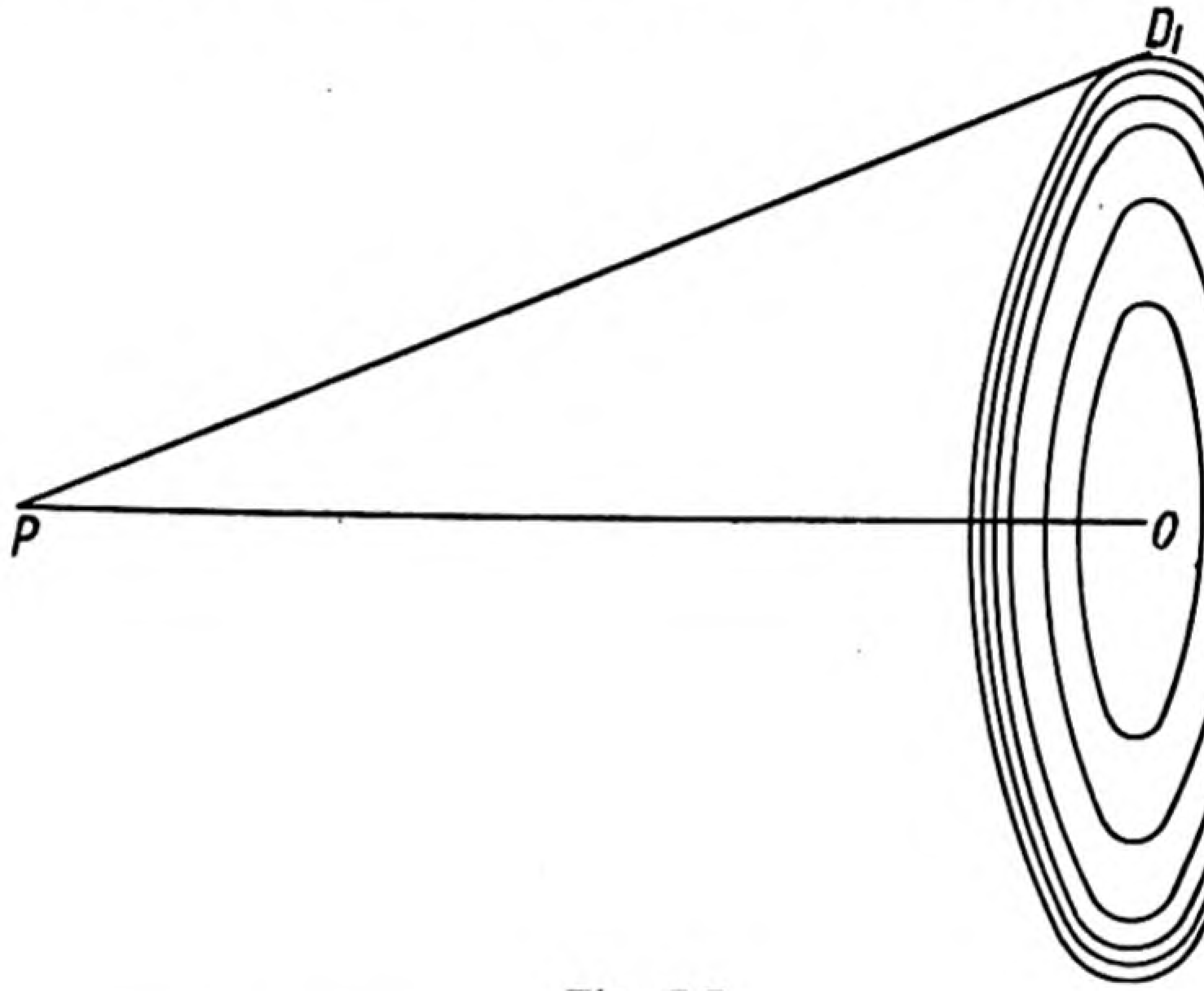


Fig. 7.7.

and the outermost ring will, of course, by definition of the half-period zone, be equal to $\frac{\lambda}{2}$, *i.e.* equivalent to π radians. The value of the resultant amplitude R (see p. 26) will then be given by $R=\frac{na\sin\theta}{\theta}$ $=\frac{na\sin\frac{\pi}{2}}{\frac{\pi}{2}}=\frac{2na}{\pi}$, when n is made very large. Hence it follows that R is the diameter AB (Fig. 7.8) of a circle of circumference $=2na$. The resultant amplitude of the second half-period zone in a similar manner will be given by BC, that of the third half-period zone by CD, and so on, since the effect due to each successive zone is π out of phase with its neighbours and the magnitude of the disturbances falls off slightly with the increasing order number of the half-period zone. If the surface area comprises a large number of zones, then it is quite evident from Fig. 7.8 that the resultant amplitude at P will be given by OA, *i.e.* $\frac{R}{2}$, where R is the magnitude of the total effect due

to the first half-period zone. The student of optics will recognise the spiral of Fig. 7.8 as a part of the Cornu's spiral (see Appendix) which is a most useful aid in the interpretation of optical diffraction problems.

From the above analysis it is seen that the vibrating diaphragm will exert its maximum effect at P (Fig. 7.6) when its diameter has a critical value defined by $PD_1-PO=\frac{\lambda}{2}$, where λ is the wave-length of the sound. This argument will apply also to a plane sound wave reflected by a plane surface, since the latter may be regarded as generating the reflected wave. Suppose, further, that a plane reflecting surface is sufficiently large to contain a large number of zones as for the surface of Fig. 7.7, but that the alternate half-period zones, one, three, five, etc., have been removed, except for a few narrow radial strips to hold the framework together. If, now, plane waves travelling from *right* to left are incident upon this surface, then it is evident that the disturbances reaching the point P from these odd-numbered zones will arrive there in phase, for the phase difference from the successive "cut-out" zones will be equal to 2π. Such a device is known as a zone plate, and the distance PO is sometimes referred to as its focal length, the value of which will obviously be a function of the wave-length of the incident waves.

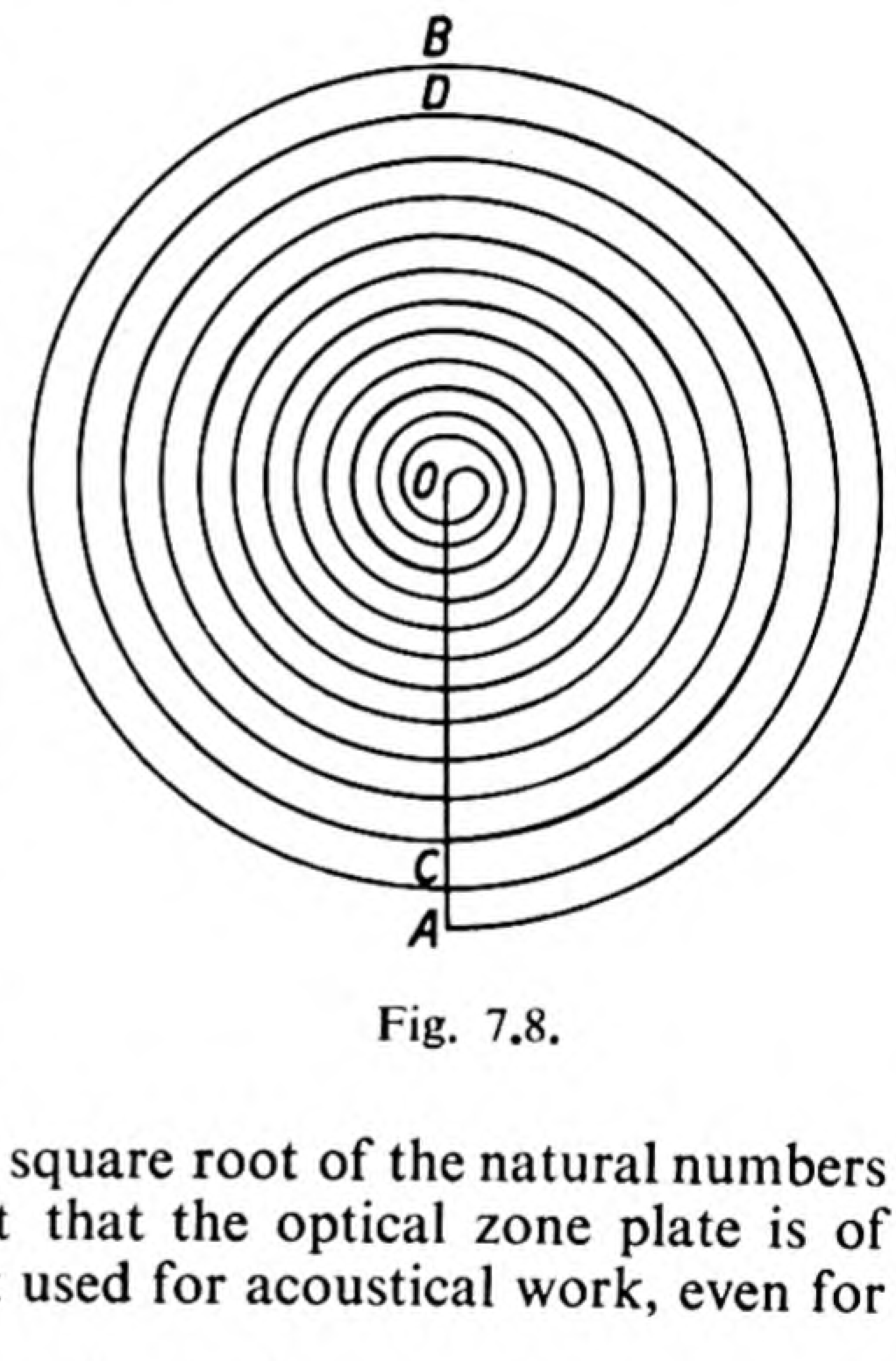


Fig. 7.8.

It follows from Fig. 7.6 that if r_n is the radius OD_n of the nth ring then, $\pi r_n^2=\pi(x+n\lambda/2)^2-\pi x^2$
or $r_n^2=nx\lambda+n^2\lambda^2/4$.

Neglecting the term in λ^2, to a first approximation then $r_n=\sqrt{nx\lambda}$, hence successive radii of the zone plate should be proportional to the square root of the natural numbers 1, 2, 3, etc. It should be evident that the optical zone plate is of much smaller dimensions than that used for acoustical work, even for ultrasonic sounds.

As the incident plane waves considered above correspond to an object at infinity, it follows by definition that the focal length (f) of the zone plate is given by the corresponding image distance x, i.e. $f=r_n^2/n\lambda=r_1^2/\lambda$. Many such foci will exist on the axis of the zone plate. The ordinary optical lens formula connecting *any* object and image distances is applicable to the zone plate. The combination of a zone plate with a small detector, *e.g.* microphone, situated on the axis at the focus forms a very sensitive means of detecting and finding the direction of a source of sound, the maximum effect obviously occurring when the axis of the zone plate points in the direction of the source.

Scattering of sound waves *

This phenomenon is brought about when a small object is situated in the path of the waves, and may be illustrated by the effect produced when a pole is set up vertically in the path of water waves. The effect with sound waves in air arises from the increased pressure variation at the point of incidence of the waves due to the smaller compressibility of the solid medium as compared with that of the air. Hence the presence of even a small obstacle O does mean an abstraction of some energy from the onward travelling waves W_1W_1', W_2W_2', etc., as indicated in Fig. 7.9, and the magnitude of the scattering ABC will depend upon the wave-length of the waves. It should be emphasised that in the phenomenon of scattering the ordinary laws of reflection are no longer obeyed, the object concerned being small by comparison with the length of the incident waves gives rise only to a single wavelet. For a given mass of particle and given amplitude of primary radiation, the intensity of the scattered waves is proportional to $1/\lambda^4$, where λ is the wave-length. This result was originally deduced by Rayleigh and confirms the observation, due to Tyndall, that blue light waves are scattered more than the longer red waves, which led to the explanation of the blue colour of the sky. Rayleigh gives an acoustic illustration by considering the scattered sound from a grove of trees, when a composite musical note is sounded near it. From the above formula it is to be expected that the relative intensity of the octave to the fundamental, should be 2^4 times as great in the scattered as what it was in the original sound. Hence it follows that the scattered sound may appear to show a rise of pitch of an octave.

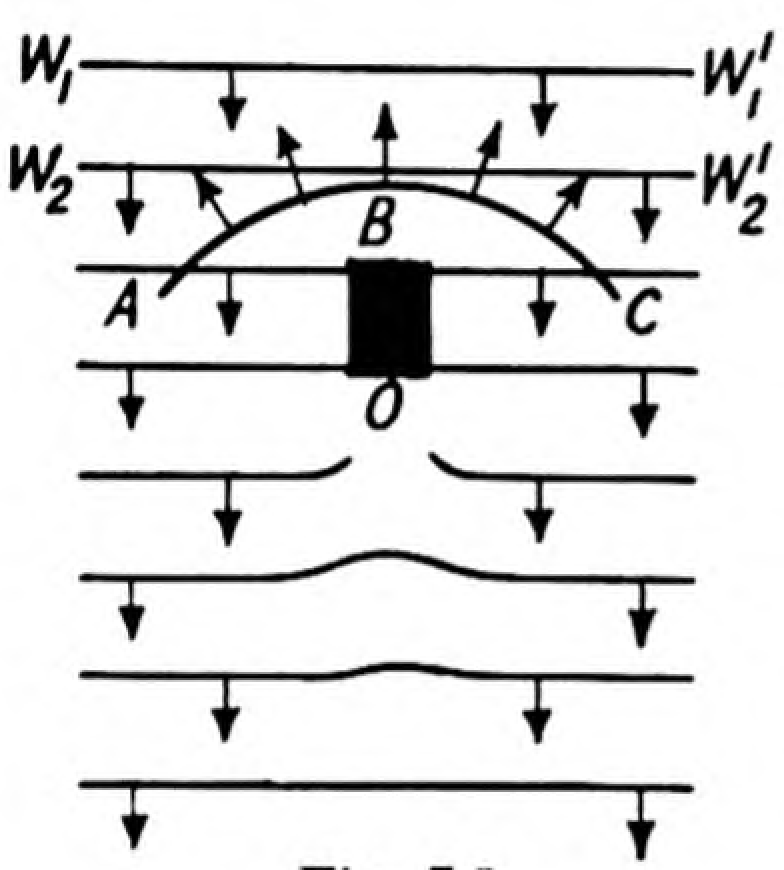


Fig. 7.9.

This phenomenon of scattering of the incident sound waves is of considerable importance in the study of the reflection of sound from the projecting ribs of ornamental ceilings, etc., for it is evident that these must be regarded as a number of small objects in the path of the waves. The simple laws of reflection of waves at smooth and plane surfaces will therefore not be applicable, and the distribution of reflected sound within the enclosure thus becomes considerably modified.

* See "The Dynamical Theory of Sound," H. Lamb. Arnold. § 81, p. 244.

CHAPTER 8

THE VELOCITY OF SOUND

It has been shown that the velocity of sound depends upon the elasticity and density of the medium in which it is being propagated. Hence it follows that a direct measurement of this velocity, combined with the theoretical formula, will provide useful information about the dynamical elastic properties of the medium, the particular elastic modulus involved being dependent on the type of wave transmitted. For example, in the case of true fluids the propagation of sound waves is governed solely by the bulk modulus of elasticity (see p. 54), and for gases this is expressible in terms of the ratio (γ) of the specific heat at constant pressure (S_p) to that of constant volume (S_v). This ratio is a function of the molecular complexity of the gas (p. 349), and it was from measurements of the velocity of sound in argon by the Kundt's tube method that Lord Rayleigh first established the gas to be monatomic. Again, it is often easier to carry out the velocity measurement than to perform a direct experiment on the determination of the elastic modulus concerned, and such is the case with the determination of the bulk modulus of a liquid. Often, also, the form of a given solid specimen renders it unsuitable for the ordinary laboratory procedure of elasticity measurements, and the indirect method of deduction from sound velocity measurements is therefore a valuable alternative.

A knowledge of the velocity of sound in atmospheric air is necessary for sound location and in the design of auditoria for the purpose of eliminating echoes, while a knowledge of its variation with composition has afforded a convenient method for the detection of fire-damp in coal-mines. Geophysical and meteorological problems, in which the velocity of propagation of mechanical disturbances plays a notable part, are mentioned elsewhere, and serve to indicate the importance of velocity measurements from both the practical and theoretical aspect.

The methods adopted are either direct, in which the passage of sound from a source of short duration, *e.g.* an explosion, is timed over a known distance, or indirect, in which a continuous source of known frequency, n, is used, and its wave-length λ being directly measured, the velocity V is calculated from the formula $V=n\lambda$. The wave-length is determined in either of two ways, (*a*) by locating two points in a progressive wave train at which the particles of the medium are vibrating in phase, their distance apart being therefore an integral number of wave-lengths; or (*b*) by locating the nodes and anti-nodes in a standing wave system, when the wave-length will be twice the distance between successive nodes or anti-nodes.

It is interesting to compare these methods with those used in finding the velocity of light. The extremely high frequencies associated with light waves do not permit their direct measurement, hence the

determination of the velocity of light is possible only by the first method, and to obtain measurable time intervals the distances employed have usually been of astronomical dimensions. The light signal itself sends out a train of waves, and it is the *group*-velocity and not the *wave*-velocity which is actually measured (see p. 76). The dispersion of sound waves is negligible in gases at audible frequencies, and so the wave- and group-velocities are equal.

Measurement of the velocity of sound in fluids

Direct methods. The earliest investigations were of this type, and were carried out on a large scale owing to the difficulty of measuring small time intervals with the apparatus available at the time. Experiments were thus necessarily restricted to the media available in bulk—air and water. In the air measurements the interval between seeing the flash of a gun and hearing the sound of the explosion was recorded by an observer situated at a large known distance from the gun. In the water measurement, Colladon and Sturm (1827) suspended a bell beneath a boat on Lake Geneva and arranged for an explosive charge in the boat to be fired at the instant at which the bell was struck. The flash was seen and the sound heard by an observer in a boat some 14 km. away, the listening being performed with the aid of a horn, the wide end of which was under water and directed towards the bell. The velocity as found was 1440 m. per sec. Modern methods are similar in principle to the earlier ones, but two or more microphones, A and B, replace the observer. These are placed in line with the explosive source S; A and B are sufficiently remote from the source S to eliminate the effect of the enhanced velocity of propagation in the vicinity of an explosion (see p. 145). The essential measurements are the distance AB and the time taken for the sound to travel over that distance. To measure this time interval an electrical method of recording is preferably employed; for example, both microphones may be incorporated in an electrical circuit, so that on the arrival of the sound wave an armature of an electromagnet is temporarily opened or closed. In this way a pen making a linear trace on a rotating drum is momentarily deviated, and if the constant speed of the drum is known, the time intervals between the two deviations may be accurately computed. Alternative to a knowledge of the speed of rotation a time base line may be traced, using an electric clock (or tuning-fork) in conjunction with a recording pen. A variation in the above procedure is to employ a galvanometer in circuit with the microphones, so that the variation of the current passing in the electrical circuit, due to the arrival of the sound, is made evident by the movement of a light beam reflected by the galvanometer mirror. The reflected beam moves over sensitised paper on a rotating drum, which has perforce to be enclosed in a light-tight box.

Indirect methods. The method just described was later modified by replacing the gun by a source of definite frequency, so that A and B (Fig. 8.1) receive a train of waves. The microphones are connected to XX and YY plates respectively of a cathode-ray oscillograph, and a 1 : 1 Lissajou figure is the result. The distance AB is adjusted

until the figure is a straight line, thereby indicating that the signals received at A and B have a phase difference of 0° or 180° (see Fig. 8.1). A is then moved to A', say, in the line SAB such that a linear Lissajou figure again appears. This displacement changes the relative phases

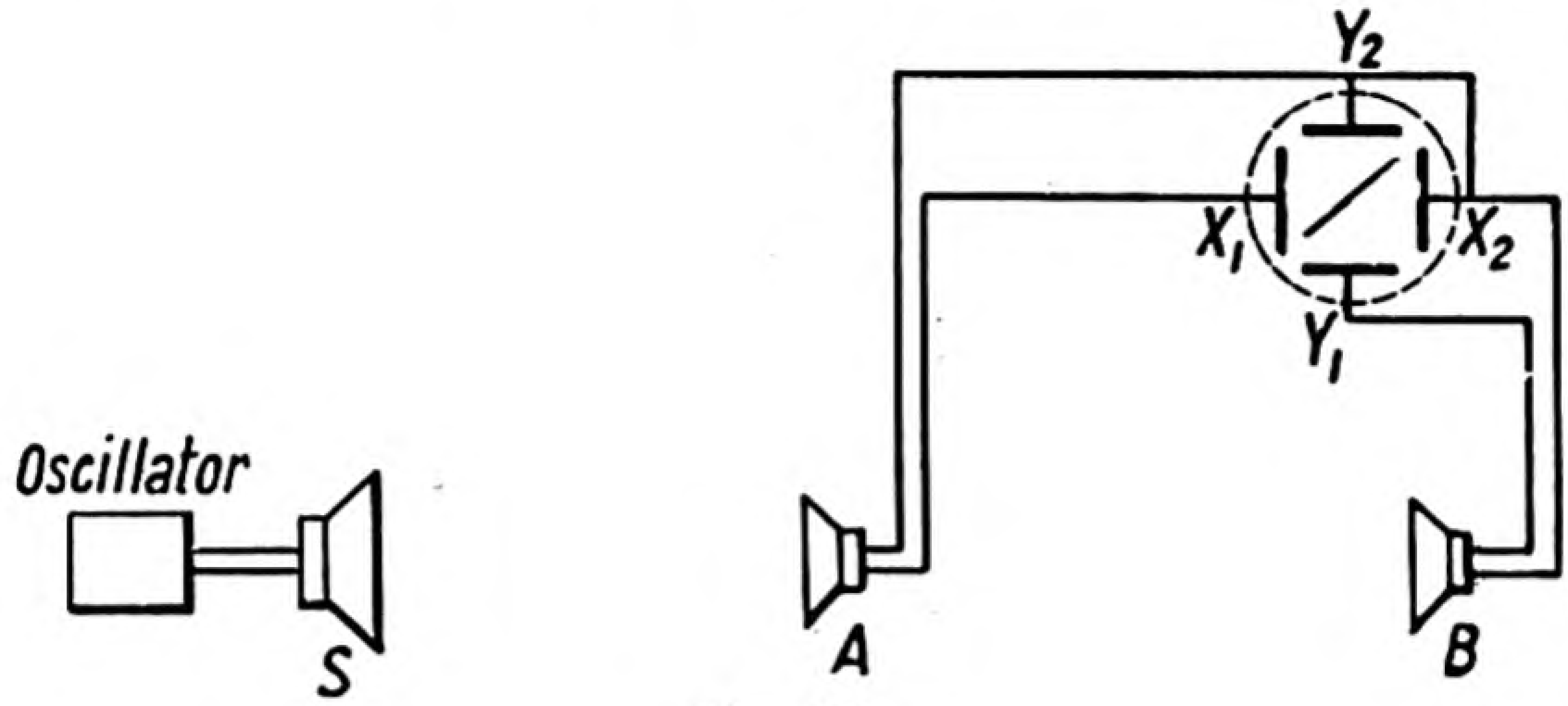


Fig. 8.1.

of A and B by 180°, i.e. AA' is one half-wave-length. Several half-waves can be identified in this way, and thus λ is obtained. In practice it is difficult to judge the point at which the elliptical figures become linear, so the arrangement may be modified as in Fig. 8.2; the loud-speaker $L.S.$, fed by an audio-frequency oscillator O, forming the source of sound, and the cathode-ray oscillograph ($C.R.O.$) equipped with two linear-time base traces is the indicating instrument. The two traces are traversed simultaneously, and the figures in the diagram

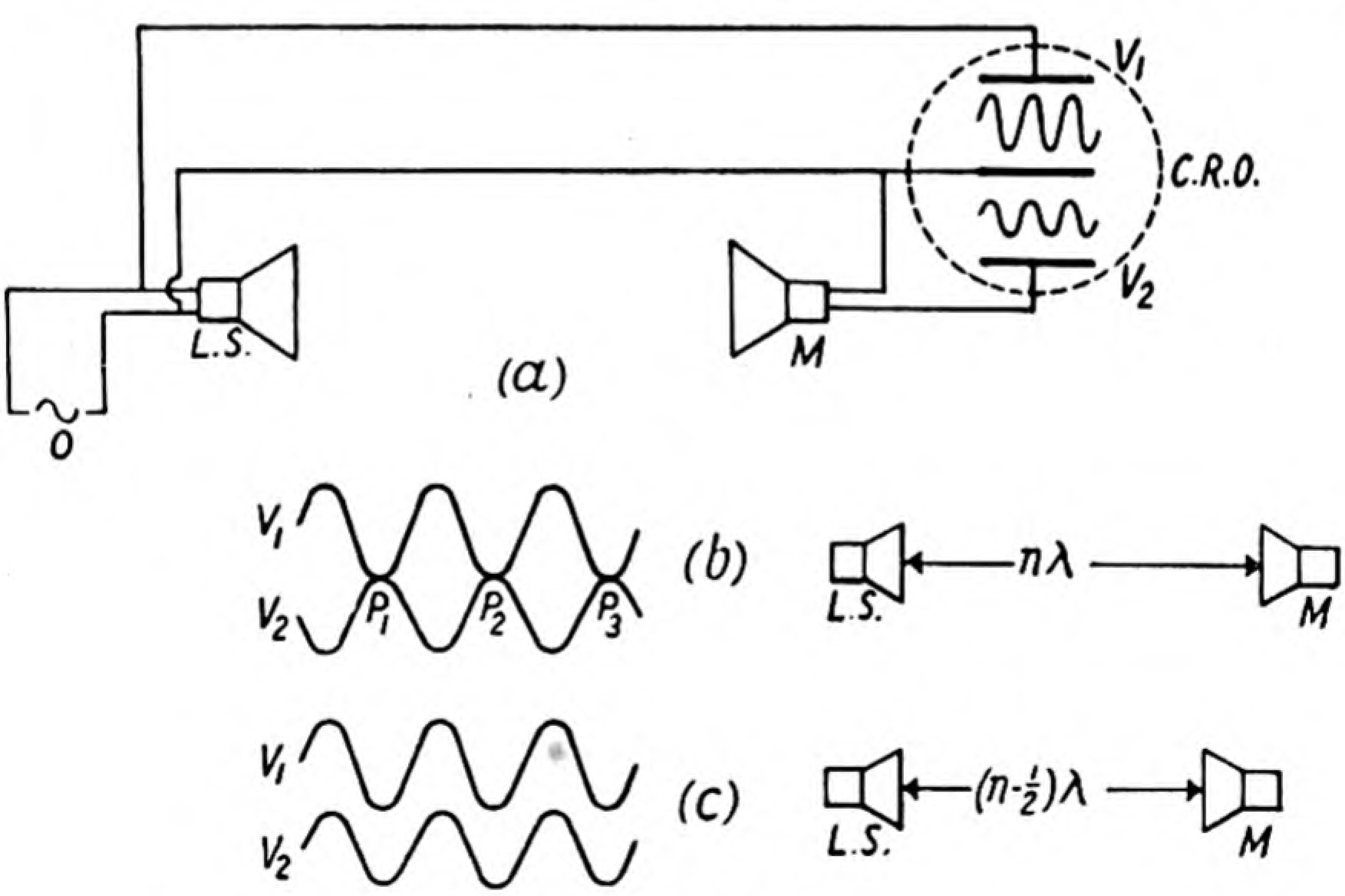


Fig. 8.2.

may be regarded as "pictures" of the waves generated and received at the speaker and microphone (M) respectively. When M is moved towards or away from $L.S.$, the displacement is shown on the lower trace of the $C.R.O.$ by a shift of the wave system. The diagram

should render further explanation unnecessary. Fig. 8.3 shows a typical C.R.O. "display" using the method just described, in which, however, "pulses" are employed instead of continuous waves. The upper traces in the figure show the transmitted pulses and the lower ones are the pulses received at the microphone. The setting of the upper and lower traces "into step" as indicated in the right-hand side of the diagram is quite a sensitive adjustment if the initial pulse is made sharp. The use of "pulses," which is in the manner of radar technique, has a disadvantage in that the velocity measured does not refer to a single frequency.

Another indirect method is that due to Kundt, whose apparatus is shown diagrammatically in Fig. 8.4. The sound source is a rod AA', clamped usually at its centre N and excited into longitudinal vibration by stroking along its length at the free end A' with a resined cloth (for glass and metal rods a damp cloth is preferable). From a

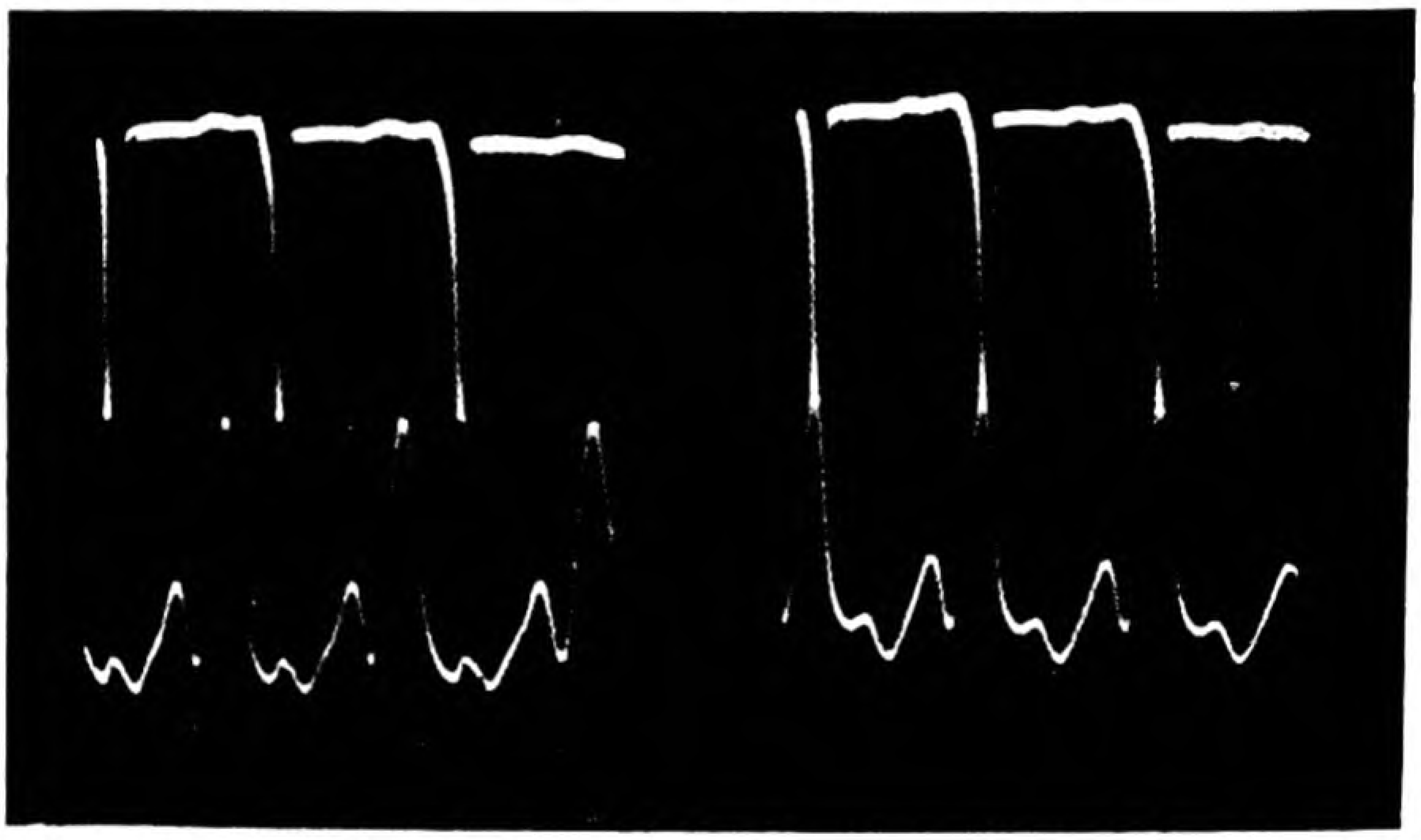


Fig. 8.3.

practical point of view, greater stability of the apparatus during stroking is obtained by clamping the rod at two points, N_1 and N_2, as shown at (*b*), the rubbing being performed lengthwise between these nodes. A light diaphragm is rigidly attached to one end of the rod, and it fits loosely into a glass tube G of 4 to 8 cm. diameter, the other end of this tube being closed by means of a tightly fitting plunger P, whose position can be varied. The fundamental mode for the longitudinal vibrations of a rod clamped at its centre is such that each end is an anti-node, and so the wave-length (λ_R) of the longitudinal vibrations in the rod of Fig. 8.4*a* is equal to twice the length of the rod. The corresponding frequency n of the note emitted by the rod will be given by $n=\frac{V_R}{\lambda_R}$, where V_R is the velocity of propagation of longitudinal vibrations in the material of the rod. If n is measured by means of a calibrated sonometer, the velocity is readily calculated, and, furthermore, a value of Young's modulus E of elasticity for the

material of the rod may be found from the formula $V_R = \sqrt{\frac{E}{\rho}}$, where ρ is the density of the medium.

The chief objective of Kundt's experiment is, however, to find the velocity of sound in the air of the glass tube, and for this purpose the air column PA is adjusted in length by means of the plunger until it resonates to the frequency of the note emitted by the rod, *i.e.* until a standing-wave system is established and the length PA is approximately equal to an integral number of wave-lengths (λ_A). This adjustment is possible, for the velocity of sound in solids is greater than in gases, and so from the relations $V_A = n\lambda_A$ and $V_R = n\lambda_R$, the velocity of propagation of sound within the air of the tube is given

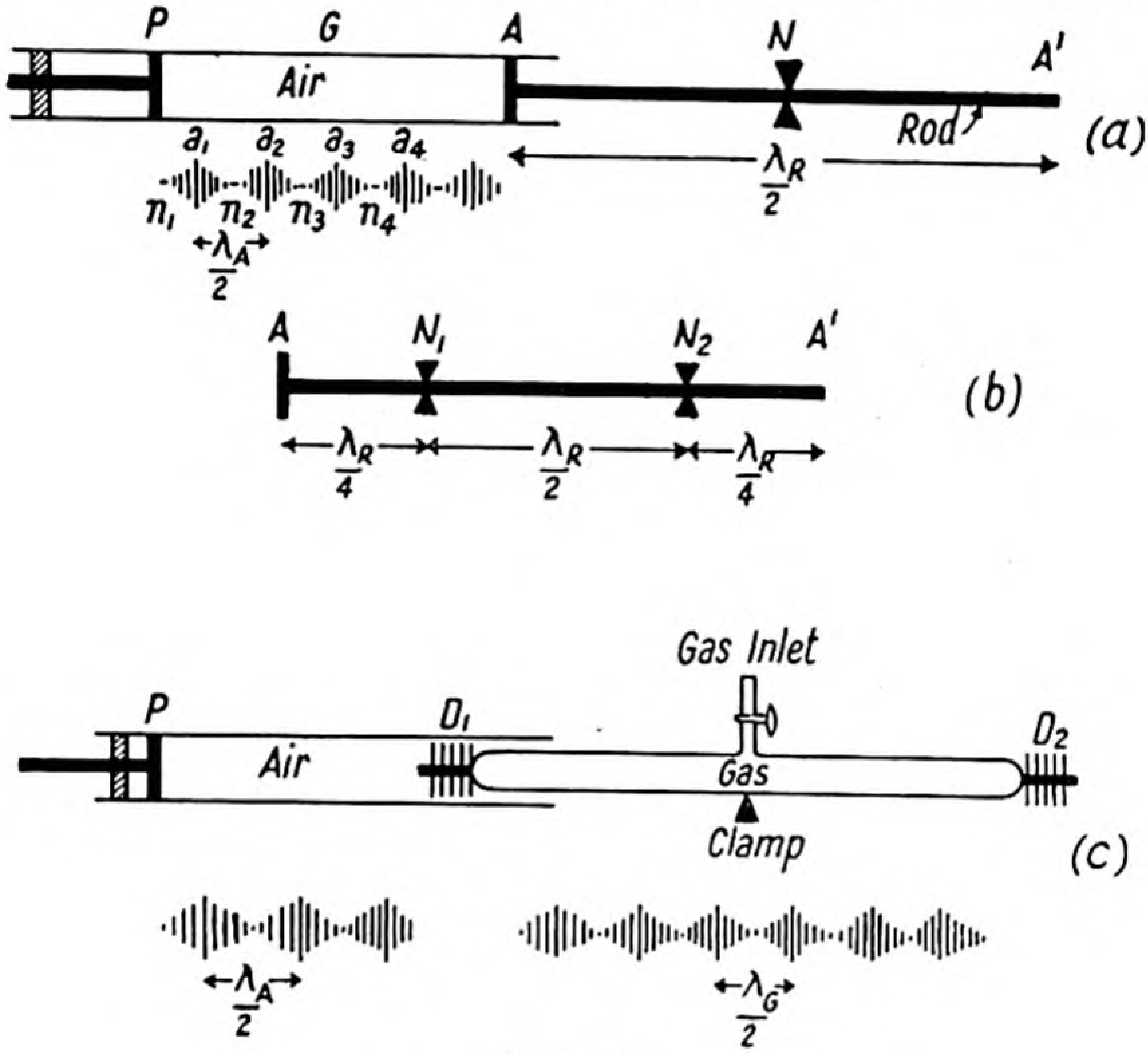


Fig. 8.4.

by $V_A = \frac{\lambda_A}{\lambda_R} . V_R$. If V_R is known (*i.e.* from direct velocity determinations or indirectly from measurements of Young's modulus and density), it is obviously unnecessary to know the frequency of the sound giving rise to the aerial vibrations.

The resonance of the air column is indicated, and λ is measured by making use of fine *dry* cork dust or lycopodium powder, which is sprinkled sparingly along the bottom of the glass tube. At the commencement of the experiment the tube is turned about its horizontal axis until the powder is on the point of slipping down, and then the plunger is moved until the dust is thrown into violent motion at certain points along the tube. When the vibration has ceased the powder will settle along the bottom of the tube, after the pattern shown in the lower part of Fig. 8.4*a*. It will be noted that both P and A

correspond approximately to nodal planes of the aerial standing-wave system, although since A is a place of maximum vibration for the rod, the air particles situated there will exhibit a small amplitude of *motion* when the rod is excited.

If the apparatus is rendered gas-tight by fastening rubber tubing to the rod at the node and passing it over the end of the glass tube, different gases can be introduced through a small side tube, and the value of λ_G, the wave-length in the gas, determined as for air. The frequency is the same in each experiment, so the velocity V_G of the sound in the gas can be compared with that in air, since $\frac{V_A}{V_G}=\frac{\lambda_A}{\lambda_G}$. It is necessary to secure the rubber tubing to the rod at a node to avoid loading the rod at or near an anti-node, as this would lower its frequency of vibration.

Behn and Geiger modified the method so as to obtain a system in which it was easier to maintain the purity of the gas under test by replacing the exciting rod with a glass tube as in Fig. 8.4*c*. This

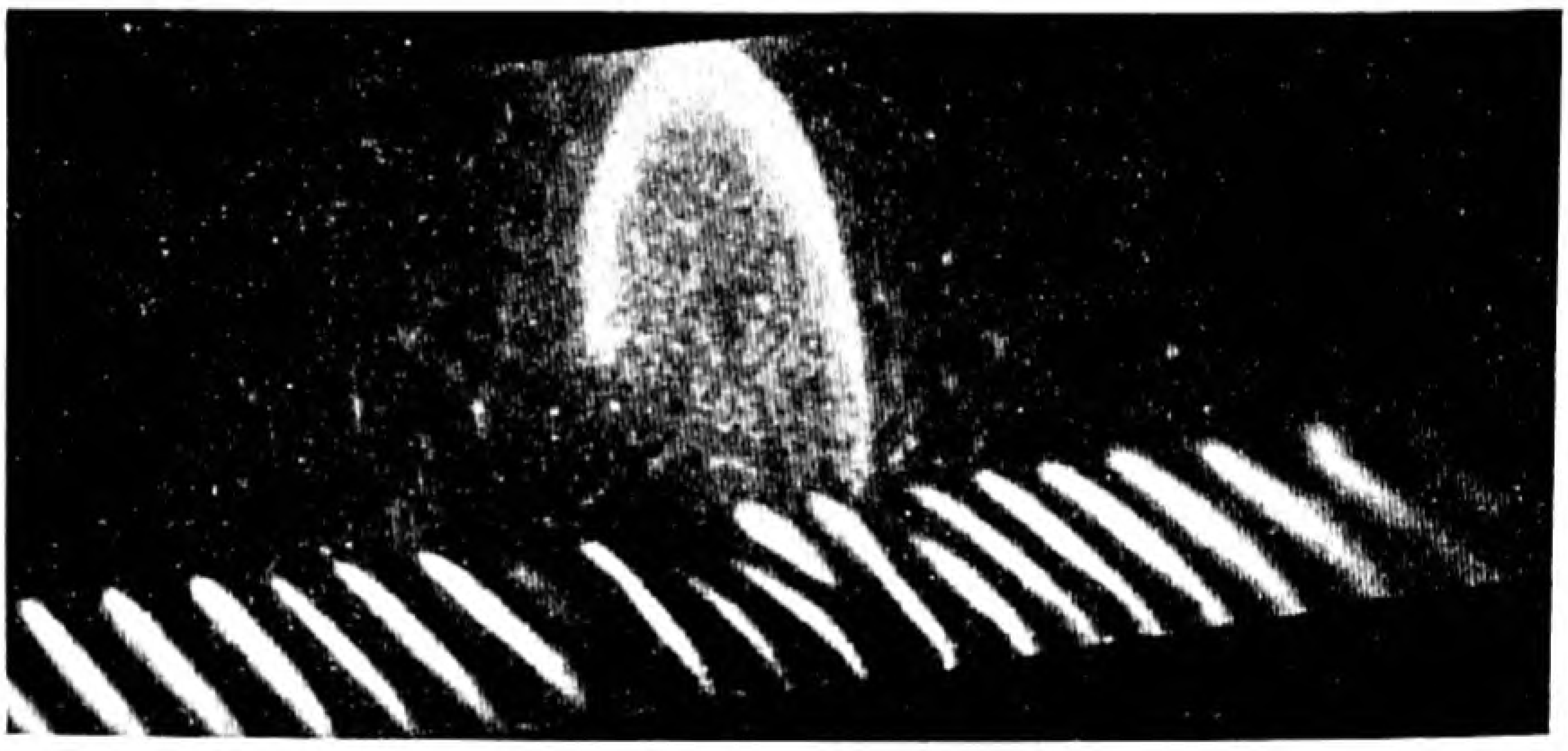


Fig. 8.5.

tube contains the fine powder and the gas, both thoroughly dried, and is loaded symmetrically at its ends by metal discs which can be screwed on to the projecting rods, if any, or otherwise stuck on by wax as required. In this way the effective length and thus the frequency of the longitudinal vibrations of the actual material of the tube is altered, until the movement of the powder indicates the resonance of the gas column with the vibrations of the tube when stroked. This tube is now used as the source of sound in the larger tube, which contains air and dust as before, and is tuned as previously by adjusting the position of the plunger. $\frac{V_A}{V_G}$ is obtained from the measured values of λ_A and λ_G. The discrepancy between the diameters of the two gaseous columns introduces an error which is discussed later in this chapter. It will be noted that in this method the length of the gas column is fixed and the frequency of excitation is varied.

Experiments on gases may be more easily carried out by adopting modern instruments and apparatus, *i.e.* by using a loud-speaker

fixed at, and closing, one end of the gas tube, the speaker being energised by means of a variable frequency valve oscillator. With this type of generator operating at fairly large amplitudes, a beautifully defined series of rings of dust are obtained when the oscillator and gas column are in resonance (Fig. 8.5).

Another method dependent upon resonance, due to Bate, and capable of considerable accuracy is indicated in Fig. 8.6. The open end of the tube, which should be about 2 m. in length and 4 cm. in diameter, is partly closed by a movable wedge L, which acts like the lip of an organ pipe, which is virtually what the system now becomes. By adjusting the distance of the wedge from the slit S, which is directly opposite, and also by suitable adjustment of the air pressure, it is possible to obtain a series of stable notes whose frequency n may be deduced from the approximate formula $n=\frac{KV}{h}$, V being the air jet velocity and K a constant.

A convenient frequency for the resonator having been decided upon, h and V are suitably chosen, and the position of the plunger P within the resonant cavity is varied until resonance is attained. A microphone

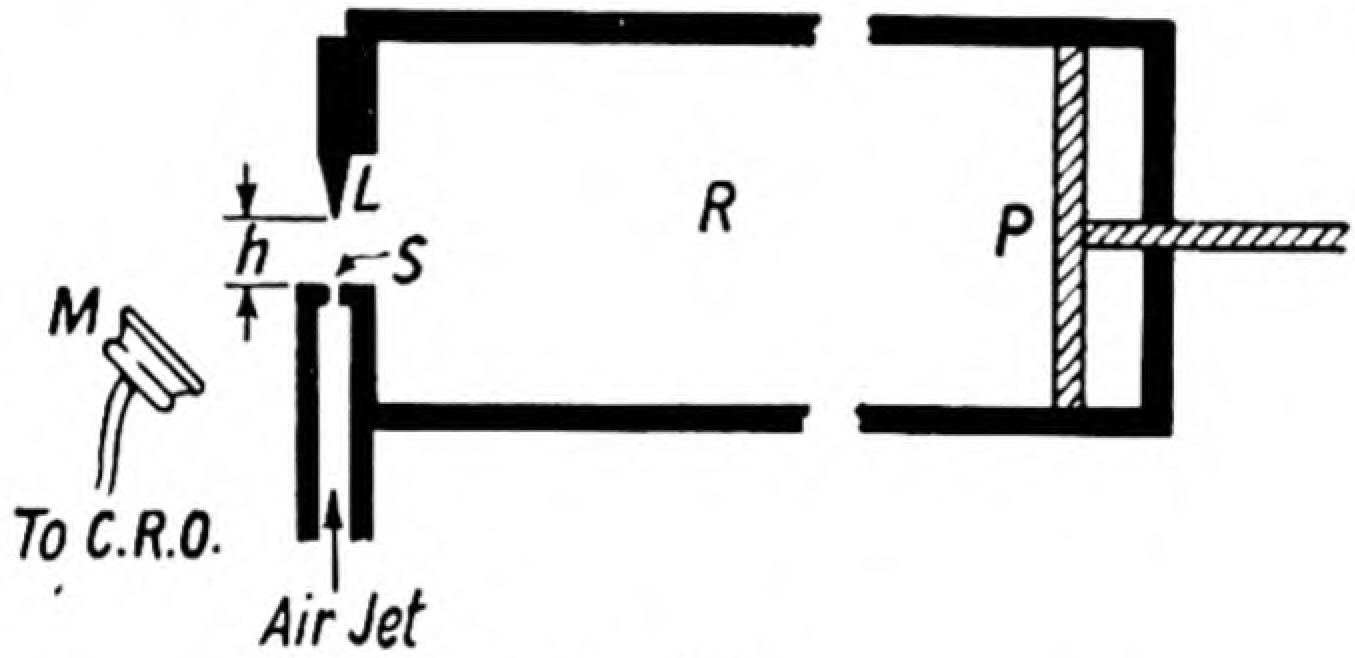


Fig. 8.6.

M is placed near the pipe mouth and its amplified output applied to a C.R.O. to determine the resonant frequency, using either of the two methods described in Chapter 13 to obtain a stationary pattern on the screen. The resonator will, in general, respond at a number of different settings of the plunger, each separated by a half-wave length, and for each position an identical stationary figure will be obtained on the cathode-ray tube screen, thus showing the resonant frequency to be constant. The advantage of this method lies in the delicate sensitivity of the setting of the plunger, which is possible because of the flexibility of the coupling between the air jet and the air column. A very small movement of the plunger from a resonant position will be made evident by a rotation of the figures on the cathode-ray tube screen.

Velocity of sound in solids

If a metal pipe is struck at one end, two mechanical wave systems will be propagated in the direction of the pipe. The faster set will travel through the material of the pipe, while the other will constitute an aerial vibration within the pipe. If V_A and V_I are the velocities of propagation of the disturbances within the air and in the metal, *e.g.* iron,

respectively, and t sec. is the time between the arrivals of the separate disturbances at the listening end of the pipe, then $t=\frac{l}{V_A}-\frac{l}{V_I}$, where l is the length of the pipe. It follows that V_I can be found if V_A is known, and the time interval can be precisely measured. Biot carried out such an experiment, and as its precision depended upon obtaining a sufficient length of pipe, he used a total length of 950 metres, so that t could be measured accurately. For an iron pipe 1000 yards long the time interval is approximately $2\frac{1}{2}$ sec. It is interesting to note that this method is an example of the use of pulses of sound, *i.e.* a limited number of waves only are propagated at a time, and a modern adaptation of this technique may be conveniently employed, utilising the advantages of a *C.R.O.* for measuring small time intervals. The specimen rod need only be 30 or 40 cm. long, and it is conveniently set in a horizontal position and clamped at its centre N (Fig. 8.7*a*). A small hammer-head is located at the upper end of a vibrating spring S which is actuated by an electromagnet M, energised by a suitable source of A.C. In this way a series of taps are given to the specimen so that a damped system of longitudinal waves are set up in

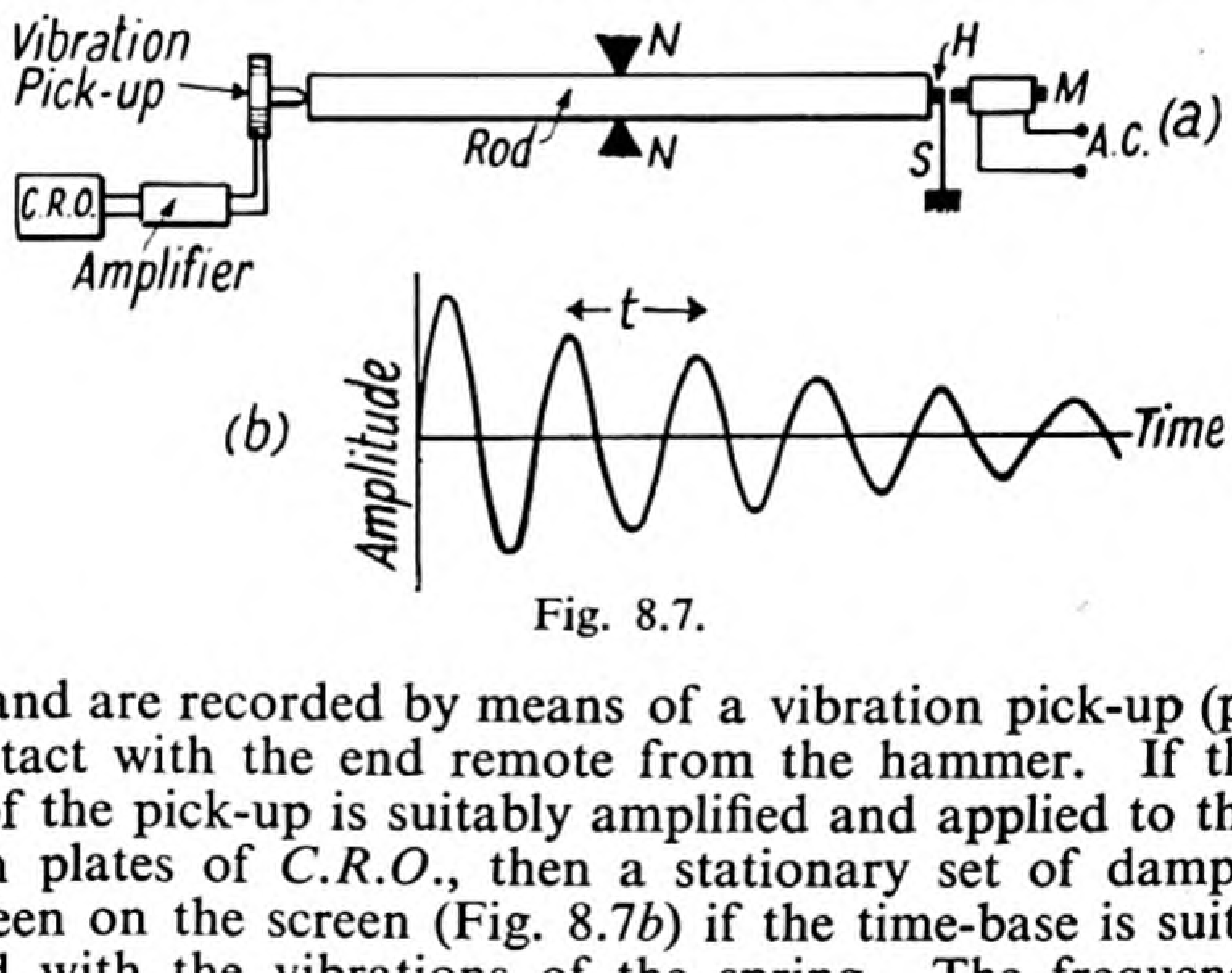


Fig. 8.7.

the rod, and are recorded by means of a vibration pick-up (p. 241) in light contact with the end remote from the hammer. If the output voltage of the pick-up is suitably amplified and applied to the vertical deflection plates of *C.R.O.*, then a stationary set of damped waves will be seen on the screen (Fig. 8.7*b*) if the time-base is suitably synchronised with the vibrations of the spring. The frequency of the time-base being known, the time (t) for the longitudinal disturbance to travel up and down the rod can be found since it will be the equivalent, on the correct time-scale, to the horizontal distance on the screen between successive maximum amplitudes. It is evident that the amplitude ratio of successive maxima will give a measure of the damping factor of the medium. It is noteworthy that this method is not restricted to solids of regular form.

If the solid specimen is in the form of a cylindrical rod or tube, it may be used in the Kundt's experiment described on p. 136, and the velocity of sound in the specimen deduced by assuming the value of the velocity of sound in air applicable under the existing conditions.

An alternative method of exciting solid rods into longitudinal vibration has recently been employed for the measurement of the

velocity of propagation of longitudinal waves. An electrotractive method of excitation is employed, which depends for its action upon the change of dimensions of a dielectric in an alternating electric field applied across the specimen; it is the phenomenon in electrostatics corresponding to that of magnetostriction in magnetism. The specimen rod may be conveniently clamped at its centre so that its lower end (Fig. 8.8) rests upon a thin metallic foil placed on the upper surface of, say, a thin sheet of mica. The latter rests on a metal block which forms the lower electrode, the metal foil constituting the upper electrode. The rod is vibrated by means of a variable frequency oscillator, the output voltage from which is applied through two capacitances C_1 and C_2 to the two electrodes, the dielectric being in addition suitably polarised from a D.C. source. Consequently, when the frequency of the oscillator (n) has been adjusted so that the rod is in resonance, the longitudinal vibrations set up will correspond to this frequency n and the wave-length λ will be equal to $2l$, where l=length of rod. Resonance is detected by means of a vibration pick-up (see p. 241) which is lightly placed on the upper end of the rod, and is connected through an amplifier to the Y plates of a C.R.O., a linear time-base operating in the X direction.

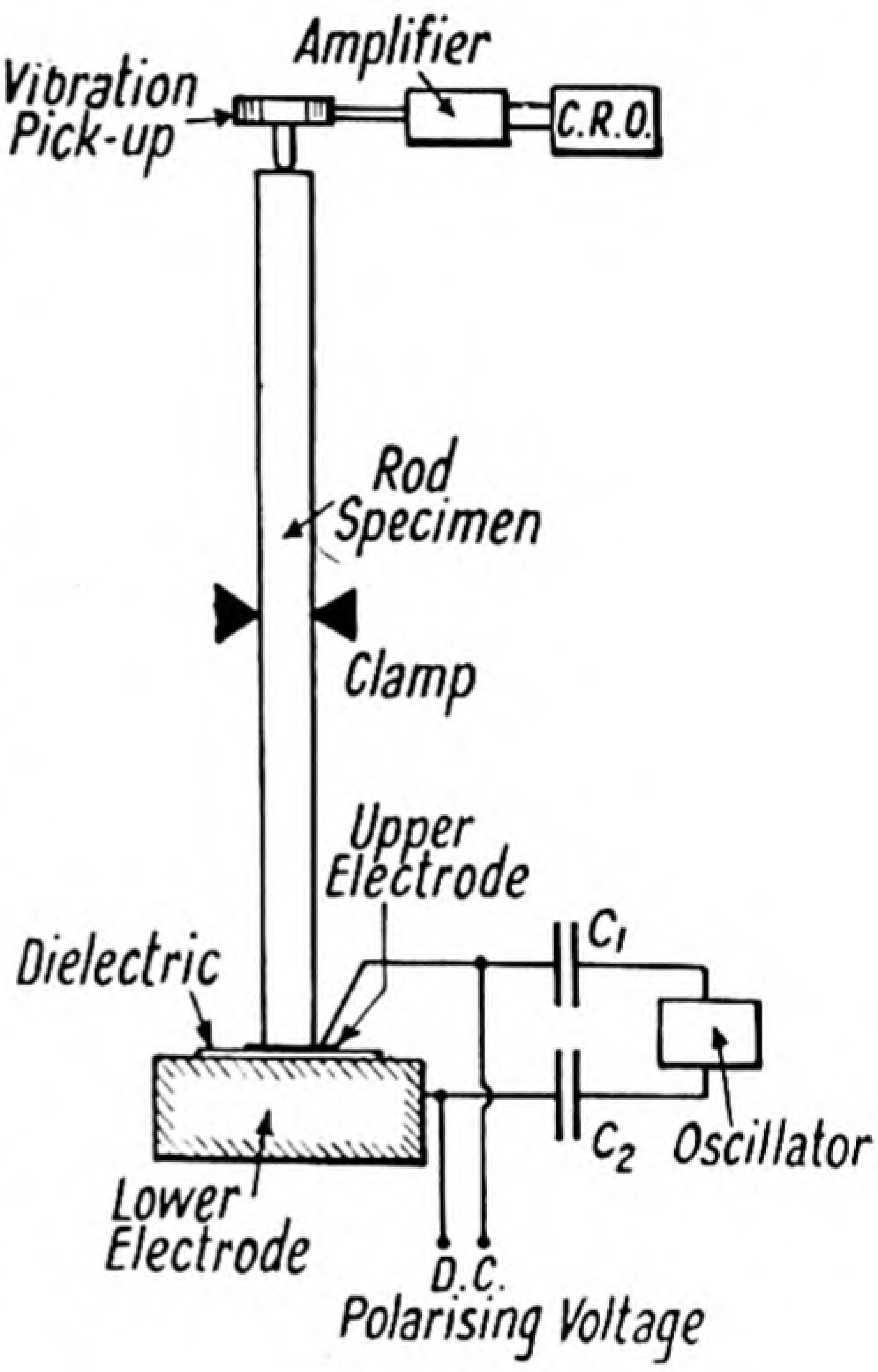


Fig. 8.8.

Sources of error in the various methods

(1) *Personal equation.* The lapse of time between the instant at which an event actually occurs and the instant at which it is recorded is termed the "personal equation" of the recorder, a term which is applicable both to human beings and automatic recorders. The effect varies with circumstances and is almost impossible to eliminate in the individual. For example, in the gun method the observer sees the flash and starts the watch about 0·1 sec. later; he then hears the explosion and stops the watch about 0·1 sec. later. If these time delays were identical they would cancel each other in the final result, but it is known that the personal equation for visual observation is less than that for aural observation, and cancellation is therefore only approximate. If the two were theoretically equal, other factors, such as fatigue, would tend to make them unequal. The only way to diminish the error is to make the actual time large, which means a long path distance for the sound to travel, but this introduces errors which are discussed below.

Microphones and other mechanical devices have personal equations which are constant when used under defined conditions, but even then the values differ for apparently identical pieces of apparatus, so that the timing error does not necessarily cancel in the experiment depicted in Fig. 8.1. In other words, *AB* is not necessarily an integral number of wave-lengths, for *A* and *B* only appear to be in phase. Cancellation is effected by interchanging the microphones *A* and *B*, and taking the mean value of the recorded intervals, or by taking two positions of *A* an integral number of half-wave lengths apart as described (p. 135).

(2) *Timing*. Ordinary methods of timing, *e.g.* by stop-watch, are suitable only for large-scale experiments, as percentage errors of measurement of time are diminished. The cathode-ray oscillograph, combined with a standard source of frequency has, however, made it possible to measure small intervals of time with considerable accuracy, thus making small scale experiments both possible and reliable.

(3) *Temperature*. In large-scale measurements the effect of variations of temperature between source and receiver cannot be assessed accurately. The mean value of the temperature at various points in the path is usually taken as the true value. Laboratory-scale experiments do no suffer from this disadvantage, since the apparatus can be designed to attain constant temperature conditions.

(4) *Humidity*. Since the density of water vapour is approximately only 0·6 of that of air at the same temperature and pressure, it follows that the effect of humidity will be to decrease the effective density of the atmosphere and so increase the velocity of propagation of sound in the medium. It should be noted, however, that the effect of humidity on the velocity of sound in hydrogen is to *diminish* it. This correction for the humidity of the gaseous atmosphere is a difficult assessment in large-scale experiments, owing to its possible variation from point to point. A mean value of the humidity at a number of points is therefore taken, and allowance made for the reduced density, by application of the law of partial pressures, so enabling the velocity in dry air to be calculated.

It is therefore important when measuring the velocity of sound in a gas, by a laboratory method, to ensure that it is thoroughly dry. If this is impossible the humidity must be determined to enable a correction to be applied. Sound waves travelling over water tend to advance more quickly near the water, where the air is saturated, than at higher points, so that a sound wave-front, initially vertical, tends to turn upwards. This elevates the direction of the sound and explains why sounds originating near the surface of the sea a little way from the shore are frequently heard clearly at elevated points inshore (see p. 119).

(5) *The tube effect*. Improvements in methods of timing have led to a reduction in the length of the sound path required, and by enclosing the gas in a tube, temperature and humidity are more easily controllable. The presence of a tube, however, causes a reduction in the velocity of propagation, due to viscous resistance of the fluid to its own vibratory motion. This reduction varies with the diameter of the tube and the frequency of the sound, and in tubes of capillary

dimensions such vibrations are damped out in a short distance. This property is used in the manufacture of sound absorbents mentioned in Chapters 9, 14 and 15.

The tube effect has been investigated by Helmholtz, Kirchhoff and others, and their results are summarised in the empirical equation,

$$V'=V\left(1-\frac{k}{r\sqrt{n}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where k is a constant (about 0·2, when r is in metres), V' is the measured velocity in a tube of radius r, V is the "free" air velocity required, and n the frequency. Alternatively, this equation may be written

$$V'=V\left(1-\frac{k'}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which $k'=\frac{k}{\sqrt{n}}$, a form which is convenient when a constant frequency is used, for k' then depends only on the roughness and material of the tube. By utilising two tubes of different radii, r_1 and r_2, but of the same material and smoothness, and by employing a constant frequency throughout, k' is made constant for both tubes, and may be eliminated. If V_1' and V_2' are the measured velocities in the respective tubes, then the simplified expressions for the two tubes are

$$r_1V_1'=Vr_1-k'V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$r_2V_2'=Vr_2-k'V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Subtracting,

$$V=\frac{r_1V_1'-r_2V_2'}{r_1-r_2}=\frac{n(r_1\lambda_1-r_2\lambda_2)}{r_1-r_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Alternatively, by writing the equation (2) in the form $V'=-\frac{Vk'}{r}+V$ or $\lambda'=-\frac{\lambda k'}{r}+\lambda$, a linear relation is seen to exist between λ' and $\frac{1}{r}$ as variables. Hence, a graph plotted between these quantities would enable k' and λ to be found from the slope ($=-\lambda k'$) and the intercept on the λ'-axis ($=\lambda$).

The tube effect is present in Behn and Geiger's method (p. 138), and is eliminated by filling the narrower tube with air and determining the apparent velocity of sound.

(6) *Yielding of walls of tube.* This is practically imperceptible with tubes containing only gas, but when filled with liquids the yielding of the tube near the nodes, in resonance tube methods, is sufficient to be detected by a stethoscope applied to the outside of the tube; its effect is to lower the wave-velocity in the liquid.

Measurements in tubes at wide ranges of temperature. These are conveniently carried out at fixed points, such as 0° C., 100° C., B.P. of sulphur (444·6° C.), F.P. CO_2 (−79·4° C.), B.P. oxygen (−183° C.), by immersing small plunger-tubes in a jacket containing the appropriate substance. In a series of experiments with a jacket containing a mixture of a suitable liquid, *e.g.* acetone and solid CO_2, a pair of identical tubes with an organ pipe mouth was used, the plunger face

in one being adjusted to give the same frequency as the other, so that one was $\frac{\lambda}{2}$ longer than the other. In this way the tubes are used differentially, for the space which is measured is well immersed in the low temperature zone, and any variations in temperature in the more exposed portions of the tube are common to both. Owing, however, to the formation of ice it was difficult to move the plungers, and it is more expedient to set them at suitable points and, after reducing the temperature, to make the frequency from each equal by lengthening each pipe at the exposed ends by the same amount of "sleeve."

To ensure uniformity in the low temperature region, an apparatus has been designed by Bate. It consists of a pair of bottle-pipe resonators in which the *bottles* differ in length, but the pipes are equal. The lengths of the latter are varied by the sleeves secured to a flange, which ensures that such adjustment is equal in both pipes. Resonance is excited in each resonator, and is adjusted until the frequency is the same from both. The difference in length is an integral number of half wave-lengths as before.

For high temperature measurements the plunger-tube is frequently used in a high temperature enclosure such as a furnace. The sound is of uniform frequency, and is generated by a telephone diaphragm energised by a constant frequency oscillator. The input energy is measured on suitable meters for different positions of the plunger, and resonance is indicated by the input becoming a minimum. Several resonant points may thus be identified, and the mean value of the wave-length obtained.

Velocity of propagation of waves of finite amplitude

In the previous derivation of the formula $c=\sqrt{\frac{\gamma p}{\rho}}$ representing the velocity of propagation of plane compressional waves in an infinite fluid medium, it was assumed that the amplitude of the particle displacement, and hence also the change of condensation and density, was always small. This assumption is no longer justifiable however, if, for example, explosive waves are being considered, and the theory requires modification as follows

It can be shown (Appendix) that for a plane progressive wave $\frac{d^2\eta}{dt^2}=-\frac{1}{\rho_0}\cdot\frac{dp}{dx}$ where η is the particle displacement in the direction of propagation (x) and ρ_0 is the value of the undisturbed density of the medium. Furthermore, the condensation (s) of the wave is defined by $\frac{\rho}{\rho_0}=(1+s)=\left(1+\frac{d\eta}{dx}\right)^{-1}$, and for an adiabatic process the relation between pressure and density is expressed by $\left(\frac{p}{p_0}\right)=\left(\frac{\rho}{\rho_0}\right)^{\gamma}$, where γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. Hence

$$\frac{p}{p_0}=(1+s)^{\gamma}=\left(1+\frac{d\eta}{dx}\right)^{-\gamma},$$

and therefore

$$\frac{dp}{dx}=-\gamma p_0\cdot\frac{d^2\eta}{dx^2}\left(1+\frac{d\eta}{dx}\right)^{-(\gamma+1)}.$$

The equation of propagation now becomes

$$\frac{d^2\eta}{dt^2}=\frac{\gamma p_0}{\rho_0}\left(1+\frac{d\eta}{dx}\right)^{-(\gamma+1)}\frac{d^2\eta}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which $\frac{d\eta}{dx}$ can no longer be neglected. It follows that the velocity of propagation (c_a) is now given by

$$c_a=\sqrt{\frac{\gamma p_0}{\rho_0}\left(1+\frac{d\eta}{dx}\right)^{-(\gamma+1)}}=c\sqrt{\left(1+\frac{d\eta}{dx}\right)^{-(\gamma+1)}}=c\left(1+\frac{d\eta}{dx}\right)^{-\left(\frac{\gamma+1}{2}\right)}$$

or

$$c_a=c(1+s)^{\left(\frac{\gamma+1}{2}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The above analysis is only approximate even for the plane waves considered, and the extension to spherical divergent waves will involve, for example, the recognition of the decrease in the value of the condensation s with increasing distance from the source. It should be pointed out here that the criterion of the "largeness of condensation" is dependent upon frequency as well as the amplitude (a) since from formulae (7) p. 15 and (3) p. 65, the maximum particle-velocity $v=2\pi na$ and the maximum condensation $s=\frac{v}{c}$, where n is the frequency of the source and c is the velocity of sound. Hence an amplitude of, say, $0{\cdot}5\times10^{-4}$ cm. at 200 c.p.s. would be considered small, whereas at 2×10^6 c.p.s. it would definitely be regarded as large. The maximum value of the condensation s may also be written as $\frac{v}{c}=\frac{2\pi na}{n\lambda}=\frac{2\pi a}{\lambda}$ where λ is the wave-length of the sound, so that the ratio $\frac{a}{\lambda}$ is the factor which is a measure of the departure from ordinary sound wave propagation. The magnitude of the increase in velocity due to a large value of condensation, as deduced for the higher frequency case cited above, is easily calculated as follows:—

$$a=0{\cdot}5\times10^{-4}\text{ cm.},\quad n=2\times10^6\text{ c.p.s.},\quad c=33\times10^3\text{ cm. per sec.},$$

$$s=\frac{2\pi\times2\times10^6\times{\cdot}5\times10^{-4}}{33\times10^3}\simeq2\times10^{-2}.$$

Hence $c_a=c(1{\cdot}02)^{1{\cdot}2}=1{\cdot}024$ c., *i.e.* the velocity shows a $2\frac{1}{2}\%$ increase. If the frequency had been that of the upper audible limit, viz. 2×10^4 c.p.s., the corresponding velocity change would have been $0{\cdot}02\%$.

Now the expression (7) indicates that the velocity of propagation of a large amplitude wave train will be greater in the regions of condensation than in those of rarefaction. Hence it follows that the crests of the waves will gain on the troughs, and the effect is very well illustrated by sea waves approaching a slightly shelving beach. Here the crests are seen to become shorter and to increase in amplitude as they get nearer in-shore, until finally the face of the crest becomes vertical and gradually breaks up by falling over into the preceeding trough. The corresponding phenomenon in air is shown by explosive waves as indicated in Fig. 8.9. The wave at its inception is indicated by t_1, but as it moves out from the explosion centre, in a short distance it will have assumed the form shown at t_2 by which time the crest has grown in size at the expense of the rear of the wave. This growth continues until the pressure gradient at the front of the wave is extremely high, and is shown by the almost vertical rise at the head of the wave at t_3 in Fig. 8.9. This form of the wave may persist for appreciable distances away from the source, despite the violent motion and consequent

energy dissipation associated with this *shock*-wave, as it is termed. At increasing distances as the wave-front expands the amplitude falls off naturally, and so the wave-crest gradually assumes a rounded form again (t_4 in Fig. 8.9) and thus the wave is ultimately propagated as an ordinary sound wave. The existence of a shock-wave is revealed in a convincing manner by the Schlieren method of photography, for the large change of density which takes place within a very short distance at the head of the wave will produce a marked deviation of the path of an incident light beam.

Explosions

The detonation of a high explosive is essentially the process of generating a large quantity of hot gases within a very small volume, and this produces in consequence a high pressure and temperature. A two-fold effect is to be observed, namely the shock-wave previously considered, and in addition, an air-blast which moves with speeds of the order of 10,000 m.p.h. or greater. The effective distance travelled by this hurricane is limited to a much greater extent by the frictional air resistance than is the shock-wave, whose presence is detected many miles away from the explosion. By reason of their larger inertia, the solid objects set into motion by the explosion will initially lag behind the shock-wave and air-blast, but on attaining the velocity of these waves, will gradually overtake them owing to their greater momenta. Fig. 8.10 shows how the air pressure varies with time *at any point* in the path of the shock-wave, and indicates that an object subjected to such a wave will experience initially a sudden *pressure*, followed by a smaller but longer-acting *suction*, *i.e.* a negative pressure.

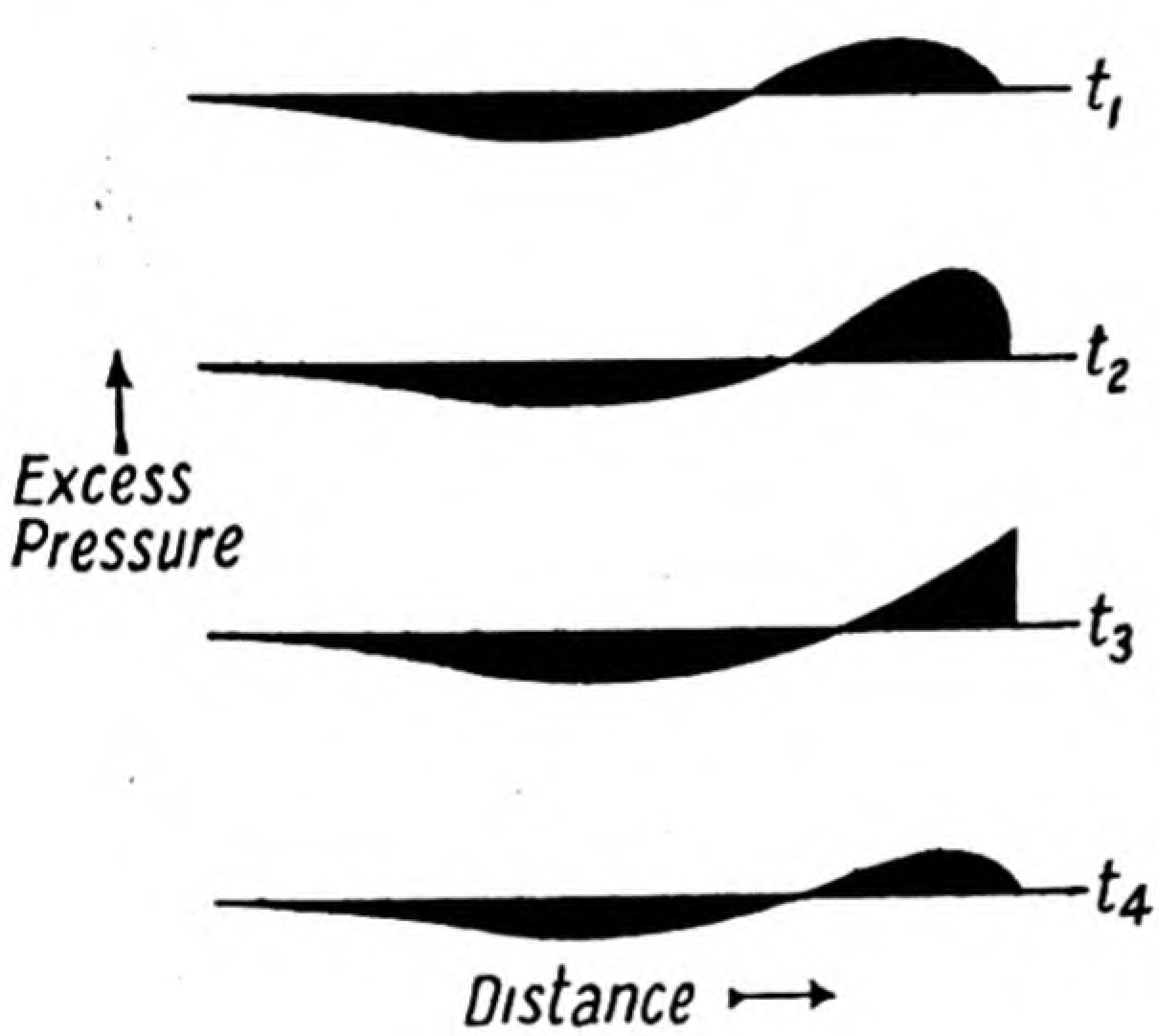


Fig. 8.9.

The Doppler effect

If a source of sound is moving relative to an observer a rise or fall of pitch is heard by the observer according to whether the distance between them is respectively diminishing or increasing. The discovery of this phenomenon is usually attributed to Doppler, who became Professor of experimental physics at Vienna University, but Scott Russell and Babinet also published independent papers on the subject, although a few years later than Doppler (1842). In a qualitative way it is easy to see that the sound waves in front of a moving source will become crowded together and shorter, while those behind will spread out and become longer. Consequently, an observer receives a larger number of waves per second when the source is approaching than

when it is receding, and the reader, when standing on a railway platform, cannot have failed to hear the drop in pitch of the whistle of an express train as it passed him. There is a fall of nearly a tone when the train exceeds a speed of 40 m.p.h.

Consider now a source of sound S, which is moving from right to left along the straight line R_2R_1 in Fig. 8.11 with a uniform velocity v_s, and let S_1, S_2, etc., be the successive positions it occupies after equal time intervals of t sec., i.e. $S_1S_2=S_2S_3=v_st$. In the position S_7 reached by the moving source, a disturbance is just about to be propagated outwards at a time $6t$ sec. after that initiated at S_1, which will have arrived at points D_1D_1' on the line R_1R_2 where $D_1S_1=S_1D_1'=6ct$, c being the velocity of sound in the medium. The crest of the disturbance from the source when it was located at S_2 will have reached points D_2, D_2' on the line R_1R_2 where $D_2S_2=S_2D_2'=5ct$. Similar reasoning will hold for the other positions D_3, D_4, etc., so that the wave system assumes the appearance of an eccentrically disposed set of circles. In front of the moving source the distance between disturbances from chosen points is given by D_1D_2, D_2D_3, etc., and in the rear by $D_1'D_2'$, $D_2'D_3'$, etc. Now $D_1D_2=(D_1S_1-S_1S_2)-D_2S_2=(6ct-v_st)-5ct=t(c-v_s)$, which is obviously equal to $D_2D_3=D_3D_4$, etc., and similarly it is easy to show that $D_1'D_2'=t(c+v_s)=D_2'D_3'$, etc.

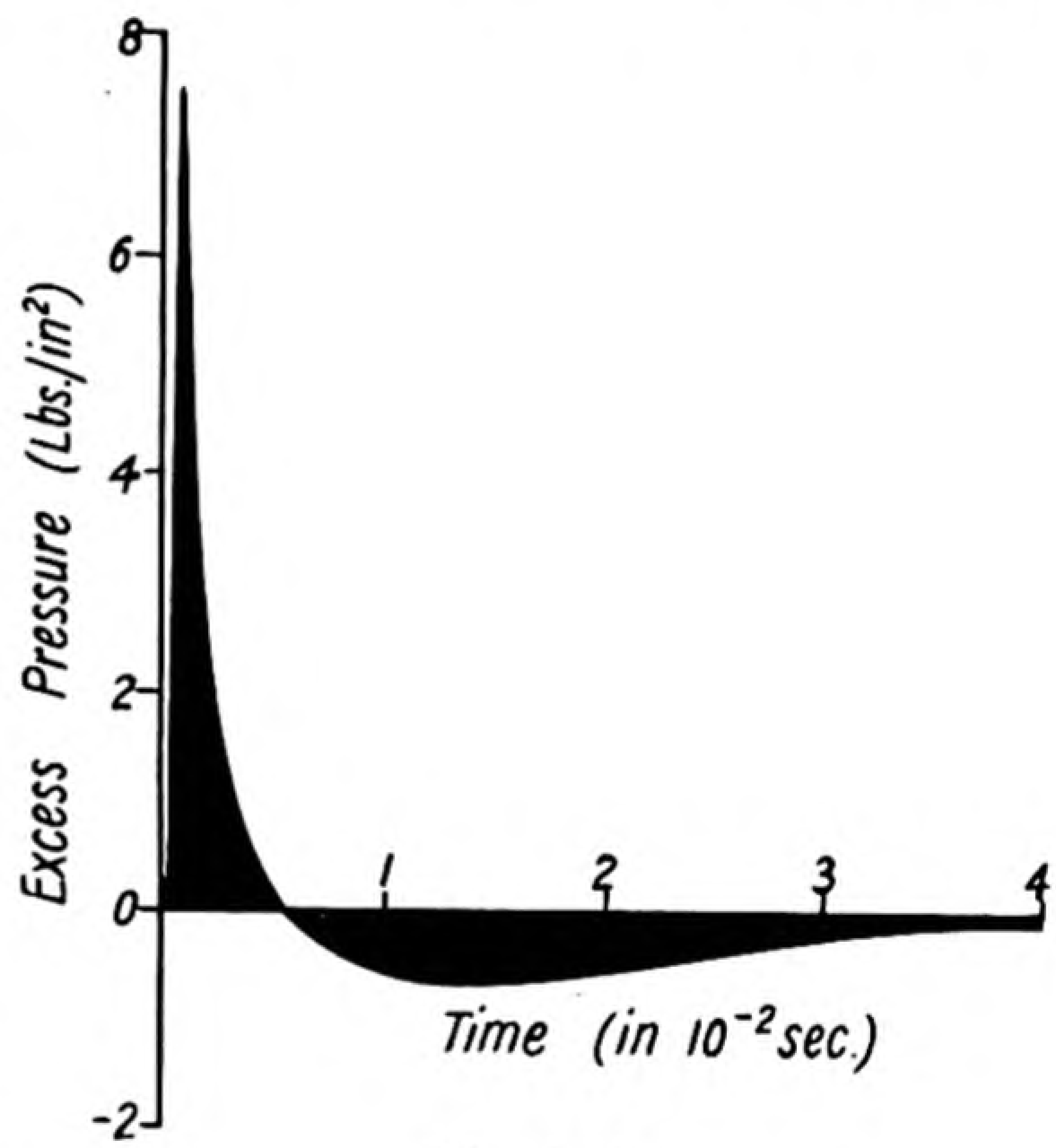


Fig. 8.10.

Let N=frequency of the waves emitted by the source, then in a time t sec. the number of waves sent out is Nt, and hence the distance between *successive* crests in front of the moving source is given by $\frac{t(c-v_s)}{Nt}=\frac{c-v_s}{N}$. It follows, therefore, that an observer at R_1 will receive sound waves at a rate given by the ratio

$$\frac{\text{velocity of sound}}{\text{distance between successive crests}}=\frac{c}{\frac{c-v_s}{N}}=\left(\frac{c}{c-v_s}\right)N \quad . \quad (8)$$

In the case of the observer at R_2 it is easy to show that the frequency of the source will appear to decrease and will be given by

$$\left(\frac{c}{c+v_s}\right)N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Consider now the case of an observer moving with a velocity v_o

towards a source S (Fig. 8.12) of frequency N. In a time τ the observer will move through a distance $R_1R_2 = v_o\tau$, and the additional number of waves received by the observer will therefore be given by $\frac{v_o\tau}{\lambda}$, where the wave-length $\lambda = \frac{c}{N}$, c being the velocity of sound. Hence the total number of sound waves received by the observer is

$$Nt + \frac{v_o\tau}{\lambda} = \tau\left(N + \frac{v_o}{\lambda}\right) = N\tau\left(\frac{c+v_o}{c}\right)$$

and hence the frequency N_m of the note heard by the listener is

$$N\tau\left(\frac{c+v_o}{c}\right) \div \tau = N\left(\frac{c+v_o}{c}\right) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

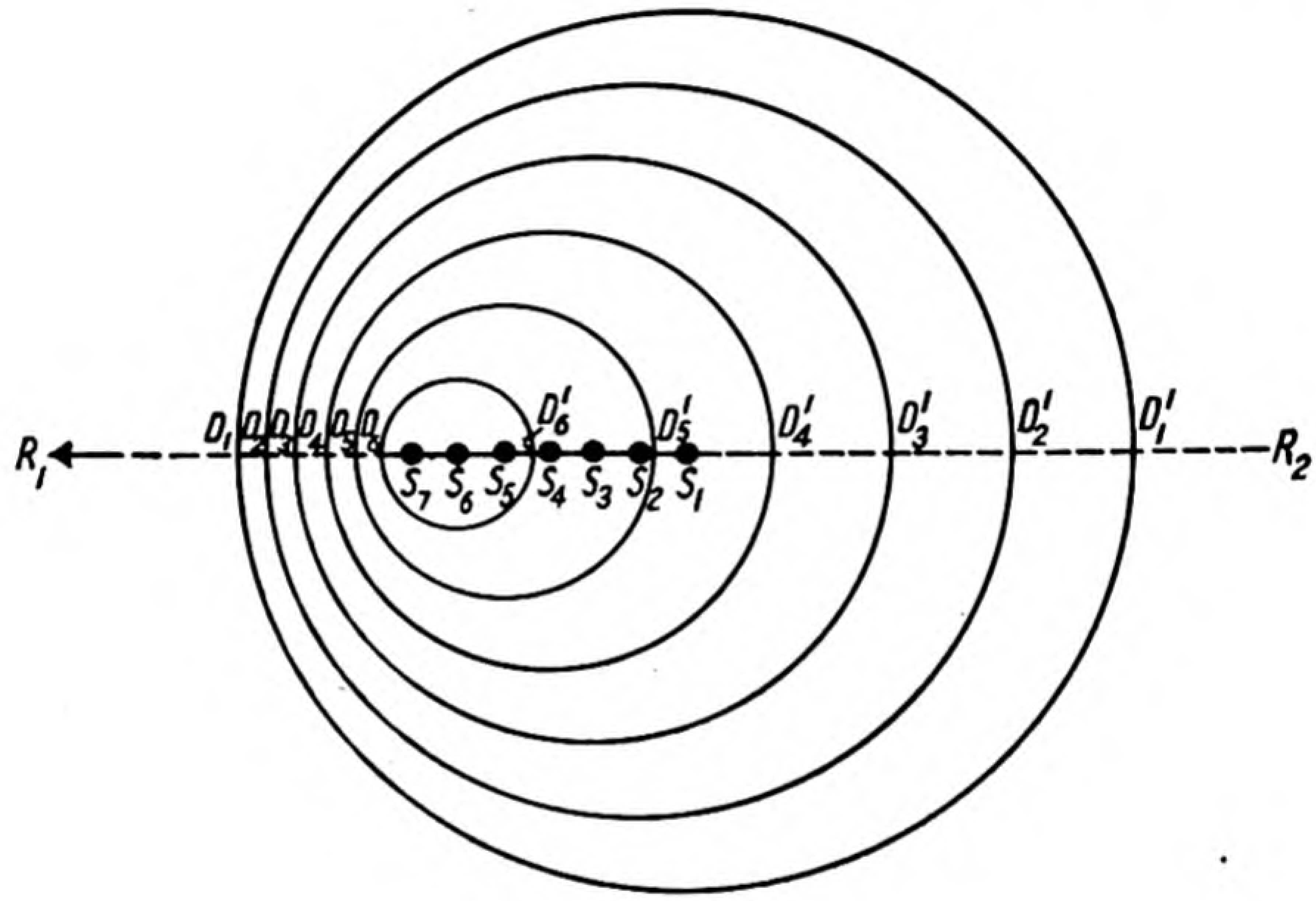


Fig. 8.11.

It is easy to show in the above manner that the frequency of the sound heard by the listener when moving *away* from the source is given by

$$N\left(\frac{c-v_o}{c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The difference between the corresponding expressions (8) and (10) is not surprising when it is considered that in the former case there is actually a change in the wave-length of the sound emitted by the source.

The expression for the frequency of the note heard by the listener when both he and the source are in motion, may be derived by combining the previous expressions. From (8) with source moving with velocity v_s, but the observer at rest, $N_r = \left(\frac{c}{c-v_s}\right)$; and from (9) with the observer moving with velocity v_o and the source with velocity v_s, $N_m = \left(\frac{c+v_o}{c}\right)N_r$, since N_r is the frequency of source when moving.

Hence $$N_m = \left(\frac{c+v_o}{c-v_s}\right) N \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It should be noted that v_s is the speed with which the source moves towards the observer, and v_o is speed with which the observer moves towards the source. If either the observer or the source is moving away from the other, then the sign of its velocity must be changed in the above expression.

If a wind is blowing with velocity v_w in the direction of travel of the sound, then the expression (12) becomes

$$N_m = \left[\frac{(c+v_w)+v_o}{(c+v_w)-v_s}\right] N \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

When the source and listener remain the same distance apart, i.e. $v_o = -v_s$, then the wind has no effect on the pitch of the note heard, for the change in the wave-length of the emitted sound is compensated by the change brought about by the wind in the velocity of sound relative to the ground and the observer.

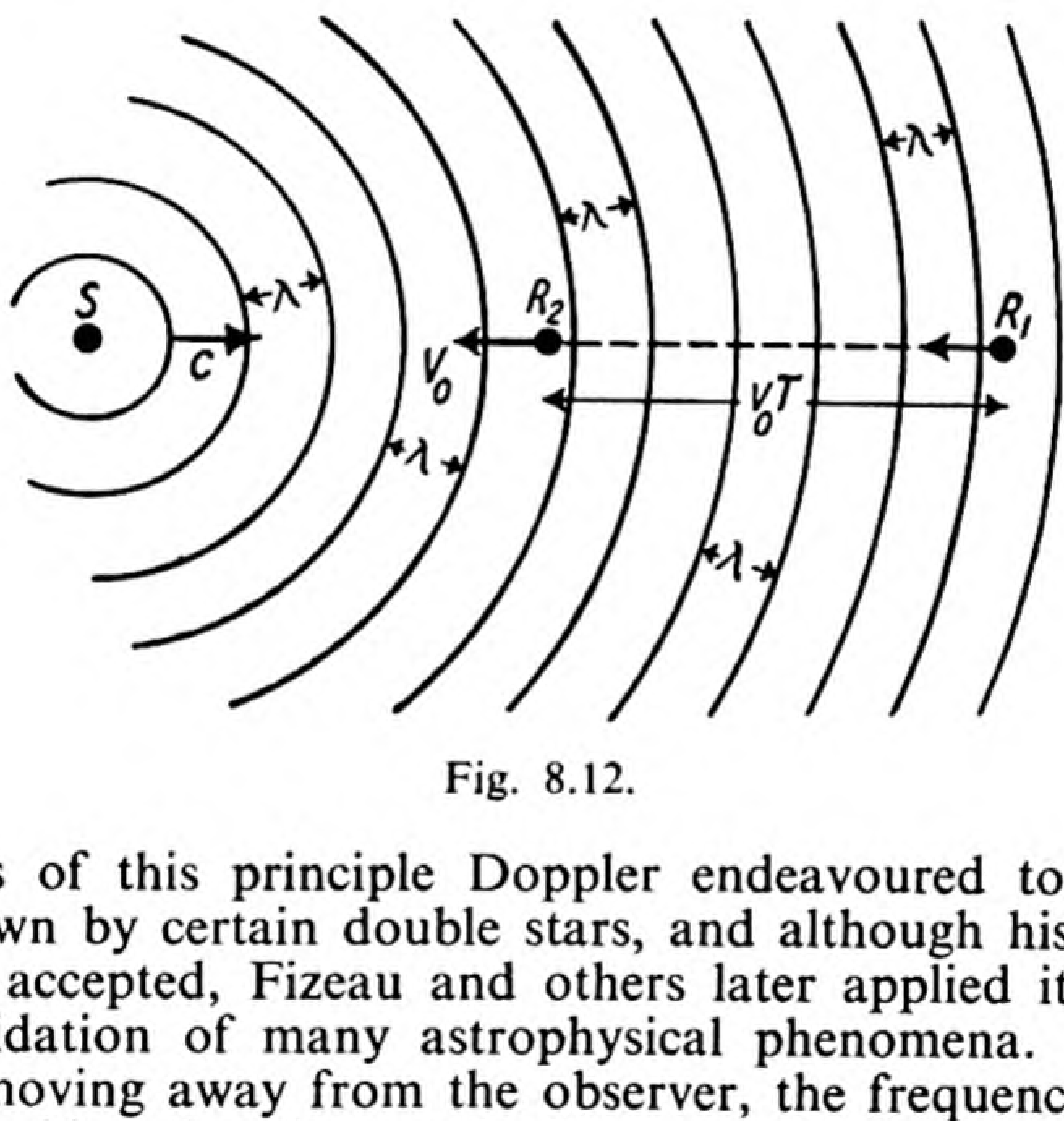


Fig. 8.12.

By means of this principle Doppler endeavoured to explain the colours shown by certain double stars, and although his explanation is not now accepted, Fizeau and others later applied it successfully to the elucidation of many astrophysical phenomena. If a source of light is moving away from the observer, the frequency of the disturbance reaching the observer is decreased, and therefore the wavelength is increased. Consequently the lines in the spectrum of the source, as compared with those obtained with the same substance in the laboratory, will be shifted towards the red end of the spectrum, and similar reasoning will lead to the deduction that for a source moving *towards* the observer the spectrum lines will be shifted towards the violet. By measuring the extent of this shift the velocity with which a star is receding or approaching the earth may be estimated. Certain stars, spectroscopic binaries, consist of two bodies revolving round a common centre of gravity, but they always appear as a single star in a telescope of the highest resolving power. A series of photographs of the spectra of the star taken at different times shows that the

spectral lines are sometimes "doubled," and from the "period" of this occurrence it is possible to determine the time of revolution of the component stars about their common centre of gravity.

The Germans made use of the Doppler principle during the recent World War as a means of controlling the flight of some of their early V2 rockets. A receiving and a transmitting set were located, both on the rocket and at the ground station, and a signal of frequency 30 Mc. per sec. was transmitted from the ground. Then it follows from equation (11) that the frequency n of the signal received by the rocket

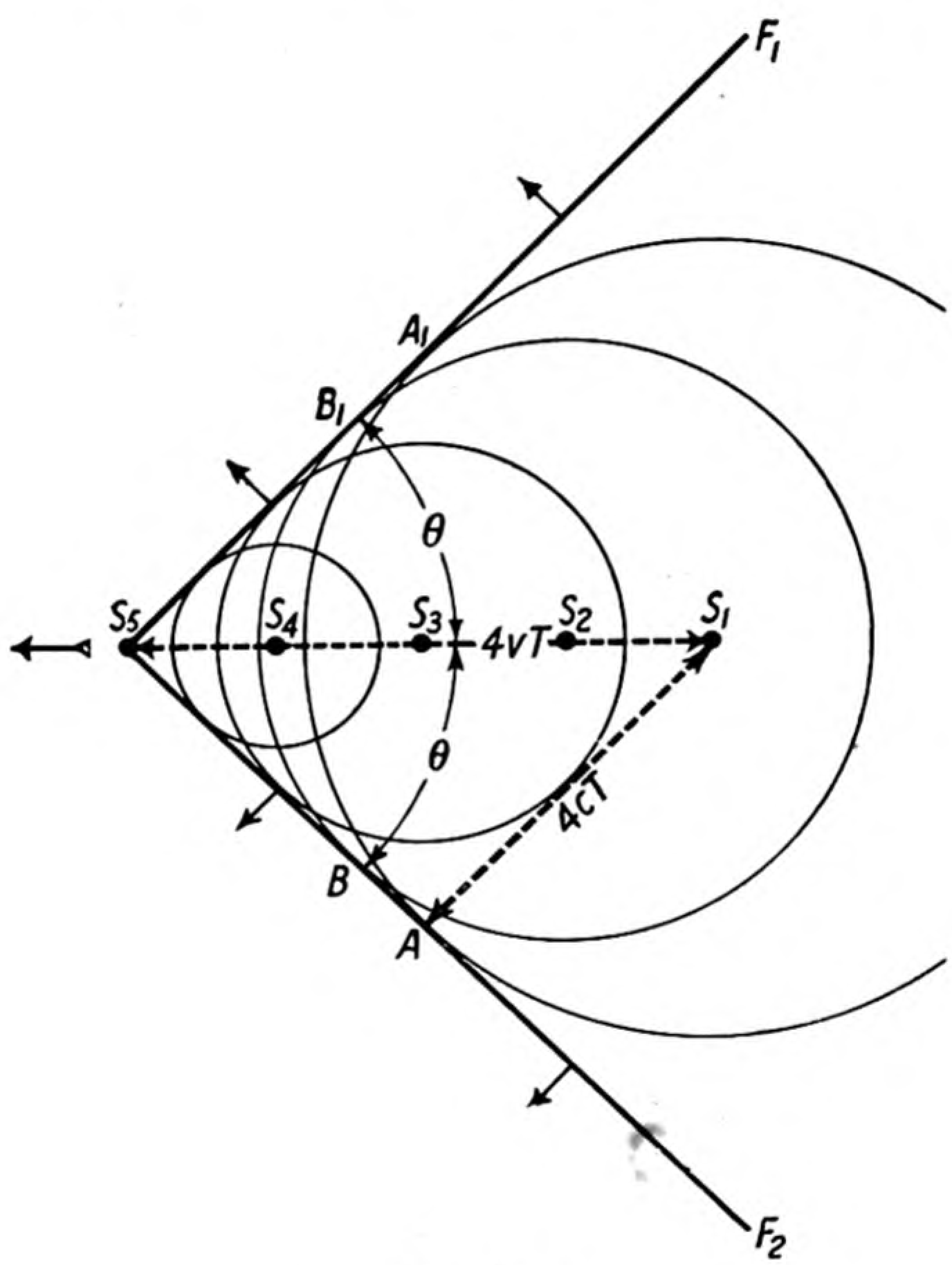


Fig. 8.13.

was given by $n = N\left(\frac{c - v_o}{c}\right)$, where $N = 30$ Mc. per sec., c is approximately 2×10^5 miles per second, and v_o is the instantaneous speed of the rocket. If $N - n = 250$ c.p.s., say, then by substitution it follows that $v_o = \left(\frac{N-n}{n}\right) . c = \frac{5}{3}$ miles per second or 6000 m.p.h. The frequency of the signal received by the rocket receiver was automatically doubled and re-transmitted to the ground. The control situated there included a suitable frequency meter, from which the speed could be measured directly, and when the correct speed was attained a signal was sent out to cut off the power supply in the projectile.

Sound source moving at supersonic speeds

At low speeds of travel the moving body will give rise to spherical sound waves of normal velocity, but when it is moving with speed greater than that of sound the air compression created by its motion can only be transmitted in a lateral direction (Fig. 8.13).

Suppose the source be moving from right to left with a velocity v greater than the velocity (c) of sound, and let S_1, S_2, etc., represent the successive positions reached by the source after equal time intervals of T sec. In the figure the body is shown to have reached the position S_5 which will be distance $4vT$ cm. from S_1. By this time the disturbance generated at S_1 will have reached points AA_1, etc., distant $4cT$ from S_1; that generated at S_2 will have arrived at BB_1, etc., distant $3cT$ from S_2, and so on.

According to Huyghens' principle the resultant wave is the tangent cone drawn to all the wavelets, i.e. S_5F_1 *and* S_5F_2 in the diagram. It is easily seen that the semi-angle θ of the conical wave produced is given by

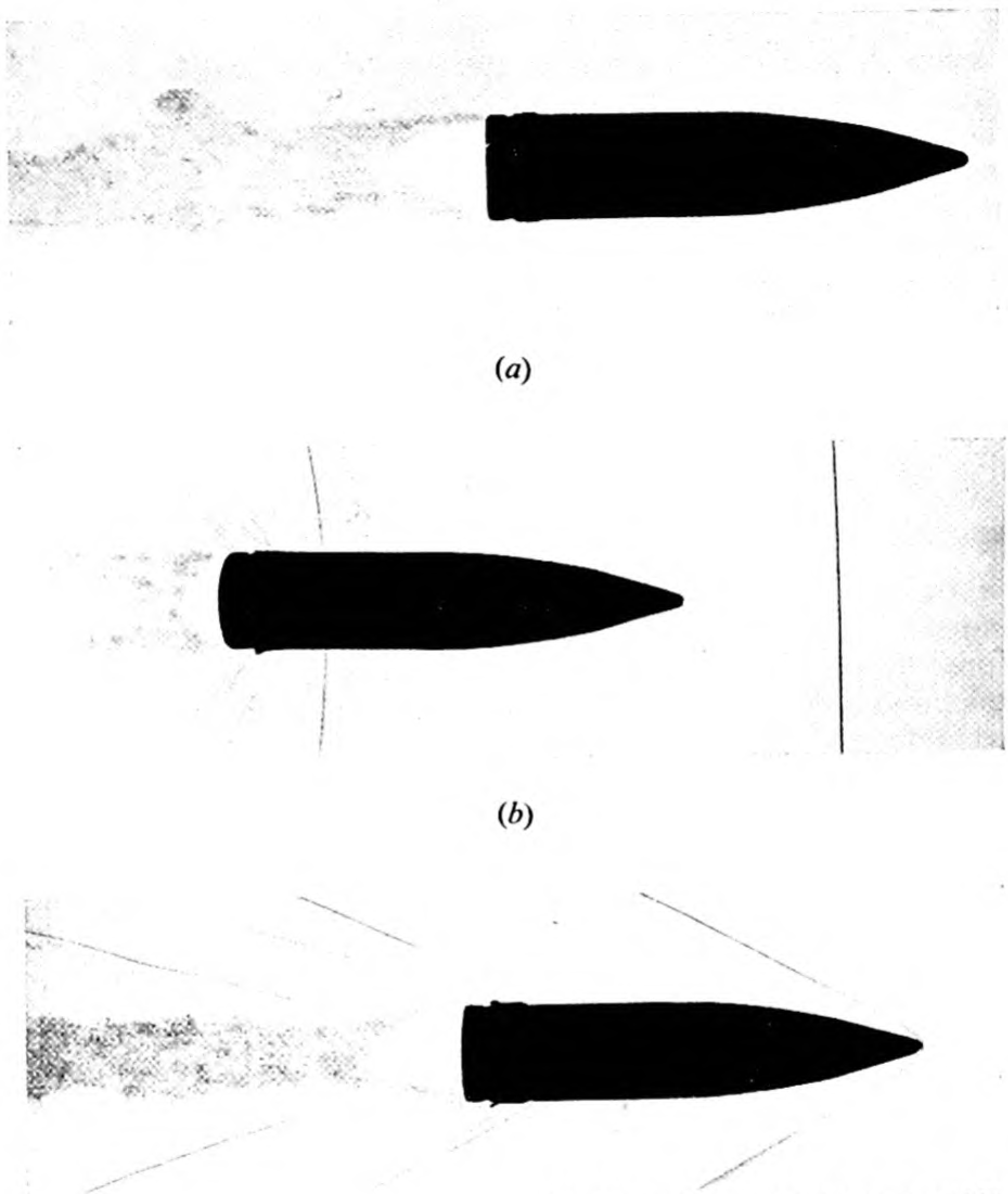
$$\sin\theta = \frac{4cT}{4vT} = \frac{c}{v}, \text{ or } \operatorname{cosec}\theta = \frac{v}{c}.$$

The ratio $\frac{v}{c}$ is known as the Mach number and the angle θ as the Mach angle. In normal aerodynamic flow the fluid is considered as incompressible, and Reynolds number R_N is the factor which controls the force experienced by a body moving through the fluid. At speeds approaching that of sound, however, compressibility effects become appreciable, and it is the Mach number which is the important factor under these conditions. A measurement of the Mach angle from a photograph of a bullet or projectile in flight (*e.g.* Fig. 8.14) provides a means of estimating its velocity, if the speed of sound is assumed. The measurement should not be made in the immediate neighbourhood of the bullet, as here the differences of pressure are so great that the "nose-wave" is propagated with a velocity exceeding that of sound, which will cause the angle θ to be increased.

Reynolds number R_N is defined by the ratio of the inertia to the viscous forces in an incompressible fluid, and may relate to the passage of a *fully* immersed body through the medium or to the flow of the fluid through an enclosure such as a pipe. These conditions imply that any free surface of the liquid does not enter into the argument, and that gravity forces are balanced by buoyancy forces. When surface control has to be considered, as in the case of surface waves due to a ship, then it is Froude's number, $v/\sqrt{lg}$, which is involved (v being the velocity and l the length dimension concerned).

It follows from the above definition that a measure of R_N is given by the ratio $(\rho l^3 . v/t)/(l^2 . \eta v/l)$, where ρ and η are respectively the density and viscosity of the fluid, v is the average fluid velocity over a fixed cross-section and l is a length characteristic of the problem such as a pipe diameter. Since l/v has the significance of a time (t), then the above expression simplifies to $R_N = \rho vl/\eta$. A low value of R_N for a fluid implies orderly or viscous flow, but as v, and therefore R_N, becomes large then inertia forces become predominant. Now the Mach number M is defined by v/c, but $c^2 = K/\rho$ for small displacements and hence $M^2 = \rho v^2/K$, which has the significant form of a ratio between inertia and elastic forces. An alternative and instructive form of this ratio

is deduced as follows. On page 67 it has been shown that $\delta p = Ks$, i.e. $\delta p = K.\delta\rho/\rho$, and therefore $c^2 = dp/d\rho$. But in an adiabatic process for a perfect gas $(p/p_0) = (\rho/\rho_0)\gamma$ and hence $\dfrac{dp}{d\rho} = \gamma RT$, where R is the gas constant and T the absolute temperature. It follows that $M^2 = v^2/\gamma RT$. Hankins



(*a*)

(*b*)

[*Hankins and Cope, Inst. Mech. Eng.* 1946.

(*c*)

Fig. 8.14. Projectile in Flight.

(*a*) Velocity	970 ft. per sec.	Mach number	0·86
(*b*) ,,	1,145 ,, ,, ,,	,, ,,	1·01
(*c*) ,,	2,830 ,, ,, ,,	,, ,,	2·51

and Cope have pointed out that absolute temperature is regarded on the kinetic theory of gases as a measure of the random energy of the molecules, and since v^2 may be considered as a measure of the directed energy, then M^2 can be regarded as determining the relative value of the directed to the random kinetic energy of the fluid.

At subsonic speeds the energy of a moving body is mainly dissipated in eddies formed at the rear of the body; at supersonic speeds, however, in contrast, the energy dissipation occurs in the waves which accompany the body and particularly in the wave emanating from the "head," *e.g.* from the nose of a bullet.

An interesting result which may accrue from the high velocity of the moving source is the inversion of the order of production of the sound as heard by an observer. If high velocity projectiles pass overhead or near at hand, it is the whine of the shell which is perceived first of all, then its explosion, and lastly the firing of the gun, although the order of the last two sounds may be reversed if the explosion takes place very far distant. An example of this phenomenon has been provided by the V2 rockets sent by the Germans against England during the past war, when the sound produced by the rocket in its passage through the atmosphere was heard after the final explosion.

Shock-waves and sound location

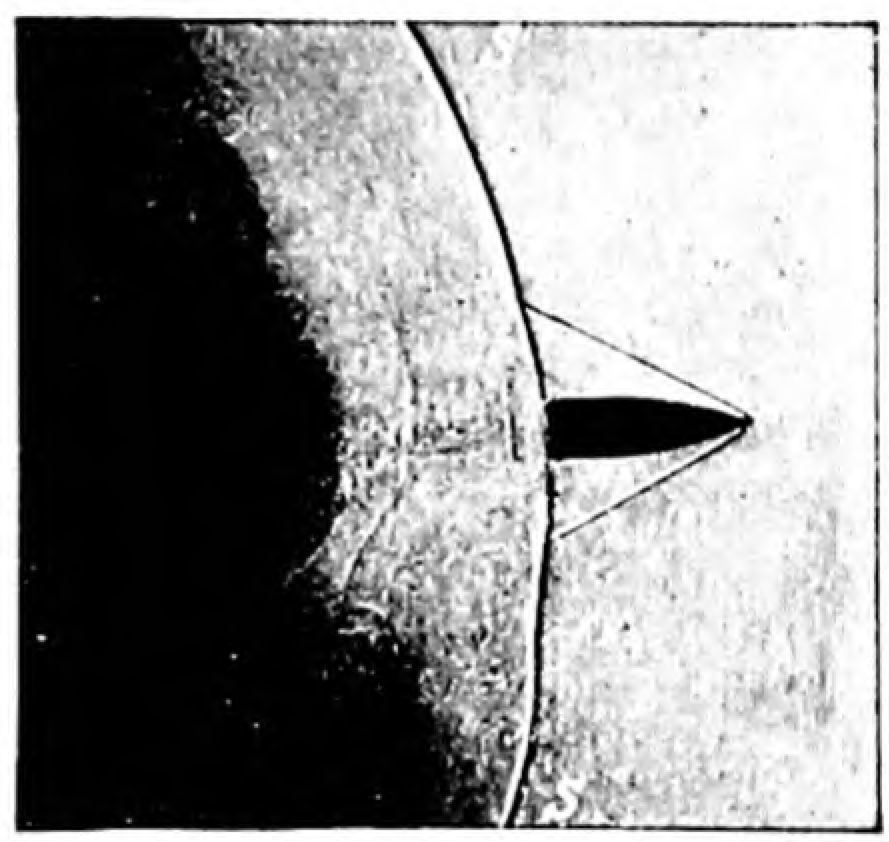


Fig. 8.15. From "Physical Principles of Mechanics and Acoustics," by R. W. Pohl. (By permission of Blackie & Son Ltd.)

In Fig. 8.15, which is a photograph of the muzzle wave of a rifle, together with the nose wave of the bullet fired, the latter is shown as having just overtaken the muzzle wave, and it provides an example of a body which is emitting compressional waves while actually in motion. In this case the body is moving faster than the velocity of sound, and the form of the wave emanating from the nose of the bullet is like that of the bow-wave of a ship, and is characteristic of all bodies moving with velocities exceeding that of sound. This **V**-shaped compression wave, as mentioned above, is the "shock" wave or "onde de choc," while the wave created by the explosion of the gun, which travels at the velocity of sound, is called the gun wave or "onde de bouche." The shock-wave is of little use for sound-ranging* purposes as its effective centre of disturbance is located in front of the actual gun position, and it is the low-pitched gun wave which is employed for sound-ranging. The sensitive form of receiver used to detect the "onde de bouche" by the British Armies in the 1914 war was developed by Tucker, and it comprised a vessel of large volume, so that its natural resonance frequency was low, and in the neck was an electrically heated grid of platinum wires. The latter becomes cooled by the incident compression wave, the effect being most marked when the resonating volume is in tune with the frequency of the sound waves.† Five or six of these microphones were spread along a base line of some 5 miles between 2 or 3 miles behind the front line, and the times of arrival of the sound waves from the

* Sound navigation and ranging are now often referred to under the general term SONAR (*cf.* RADAR).

† For hot-wire microphone, see also p. 263.

gun G (Fig. 8.16) to be located were suitably recorded by a short period galvanometer. In the diagram three microphones only, R_0, R_1 and R_2, are shown, and it is supposed that R_1 and R_2 have received the sound respectively t_1 and t_2 sec. after R_0. A diagram is then drawn to scale, and with centres at R_1 and R_2, two circles are drawn of radii ct_1 and ct_2 units respectively, where c=velocity of sound in air. It is easily seen that the gun position is fixed by the fact that it will lie at the centre of a circle passing through R_0 and touching the other two circles drawn.

Photography of sound waves

If suitable lighting is employed the progress of the condensation of a sound wave in air may actually be observed by making use of the change of the refractive index of a gas with the density. The effect is seen, for example, in the appearance of the hot air stream above a Bunsen burner, and this when illuminated may be thrown as a shadow

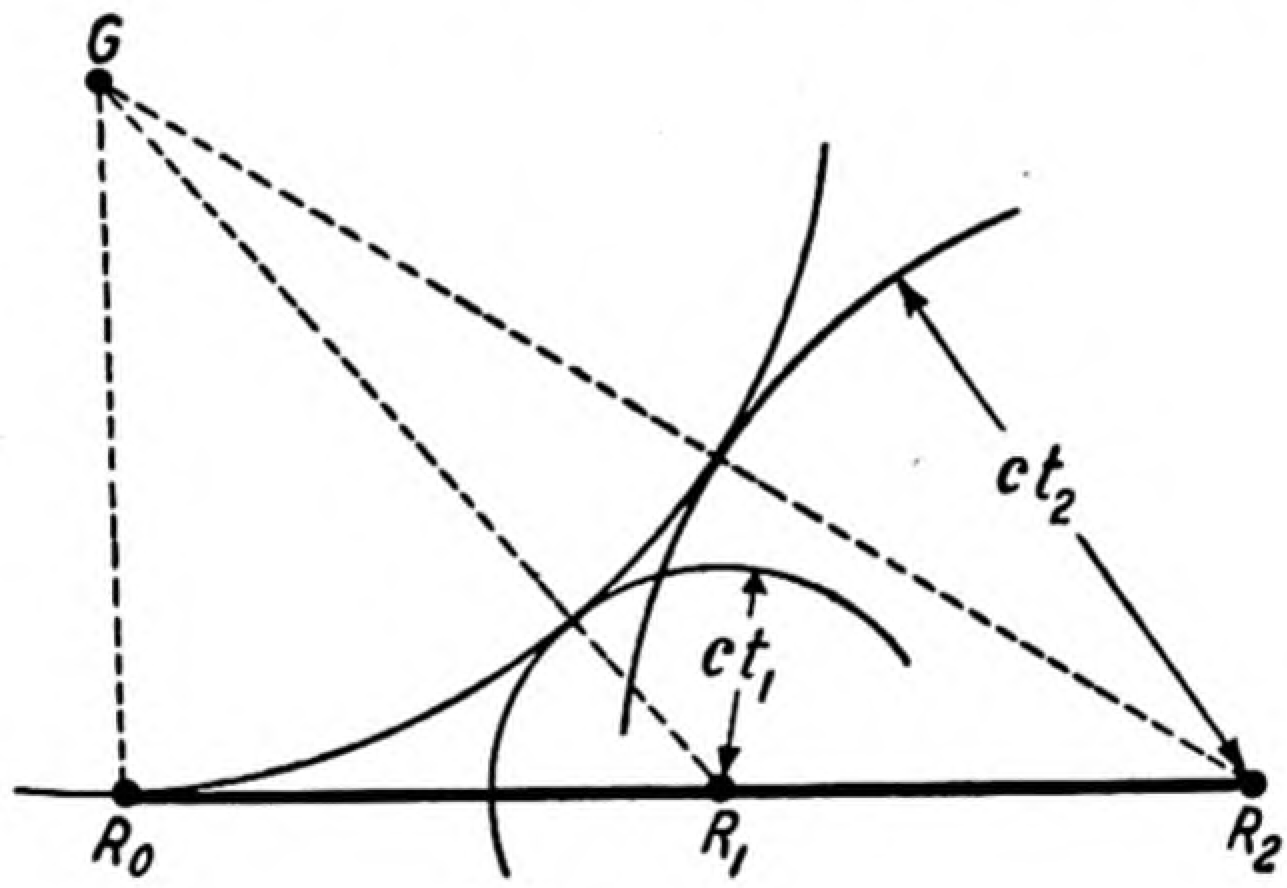


Fig. 8.16.

on a screen. Dr. Foley, of America, to mention one worker, has obtained many beautiful photographs of sound waves in air, showing their reflection and refraction at various boundary surfaces; the reader himself, during the late war, may have seen condensation pulses spreading upwards into the atmosphere from distant exploding bombs.

The photography of these sound pulses demands a special technique for, owing to their rapid speed of propagation, instantaneous exposures are necessary so that a light source of high actinic value is required, and this is usually provided by a spark discharge between magnesium wire electrodes. The general principle underlying the "shadow" method of photography due to Dvorak, is to cast a shadow on a screen of a sound pulse produced by an intense electric spark discharge. The momentary illumination required is obtained by means of a second spark, termed the "light spark," which is created at a known short interval of time after the sound. The shadow is received directly on a photographic plate or ground-glass screen, and the sparks are initiated by moving the glass plates in or out of the triggering gaps (Fig. 8.17), so producing the discharge of the condensers connected across them. These condensers are charged up to a potential difference of the order

of 100,000 volts by means of an electrical machine. The small delay time between the sound and light sparks is adjusted by varying the condenser *C*, so that by altering the time interval a succession of exposures may be obtained which will indicate the progress of the sound pulse. The diagram shows schematically a plan of the arrangement adopted by A. H. Davis, of the National Physical Laboratory,

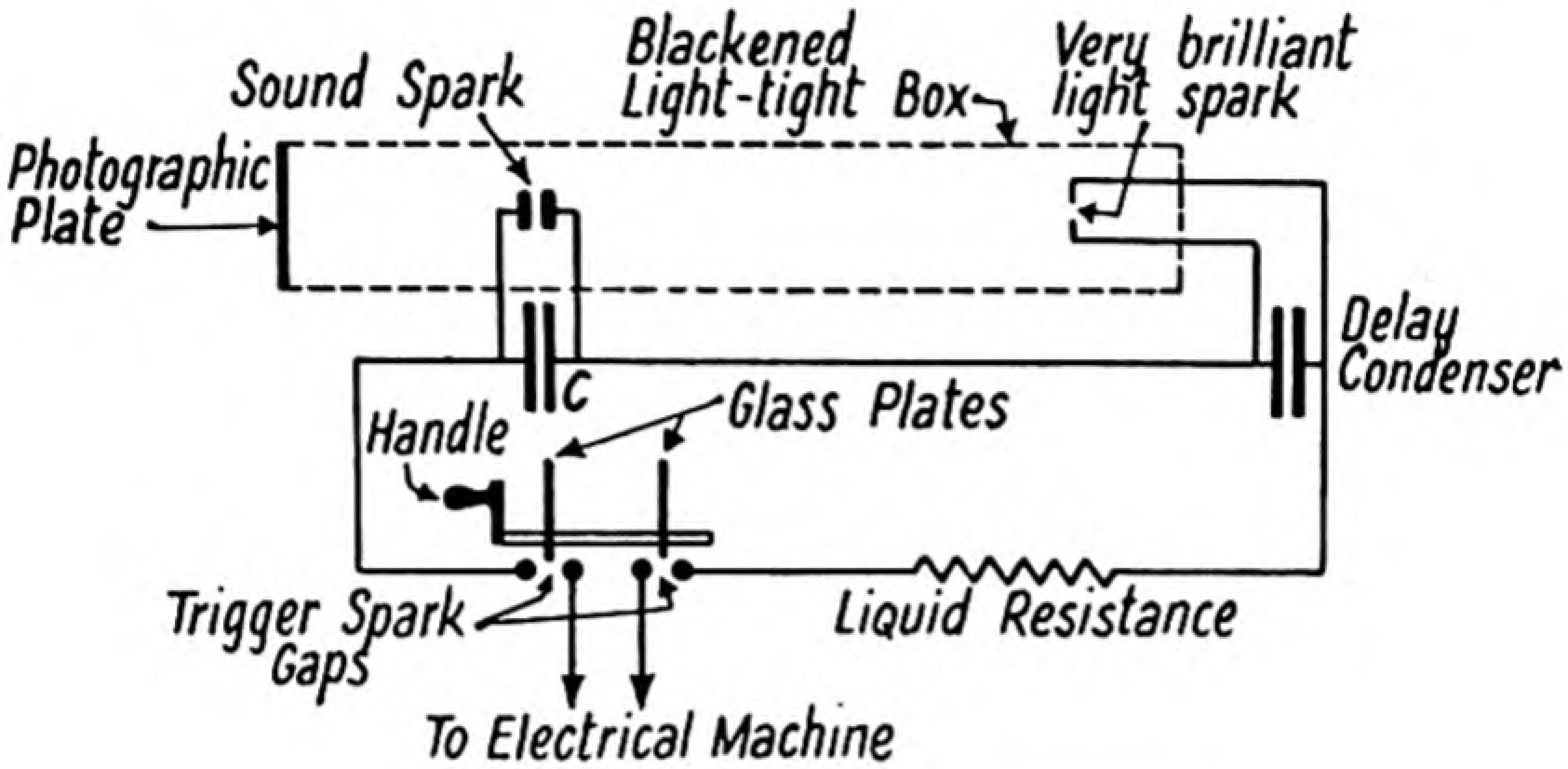


Fig. 8.17.

for studying the acoustical properties of an auditorium, a model section of the latter surrounding the sound source and being contained within the light-tight box. The distance between the light and sound spark-gaps was of the order of a few feet.

The Schlieren method due to Töpler is dependent upon the use of a concave mirror of high optical quality and is, in fact, an application of a method due to Foucault for the testing of mirrors and lenses for spherical and chromatic aberration. The light from the source *S* (Fig. 8.18) is focused, by means of the mirror *M*, on the object glass of a telescope (or camera *C*), whereas the latter is focused *on the surface of the mirror*. Any appreciable departure from uniformity of this surface will therefore be rendered very evident to an observer

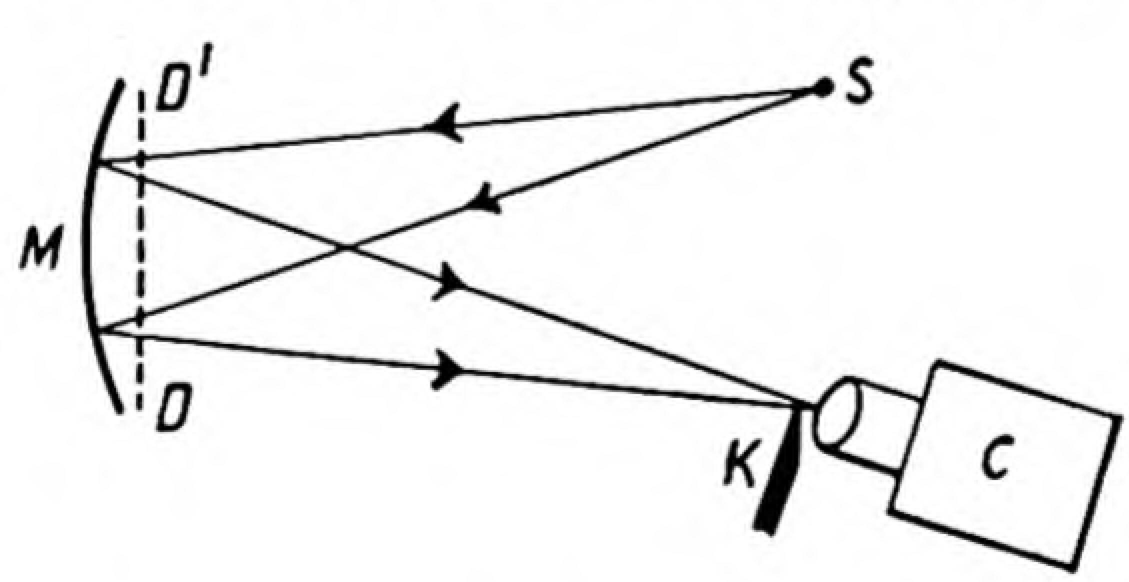


Fig. 8.18.

looking into the telescope, and might possibly mask the effect to be studied. *K* is a knife-edge which may be gradually adjusted into position until it covers approximately half of the area of the object glass. This critical position is revealed by the fact that a further small movement into the focus of the light beam will result in the mirror appearing dark when viewed through the telescope. A disturbance

passing in front of the mirror in the direction DD' will cause the rays of light to be deviated to an extent depending on the gradient of density created perpendicular to the line of the knife-edge. The Schlieren method has been used by various workers for the photography of explosive waves, etc.

A form of optical interferometer, the Mach, may be conveniently employed for quantitative measurements of gas flow phenomena in supersonic wind tunnels; the light beam traversing the tunnel "interfering," if any density change occurs, with the light beam external to the gaseous medium under observation. If the optical paths are initially unequal so that an intial fringe system is obtained, then any fringe displacement will be proportional to the density change resulting from the gas flow. An extension of the technique, due to Timbrell, projects the fringe pattern on to a slit in front of an electron multiplier cell, and the wide frequency response of the latter permits *quantitative* observations to be made on the development of shock waves within a pipe of suitable dimensions.

It is worthy of note that whereas the optical interferometer measures the total density change in the medium, the Schlieren and shadow methods measure the change in density *gradient* and in its first derivative respectively.

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CHAPTER 9

RESONANCE

If several pairs of pendulums are supported from a string as in Fig. 9.1, and are adjusted in length so that the periods of the members of a pair are identical, it is found, on displacing any one pendulum, that the second member of its pair commences to swing, but that the others remain almost undisturbed. This is an example of resonance, in which a body, vibrating at a definite frequency, causes a second body of the same frequency to move in sympathy. Other examples of resonance occur in pure and applied science. When crossing a bridge a company of soldiers is ordered to "break step" to avoid the possibility of the natural period of vibration of the bridge synchronising with the intermittent force due to the combined steps, for this might cause the bridge to collapse. Some years ago a bridge in U.S.A. was set into resonant vibration (Fig. 9.2) by the wind, which increased the amplitude of vibration until it collapsed. Wing flutter in an aeroplane is due to an external periodic force caused by turbulence in the air which synchronises with the natural period of vibration of the wing. This turbulence is caused by the passage of the wing through the air, and becomes resonant at a certain speed when fluttering commences. Such a speed is usually termed a critical speed. A radio receiver responds to a distant broadcasting station when the electrical inductance and capacity of the receiver are "tuned" to the frequency of the broadcast electromagnetic waves.

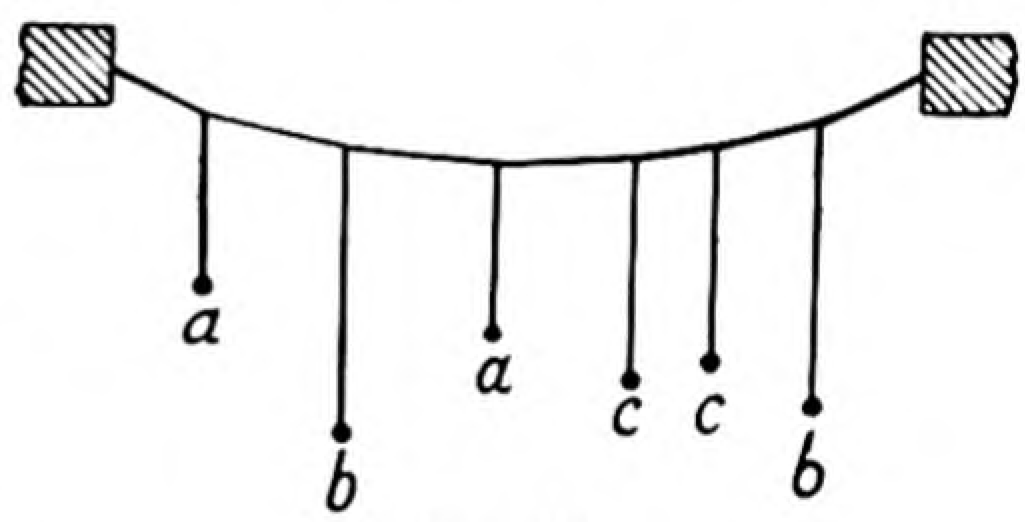


Fig. 9.1.

For the resonant effect to persist, however, the external stimulus must be continuous, for the vibrations in the receiver will be "damped out" unless maintained by a periodic force of suitable frequency and magnitude. A vibrating tuning-fork loses its energy mainly in generating sound waves, and gradually ceases to vibrate unless the loss is made good. This can be done by maintaining it electrically. A simple pendulum is a good example of a decaying S.H.M.—it dies down unless it is maintained by an extraneous source of energy. Similarly, the energy of a clock pendulum is "maintained"—it has to be started by hand, but the vibrations are maintained by impulses, given as in a child's swing, the push synchronising with the swing. The energy for maintenance comes from a spring through a train of wheels, the clockwork (or from falling weights).

It should be noted that a finite time is required to build up the resonant vibrations. Further, the work done against friction depends on the distance moved by the vibrating body. If x be the total

distance per vibration and k the average force of friction, then the frictional work done per oscillation is kx. Now if the work put in during each impulse be W, the energy available for building up the amplitude will be $W-kx$, and at resonance the flow of energy into, as well as the rate of dissipation by, the system is a maximum.

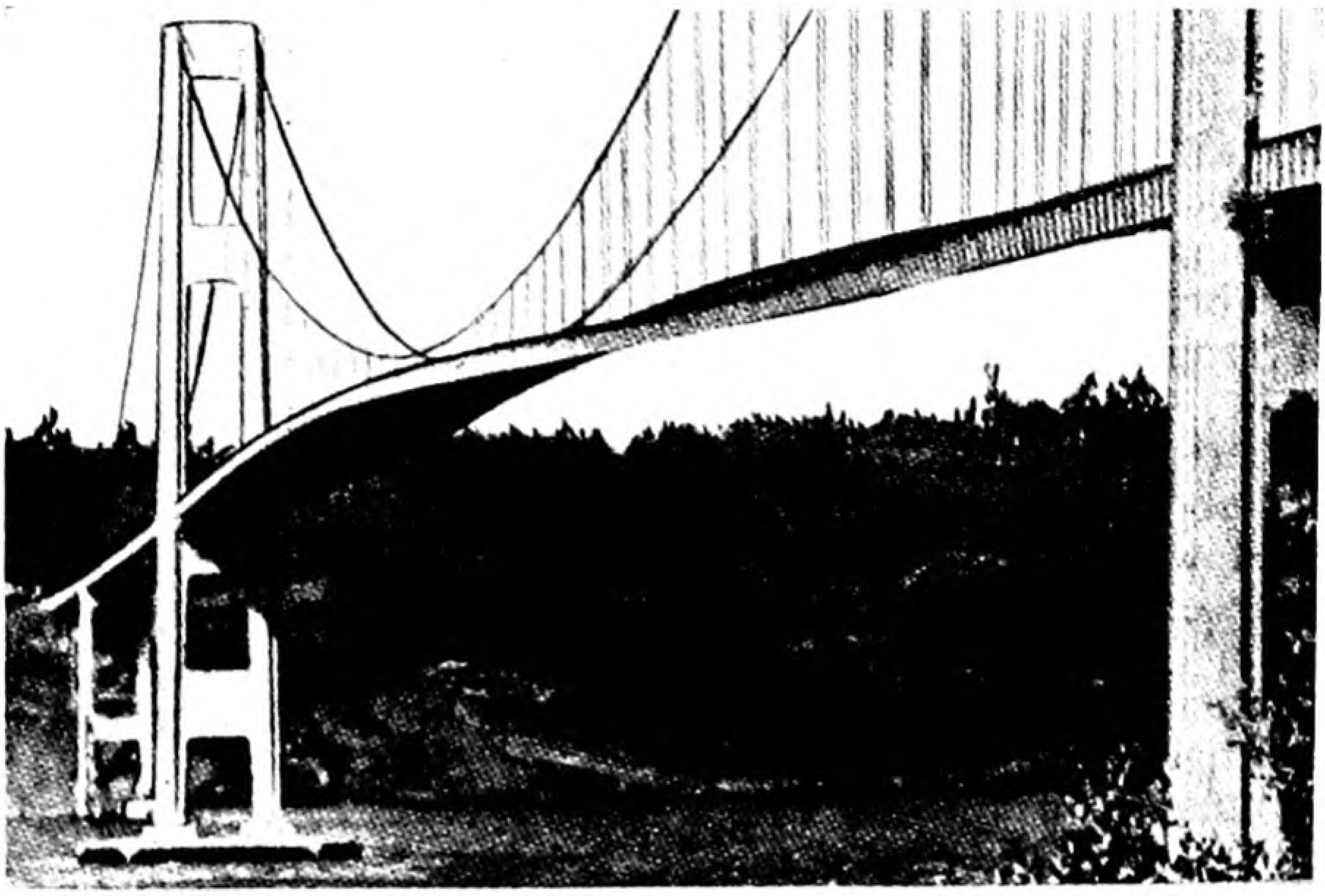


Fig. 9.2.

Roget's dancing spiral is instructive. It consists of a helical spring suspended from one end (Fig. 9.3), the other just dips into a mercury cup. Leads from an accumulator are attached to the upper end and to the mercury cup, and when the current is switched on the coils attract each other by the electromagnetic field set up. The resulting contraction of the spiral cuts off the current, the attraction ceases and the action is repeated. The natural period of vibration of the helix controls the time of passage of the current, and the current supplies the energy necessary to build up and maintain the vibration. When the wire leaves the mercury a spark occurs which wastes energy; this may be prevented by pouring a thin layer of oil on to the mercury. By increasing the depth of oil, the period is increased owing to increased viscous resistance, and a larger current is necessary. If the coil itself be immersed in another medium the operating forces will be modified appreciably.

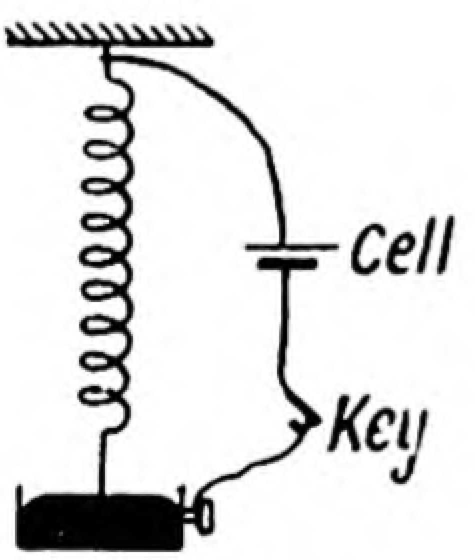


Fig. 9.3.

Pipe resonance

Pipe resonance may be explained as follows. When a source of sound, *e.g.* a tuning-fork, is vibrating over the end of a pipe, waves are set up in the air, and a succession of compressions with alternate rarefactions arrive at the mouth of the pipe, whose length has been adjusted

for resonance. Consider one compression or pulse arriving at the open end; it travels down the pipe to the stop, is reflected, and gives rise to an emergent compression at the mouth. It has been shown that such a wave travels by causing the air particles to oscillate in S.H.M. about a mean position in the direction of the sound. A particle near the mouth, then, moves to the right with the initial compression, and to the left with the reflected compression. The next compression from the source towards the right causes this particle to move to the right again. Thus in resonance the sequence of rarefactions and compressions synchronise with the vibrations of the particle, which is moving with the period of the fork. In this way the movement of the particles near the mouth are controlled jointly by the fork as the source of sound, and by the length of the pipe. At resonance these particles attain a large amplitude of vibration which results in a loud sound.

The diagrams (Fig. 9.4) indicate the movement of a particle near the mouth, in relation to that of a pendulum of the same period T as that of the fork. Actually, of course, a huge number of such particles are concerned in the motion. It is important to note that the waves in the pipe are plane, as they are moving in one direction only (*i.e.* in the direction of the axis of the pipe), and further, that the particles constituting each plane wave-front are in phase. Outside the tube the waves diverge, becoming spherical, and this change in shape indicates the reason for the end-correction, for this is the distance over which the change takes place.

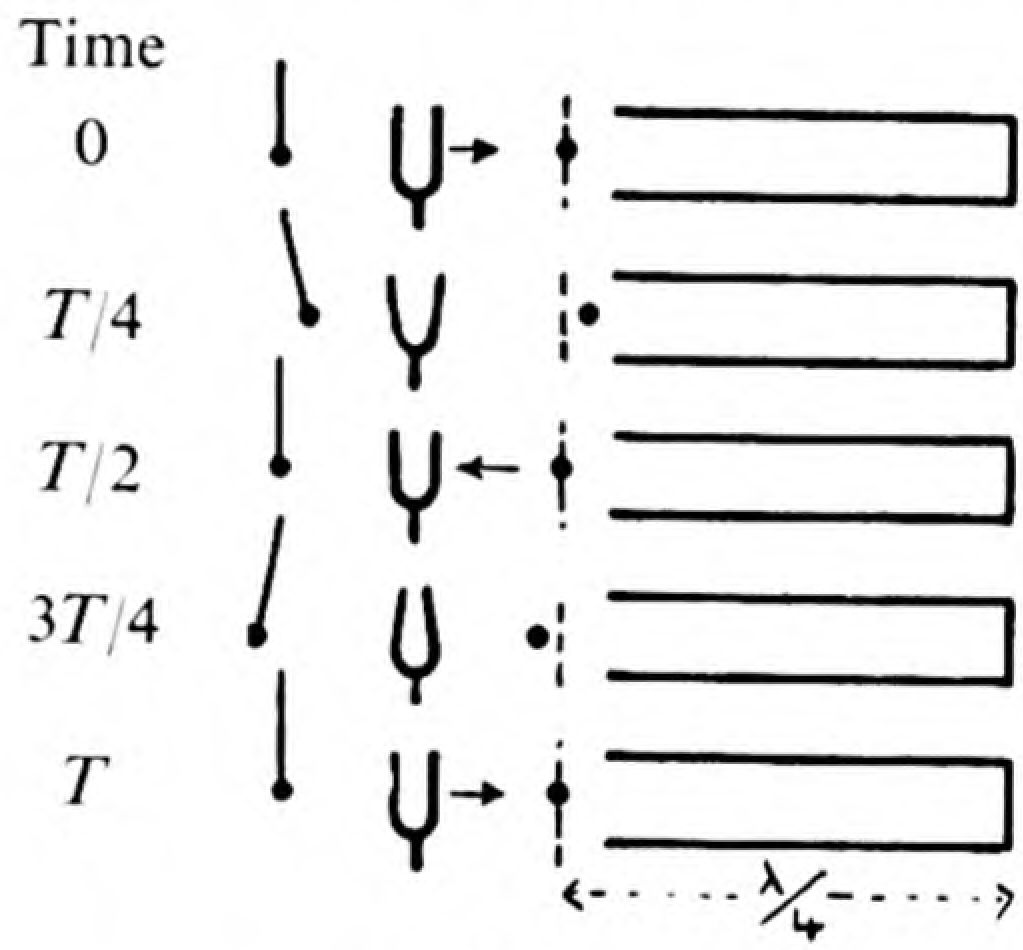


Fig. 9.4.

Within the pipe therefore the vibrations are constrained, but outside they become "free," at distances which depend on the diameter of the pipe and, to a certain extent, on the wave-length. An identical state of affairs arises in a wire carrying a current into a large block of the same metal; the actual resistance does not end in the junction. If the wire be gradually increased in diameter as it approaches the block the resistance diminishes, and becomes negligible at the junction with the block. Similarly, by gradually enlarging the diameter of a pipe the end-correction is diminished. This accounts for the bell or "flare" in musical instruments, and is mentioned again in Chapter 12.

An experiment to demonstrate pipe resonance may be performed with a spring-balance supporting a load. If the latter is displaced in an upward direction and is then released, it oscillates vertically with a definite period for a short time, finally coming to rest. If, now, a ruler assists it by giving the load a tap each time it moves upwards, the oscillations persist. In this the spring-balance corresponds to the air column, and the movement of the ruler to the compression from the fork or other source. The time taken for a complete cycle

is made evident by reference to the pendulum depicted in Fig. 9.4; the compression travels a distance equal to the theoretical length of the pipe, $\frac{\lambda}{4}$, in time $\frac{T}{4}$. Hence the speed of the compressional wave $c=\frac{\lambda}{T}$, and as $T=\frac{1}{n}$, the velocity of the sound waves $c=n\lambda$.

Take a cylindrical tube about 100 cm. in length and 3 to 4 cm. in diameter; close one end by a closely fitting plunger with a plane face (Fig. 9.5). Push the plunger along the tube until the face coincides

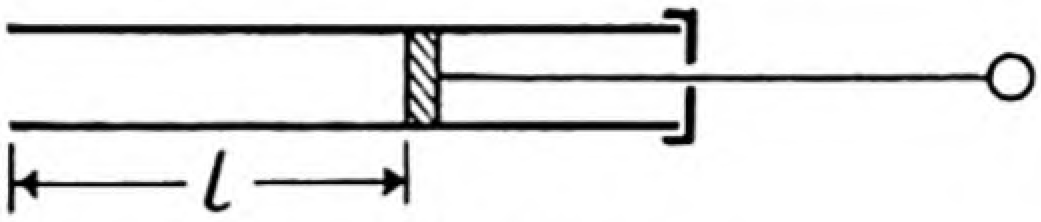


Fig. 9.5.

with the open end, mark the plunger where it enters the end plate. Next adjust the position of the plunger until the pipe resounds to a C' fork of frequency 512 c.p.s., and again mark the plunger as before. Repeat with the other forks C, D, E, F, G, A', B', in the octave 256-512 c.p.s., and obtain the resonant length of the tube for each from the marks on the plunger. Repeat the experiment to check the results, and, if satisfactory, tabulate them as in the first two rows of the table, n being the frequency.

n (c.p.s.)	256 (C)	288 (D)	320 (E)	341·3 (F)	384 (G)	426·7 (A′)	480 (B′)	512 (C′)
l cm.								
$(l+a)$ cm.								
$n(l+a)$ cm. per sec.								

Plot these results graphically with the dependent variable l as ordinate, and the independent variable $\frac{1}{n}$ as abscissa. A straight line should result which does *not* pass through the origin, but cuts the l-axis below the origin by an amount a, say. Now $c=n\lambda=4n(l+a)$, but the equation of a straight line is $y=mx+A$, where A and m are constants, and it follows in this case that $l=\frac{c}{4}\cdot\frac{1}{n}-a$, so the slope of the line is $m=\frac{c}{4}$, where c, the velocity of sound in the gas, is independent of frequency. If a be added to each value of l, the constancy of the product $(l+a).n$ may be verified by completing the third and fourth rows of the table shown above. The frequency n is measured in cycles per second, so the product is in centimetres per second, the units of velocity. Now c, the velocity of sound of frequency n, is given by $c=n\lambda$, in which λ is the wave-length. The velocity of sound in gas is known to be independent of frequency, hence λ must be proportional to $(l+a)$.

Actually $l+\alpha=\frac{\lambda}{4}$, as deduced from Fig. 9.4. It follows that $v=n\lambda=n4(l+\alpha)=4m$; *i.e.* the velocity of sound in air is four times the slope of the graph (expressed in the proper units), and may be obtained by multiplying the mean value of the results given in the fourth row of the table by four.

Further, each of the values of l are short of the appropriate quarter-wave-length by the quantity α, which is clearly independent of the frequency. It varies, however, with the radius of the tube; typical results obtained in an actual experiment are given in the table.

r, Radius of tube, cm.	2·0	3·0	4·0
α cm.	1·15	1·7	2·3
$\frac{\alpha}{r}$	·58	·57	·58

It will be seen that α is directly proportional to the radius in each case. As the length of the tube should, from elementary theory, equal one-quarter wave-length, the quantity $0{\cdot}58r$ is termed the end-correction of a tube of radius r. Bate has obtained the value $\frac{r}{\sqrt{3}}$, $[0{\cdot}577r]$, for the correction on theoretical grounds (for proof see *Phil. Mag.* **10**, 624 (1930)).

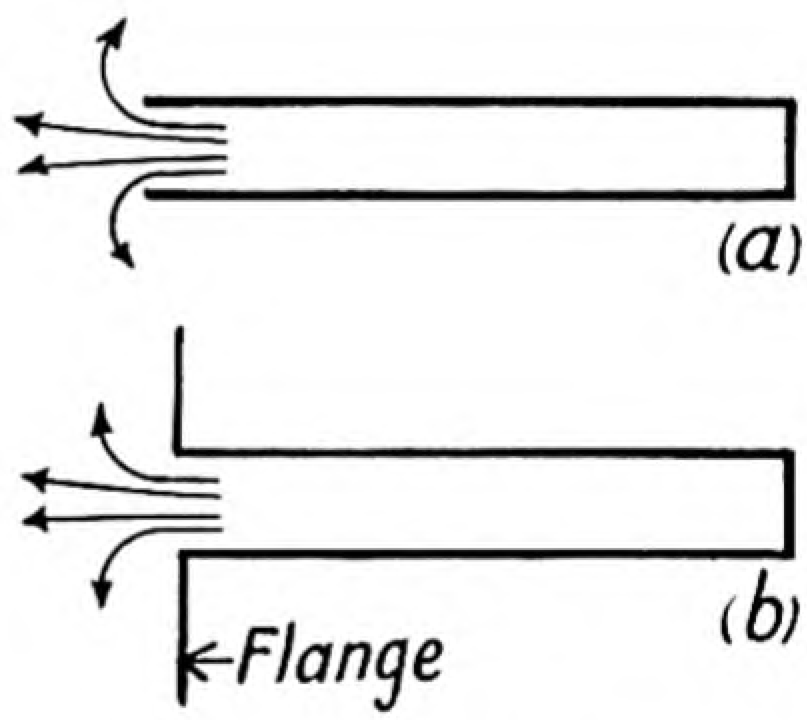


Fig. 9.6.

If the pipe terminates in an "infinite" flange or baffle, the conditions at the end are altered, as the emergent sound waves are restricted to one-half of the space which is available without the flange, as indicated in Fig. 9.6. The flange increases the end-correction by $0{\cdot}24r$ to $0{\cdot}82r$; theoretical values obtained by Helmholtz, Rayleigh, King and Bate respectively, are $\frac{\pi}{4}r=[0{\cdot}785r]$, $0{\cdot}824r$, $0{\cdot}821r$, $r\sqrt{\frac{2}{3}}=[0{\cdot}816r]$.

By increasing the length of the air column l (Fig. 9.5), a second series of positions is obtained at which resonance occurs with each of the forks used in that experiment. On plotting the resulting lengths against the appropriate reciprocal frequency, a straight line graph is obtained which intersects the l-axis in the same point as in the former graph, showing that the end-correction is the same as before.

The displacement curve indicates what is happening in the pipe. At the instant represented by the full line the particles at the enclosed anti-node are moving left, and at the open anti-node, to the right, leaving the particles at the central node diminished in number, but the node itself is not displaced. After half a period the dotted curve

represents the wave formation, when the conditions at nodes and anti-nodes are reversed. The actual displacements of normally equidistant particles at the first instant considered are shown in Fig. 9.7*d*, and Fig. 9.7*e* depicts the displacements half a period later. The nodes are

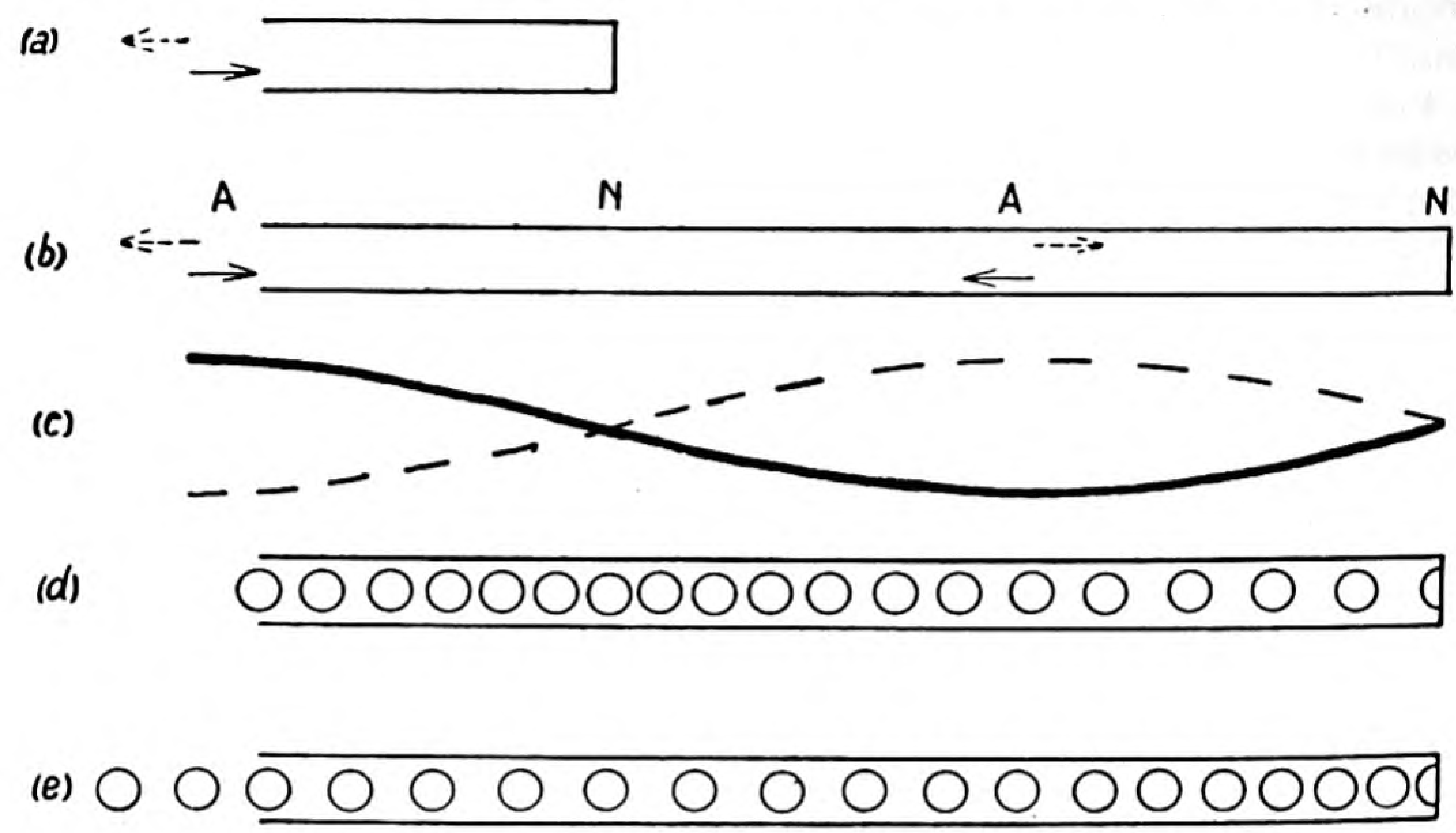


Fig. 9.7.

positions of maximum compression and rarefaction and of zero displacement, and the anti-nodes are positions of maximum particle displacement, but of minimum compression and rarefaction.

In a pipe open at one or both ends it is possible to have various arrangements of nodes and anti-nodes, and these can be constructed in a diagram by indicating an anti-node at an open end, a node at a closed end, and drawing nodes and anti-nodes alternately. The tubes represented in Fig. 9.8 have the same nodal spacing when the resonant frequencies are the same, but when resounding to the fundamental the frequency of the open pipe is $\frac{3}{2}$ that of the shorter.

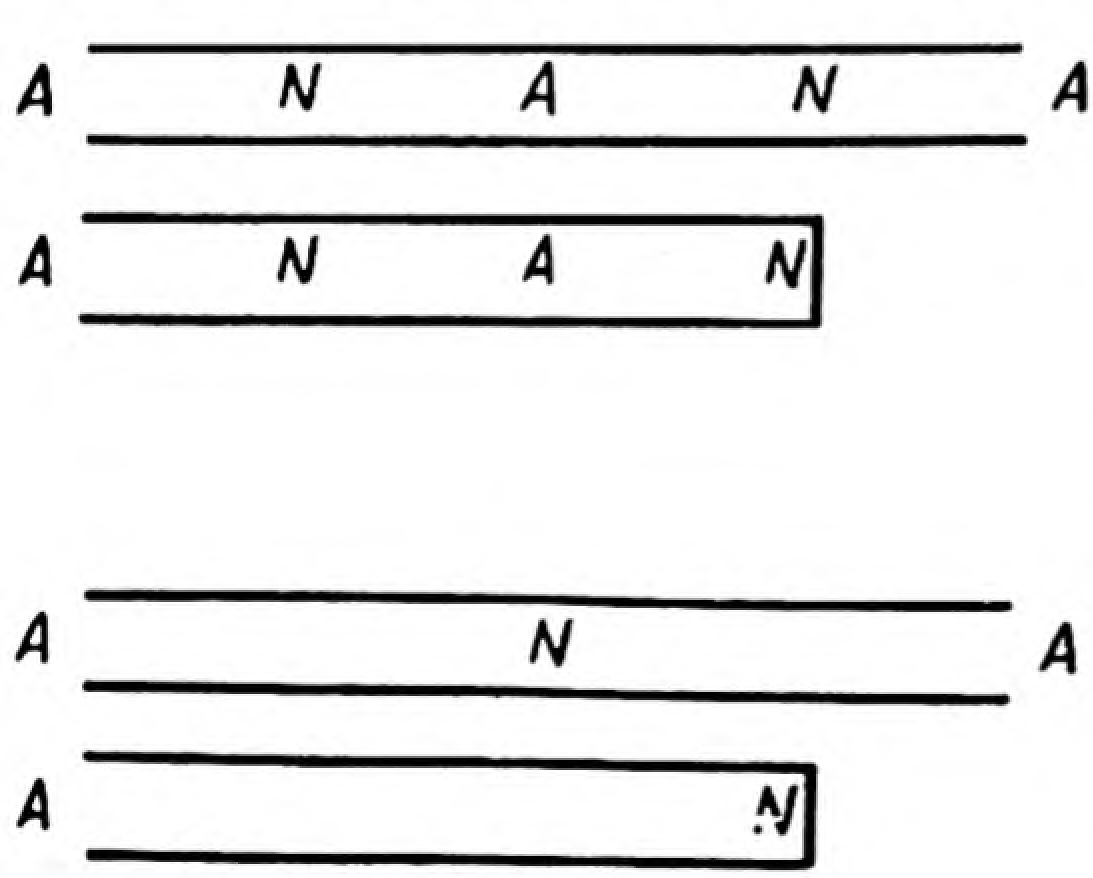


Fig. 9.8.

In the foregoing, temperature has been assumed constant, for the velocity of sound in a gas varies as the square root of its absolute temperature.

E. G. Richardson has found that there is a "stagnant" layer of air about 1 mm. in thickness next the wall of the tube when the remainder of the air is vibrating, so that the effective diameter of the air column is slightly less than that of the tube. This layer, which varies in thickness with the physical conditions, is of little consequence in organ

pipes, but for pipes of less than about 1 cm. diameter it becomes considerable, and in tubes of about 1 mm. radius and less the vibrations are soon damped out. The effect of (*a*) carpets in an empty house, (*b*) the presence of an audience—or rather their clothes—in a hall in reducing echoes is due to the absorption of sound vibrations by the pores of the materials, viscosity being a contributory factor. The diameter of a pipe affects the velocity of sound along that pipe, and is discussed later.

Cavity resonance

Take a bottle and add water to it until the air enclosed responds to a tuning-fork of known frequency. Tilt the bottle, thereby altering the shape, but not the volume, of the enclosed air, and test for resonance; as before, the air responds to the fork. Next, partly cover the mouth with a piece of glass or other hard substance—an iris diaphragm is suitable—and again test for resonance. The resonant frequency is diminished because the vibrations at the mouth are impeded by the partial closure. The original value is restored by adding to the water in the bottle, which decreases the volume of the air within.

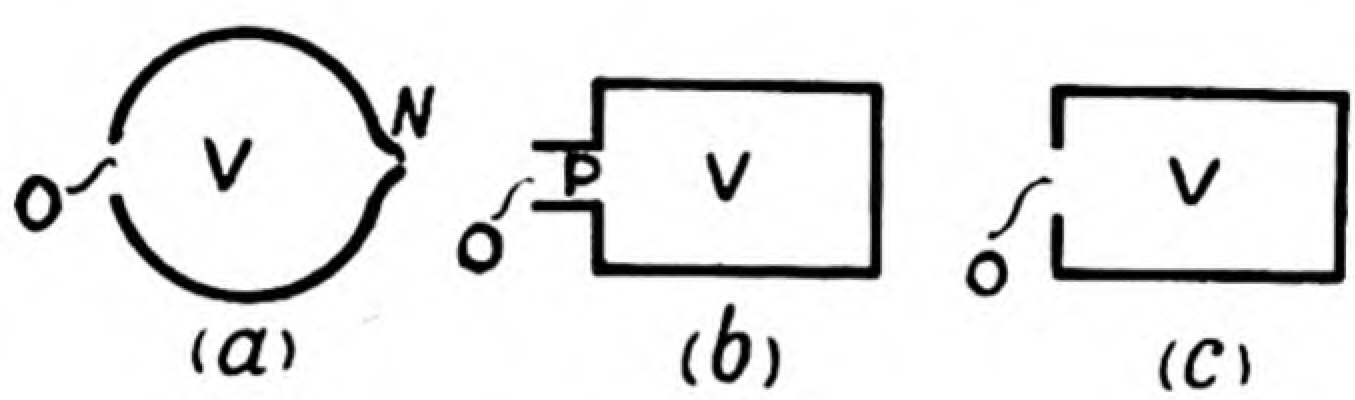


Fig. 9.9.

The experiments show that the natural frequency of vibration of a cavity (*a*) increases with the area of the mouth, (*b*) decreases with increase of volume, (*c*) does not depend on the shape.

Resonant cavities are useful for sound analysis, and special types are constructed to respond to definite frequencies. They are termed resonators, and are usually in one of the forms shown in Fig. 9.9.

Type *a* is fitted with a nipple *N* to place in the ear. Faint sounds of a particular frequency can be heard in this way to the exclusion of others, as the response is very critical. This form of resonator is usually named after Helmholtz who constructed a number of hollow brass spheres, of different sizes, which he used in his notable pioneer work on the analysis of complex sounds. A fixed volume resonator with a neck is shown as type *b*, but the volume can be made adjustable by fitting a plunger as in Fig. 9.11. This cylindrical form of resonator is often used in conjunction with tuning-forks. Type *c* is similar to *b*, but is without a neck.

The resonator formula. Consider a resonator of volume V fitted with a short neck of *effective* length l, and sectional area a. The air in the neck is assumed to act as a plug P, which moves to and fro under the influence of the external stimulus and of the resulting pressure variation within the volume of the resonator. The "effective mass"

of the plug is assumed to be ρal, where ρ is the average density of the air in the neck.

In the displaced position of the plug shown in right-hand side of Fig. 9.10, the volume of air to the right of the plug is $V+ax$, and the

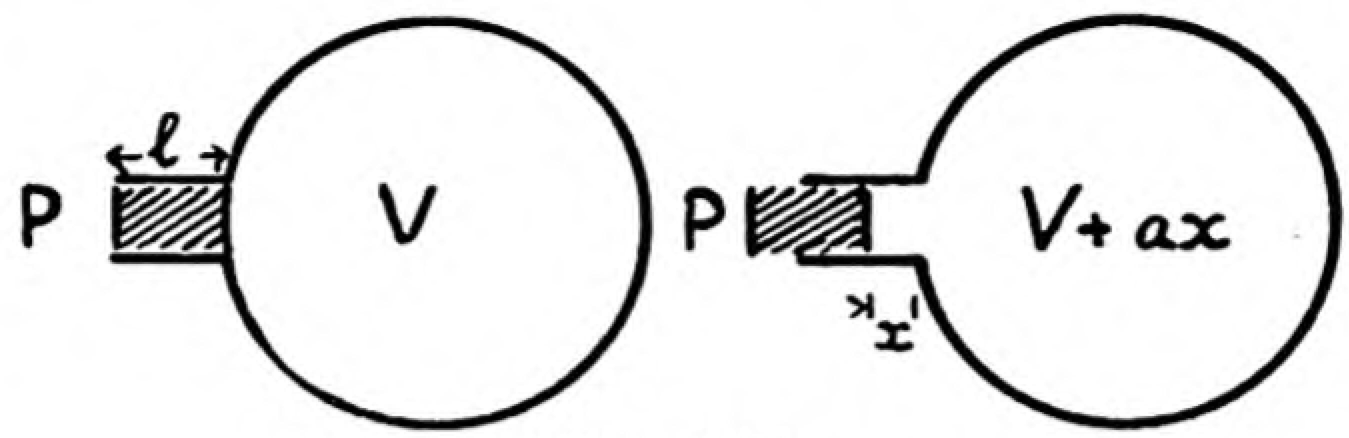


Fig. 9.10.

pressure is reduced by an amount δp to $p-\delta p$, the original value being p. As the change is adiabatic, the relationship between pressure and volume is

$$pV^{\gamma}=(p-\delta p)(V+ax)^{\gamma},$$

whence
$$\delta p.V^{\gamma}=\gamma p.V^{\gamma-1}.ax+\frac{\gamma.\gamma-1}{1.2}pV^{\gamma-2}a^2x^2+\ldots$$
$$-\gamma.\delta p.V^{\gamma-1}.ax+\ldots.$$

The terms on the right except the first are very small, and are ignored, and so

$$\delta p=\frac{\gamma p.ax}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The quantity δp is the pressure difference acting on the "plug," so the actual restoring force$=a.\delta p$, and this equals $m(-\ddot{x})$, where m is the "effective" mass of the plug of air, and hence

$$\delta p=-l\rho\ddot{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2),
$$-l\rho\ddot{x}=\frac{\gamma pa}{V}.x,$$

which is characteristic of S.H.M.

$$\therefore \; n=\frac{c}{2\pi}\sqrt{\frac{a}{l}\cdot\frac{1}{V}}, \text{ since } c^2=\frac{\gamma p}{\rho} \quad . \quad . \quad (3)$$

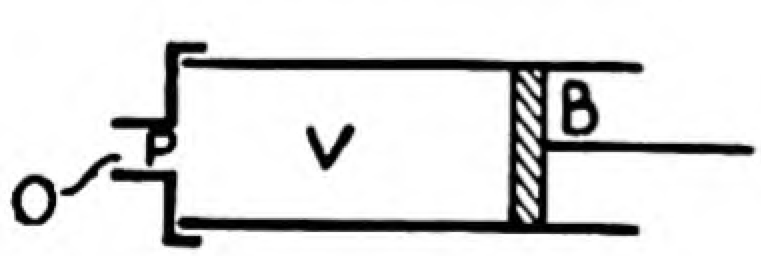


Fig. 9.11.

The term $\frac{a}{l}$ is known as the conductivity of the neck. Actually the term is not so simple as it appears, for it is complicated by end-corrections for l, and no satisfactory theory has yet appeared. The formula does, however, show that n^2V is constant for a given resonator, and this may be tested by means of a pipe closed at one end with a plunger and at the other with a cap as in Fig. 9.11. The neck should be short in comparison with the wave-length, as a long neck will constitute a pipe, and would complicate the problem. The cap with the neck is fitted on to the cylinder and the plunger adjusted until the resonator responds to a fork of frequency 512 c.p.s. Note the position

of the plunger, and then repeat with forks of lower frequencies down to 256 c.p.s. Estimate the volumes of air enclosed and tabulate:—

n	512	480	. . .	256	c.p.s.
n^2					
V'					c.c.

Plot V' as ordinate against $\frac{1}{n^2}$; a straight line graph as in Fig. 9.12 should result, which does not pass through the origin. This apparent discrepancy is due to the volume V' being made up of the volume of the resonating portion of the air in the bottle, plus that of the neck, so that the intercept ON on the V'-axis is to be regarded as the effective volume of the neck C, say. Thus the volume of the resonator

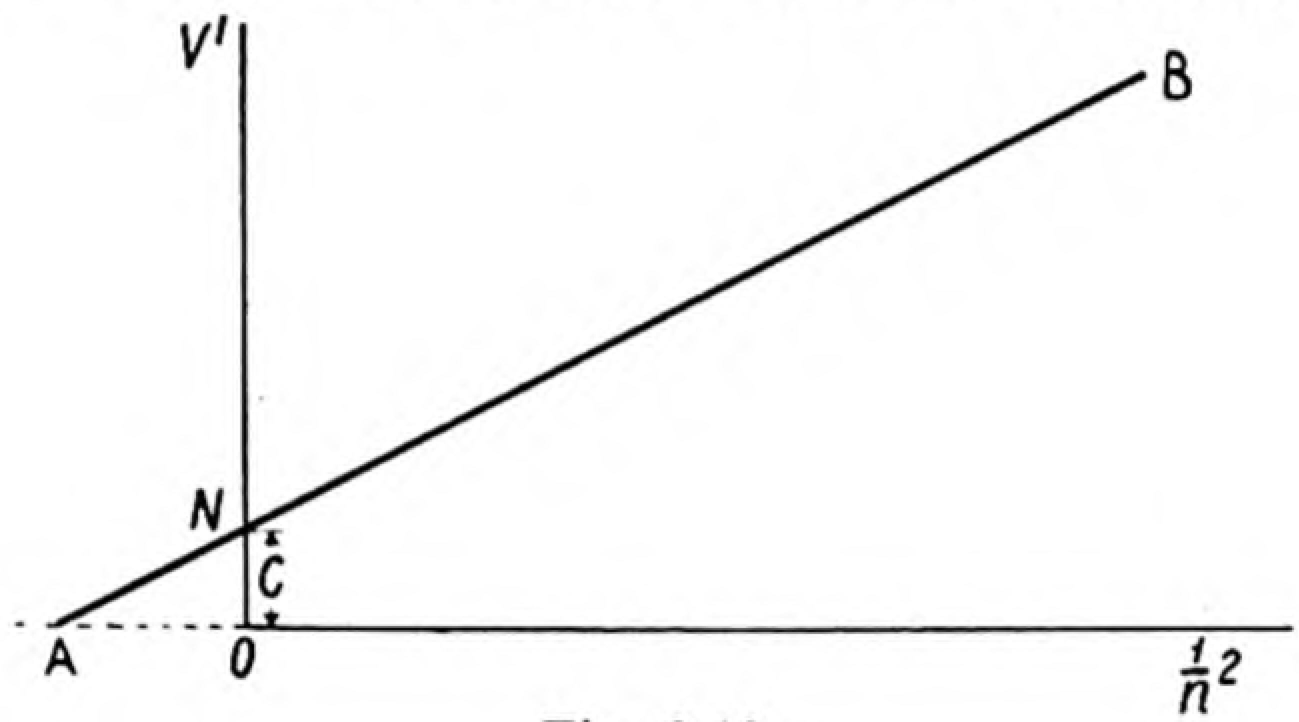


Fig. 9.12.

V equals $V'-C$. The equation of the straight line ANB, from $y=mx+D$, where D is a constant, is

$$V'=m.\frac{1}{n^2}+C,$$

i.e.
$$V=V'-C=m.\frac{1}{n^2};$$

$\therefore$ $n^2V=m$, the slope of the graph. n^2V, however, equals $\frac{c^2}{4\pi^2}\cdot\frac{a}{l}$, and this may thus be evaluated from the slope of the graph. Substitute values for v and a to find l, and hence estimate the end-correction for the neck.

To adapt the formula for a neckless resonator, it is rewritten as

$$n=\frac{c}{2\pi}\sqrt{\frac{a}{l'+\beta}\cdot\frac{1}{V}},$$

in which l' is the measured length of the neck, and β is its end-correction. If the resonator is neckless, l' disappears, leaving

$$n=\frac{c}{2\pi}\sqrt{\frac{a}{\beta}\cdot\frac{1}{V}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To evaluate β, refer to Fig. 9.13, which represents the orifice, each side of which may be regarded as an infinite flange, so that the sound waves diverge and converge on each side alternately, and this indicates an equivalent length of neck equal to twice the end-correction of a

flanged pipe of radius r when the pipe is large. Taking $\beta=2\left(\frac{\pi}{4}.r\right)$, the Helmholtz value, eqn. (4), may be written:

$$n=\frac{c}{2\pi}\sqrt{\frac{\pi r^2}{\frac{\pi}{2}.r}\cdot\frac{1}{V}}=\frac{c}{2\pi}\sqrt{\frac{2r}{V}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Comparing this with (3), the conductivity (or acceptance) of a circular orifice in an infinite flange of negligible thickness equals its diameter. The conductivity of an orifice is, then, given by its area $\div$ its end-correction. If a cylindrical resonator without a neck has the same diameter as its orifice, there is only one correction, and if fitted with a flange, this is $\frac{\pi}{4}.r$, giving its conductivity as $\frac{\pi r^2}{\frac{\pi}{4}.r}=4r$. The limiting conductivities of a cylindrical resonator of diameter D with an orifice

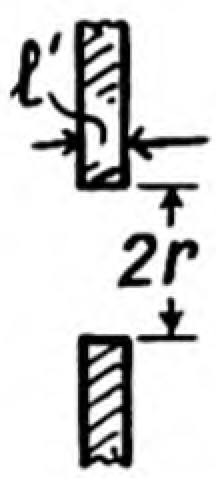


Fig. 9.13.

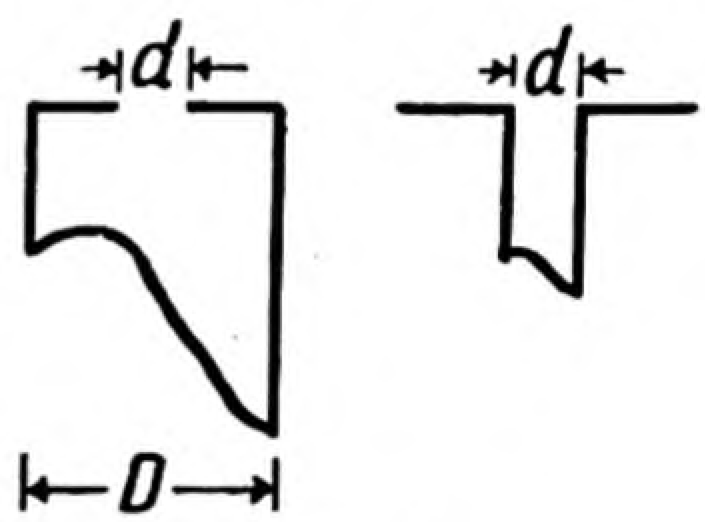


Fig. 9.14.

of diameter d are $2d$ when $D=d$, and d when $\frac{D}{d}\rightarrow\infty$. An empirical formula* embracing these limits put forward by Bate is

$$\sigma=2^{\frac{d}{D}}.d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

σ being the conductivity.

Repeat the previous experiment, but instead of the cap with the neck, replace it by one with a circular orifice. Perform the experiment several times, using different orifices ranging from 1 to 5 cm. diameter. Arrange the results as in the table:

Diameter of Orifice (cm.)	1	2	3	4	5	Open	n c.p.s.
Length of the air column (cm.)							256
							288
							320
							340
							384
							426·7
							450
							480
							512

* Recently justified theoretically for intermediate values by Nielsen.

On the same axes plot curves showing the relationship between the length of the air column (which is proportional to volume) and $\frac{1}{n^2}$. From the results estimate the diameter at which the resonator formula becomes inapplicable, when the arrangement is to be treated as a pipe. Select results which may be regarded as applying to the resonator, and substitute values of σ obtained from (6), in $n=\frac{c}{2\pi}\sqrt{\frac{\sigma}{V}}$. Compare the results with the actual values of n.

Damping of a resonator. The sharpness with which the pipe resonator of Fig. 9.5 may be tuned to a given frequency will depend on the nature of the surface of the plunger. With a heavy metallic plunger the resonance setting will be sharp, but for a surface in the nature of felt the response will be less and the tuning broader. These facts are illustrated by Fig. 13.9, in which the "*small damping*" *curve* corresponds to the metal plunger and $\omega/2\pi$ is the resonant frequency.

Normal modes of vibration of air in a spherical cavity having hard rigid walls.* The air in immediate contact with a smooth rigid wall can move only parallel to the surface, and consequently when the motion of the air within a spherical resonator is entirely radial, the boundary must be a nodal surface. In addition, for the fundamental radial mode, there will be a nodal point at the centre of the sphere, so that the complete motion will consist during alternate half-periods, of air motion respectively to and away from the centre. But the pressure change at the centre, as to be expected, is greater than that at the walls (actually it is nearly five times as large), and the radius of the anti-nodal surface, *i.e.* region of no pressure change, is approximately 0·7 of that of the cavity. In the next radial mode a second nodal surface appears at a radius 0·58 of that of the cavity, the air motion in one half-cycle being now towards this surface, both from the wall and the centre, which direction is reversed in the next half-period.

Besides the radial modes the air may vibrate *to and fro* with *nodal points*, at N_1 and N_2 say, at the ends of a diameter. The plane through the centre perpendicular to N_1N_2 will be the anti-nodal surface, and the corresponding frequency is nearly an octave lower than the fundamental radial mode. In the second diametral mode this perpendicular plane becomes a nodal surface, N_1 and N_2 being still nodal points. The similarity of the problem of the electrical cavity resonator to that of the acoustical one should be noted here. In the diametral acoustic mode the surfaces of equal pressure will correspond to the equipotential surfaces of the electric field (at a time when the magnetic field is zero) within a spherical conducting shell vibrating in a transverse magnetic (E waves) wave mode.

* See Lamb, H.: "The Dynamical Theory of Sound." Arnold, 1925.

CHAPTER 10

PHYSIOLOGICAL ACOUSTICS

The ear

The human ear is essentially the ultimate judge of the quality and intensity of music and sound, and the nature of its final assessment must depend largely upon the personal characteristics of the ears of the particular individual concerned. Any physical measurements made with acoustical apparatus allow only quantitative *comparison* between the various units involved, and in order to obtain some measure of the physiological effects it is necessary to acquire some knowledge of the characteristics of the human ear. The mechanism of the ear, by which the aerial vibrations become transformed into nerve currents which stimulate the brain, involves the application of a number of physical principles.

The ear is comprised of three main compartments, known respectively as the outer, the middle, and the inner ears, shown diagrammatically, but not to scale, in Fig. 10.1. As will be shown subsequently, the sound vibrations are propagated in turn through the *air* of the outer ear, the *solid* bone of the middle ear, and finally the *liquid* of the inner ear. The outer ear consists of a tube about one inch long, called the meatus, which is closed at the lower end by a membrane, the ear-drum, and terminates at its open end in a visible concave surface called the concha (from the Latin word meaning a shell). Now it should be noted that a concave surface will act in the manner of a light-reflecting mirror only if its dimensions are comparable with the wave-length of the incident radiation. It follows, therefore, that the action of the concha towards sound waves of, at least, medium audio-frequency will be, not to reflect, but to *scatter* some of the waves, so that they enter the meatus. Additional acoustic energy will reach the ear-drum by virtue of the fact that the meatus will act as a resonator to certain frequencies, an effect which may be shown to exist by placing a "cupped" hand around one ear, and if it is actually in contact with the head the sound, say of a ticking clock, will be heard as if considerably enhanced.

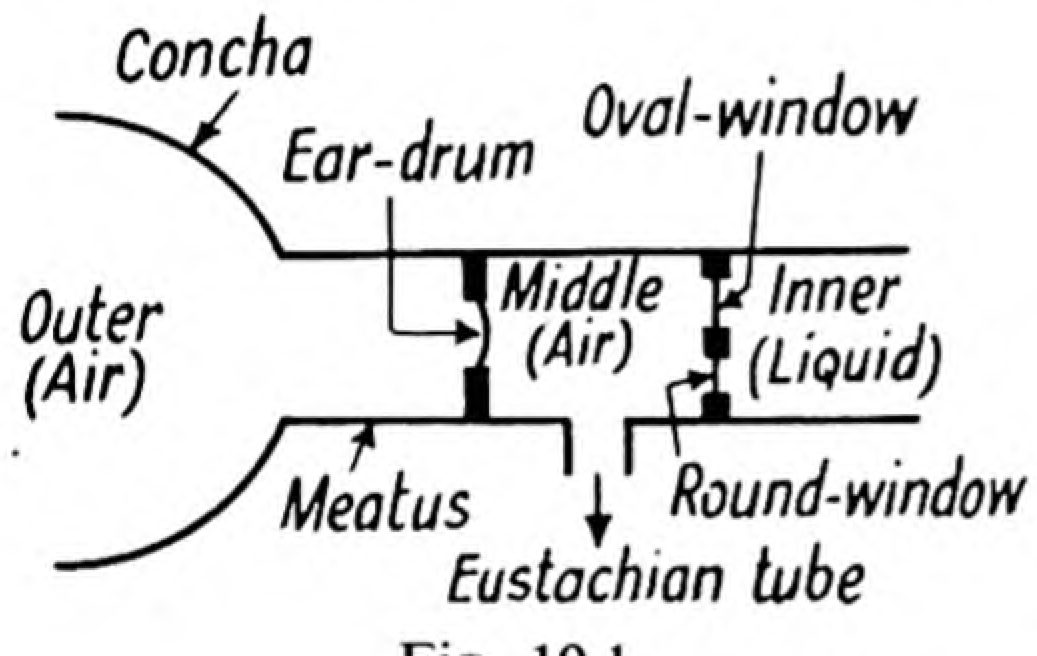


Fig. 10.1.

Assuming the ear-drum to have been set into vibration by the incident sound waves, the question arises of the transfer of this sound energy to the *inner* ear, which contains a liquid medium, with as small a loss of energy as possible. Now the most efficient transfer of energy from one medium to another of different density involves the problem of matching impedances, a problem common to all types of energy

transfer. In this case the desired coupling between the collector, the outer ear, and the analyser, the inner ear, is performed by means of three very small bones called ossicles. These bones, known respectively as the hammer, the anvil and the stirrup (Fig. 10.2) are situated within the small volume of the middle ear which communicates to the rear of the nasal cavity by the Eustachian tube (named after its discoverer, an Italian anatomist, Eustachio). This tube is about one and a half inches long, and its purpose is to enable the pressure of the air within the middle ear to be rapidly adjusted to any appreciable changes of pressure in the outer ear, such as those which are experienced when rapidly descending or ascending a deep lift shaft. By the provision of this means of equalising the air pressure on both sides of the ear-drum the latter is protected against possible disruption, although free vibration is in no way restricted. The Eustachian tube also serves to drain into the mouth any liquid which may collect in the middle ear, but it is only open for this purpose during the act of swallowing. The buzzing noises heard in the head during an attack of catarrh are due to the presence of mucus in the *middle* ear and the consequent viscous drag on the movements of the ossicles.

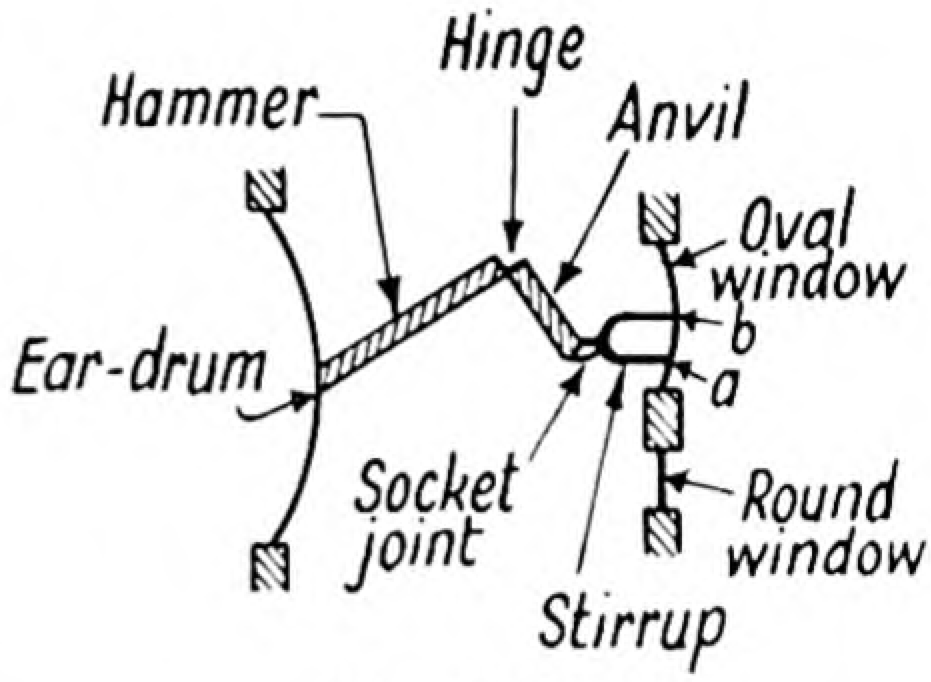


Fig. 10.2.

The hammer bone is attached to the ear-drum, so that the latter is slightly "flexed" inwards, even in quiescent periods, and so the drum is essentially an asymmetrical vibrator (Appendix 8). As a result of the lever action of the ossicles, the force exerted by the stirrup bone on the membrane set in the wall separating inner and middle ears, is about three times that exerted on the ear-drum, *i.e.* referring to Fig. 10.3, $\frac{F_1}{F_2}=\frac{1}{3}$ where P represents a hypothetical pivot point. Furthermore, the area of the oval window A_2 is much smaller than A_1 of the drum, and hence the pressure, *i.e.* force per unit area, exerted on the *inner* ear is much greater than that produced on the ear-drum by the factor $\frac{F_2}{F_1}\times\frac{A_1}{A_2}$. The value of this ratio may lie between 50 and 60. By analogy with the electrical case, therefore, the middle ear acts as a *step-up acoustic transformer*, the small pressure variations of the incident sound waves being stepped up by 50 or 60 times, and at the same time the low acoustic impedance of the air is effectively matched to the high impedance of the liquid of the inner ear. It should be noted that the stirrup bone will execute a rocking movement about its lower end a as this is attached to a point near the edge of the

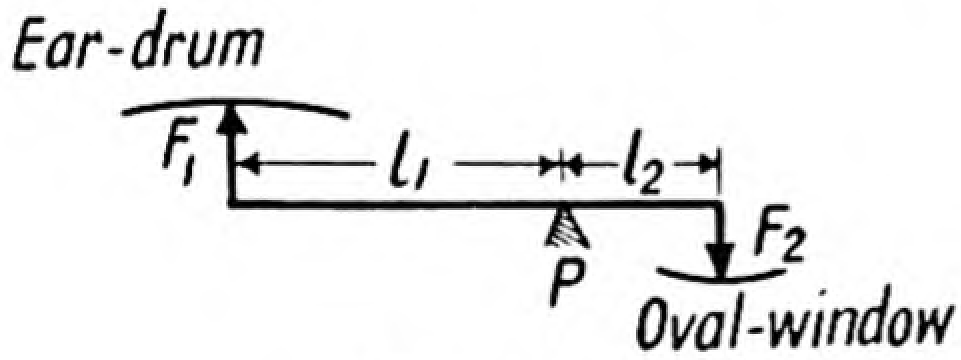


Fig. 10.3.

oval window, while the upper end *b* is located near to the centre of this window.

A complicated set of small hollow tubes and chambers imbedded in solid bone constitutes the inner ear, and is referred to as the *labyrinth*. It serves three distinct functions, and the part primarily concerned with the function of hearing is a tube coiled after the fashion of the shell of a snail, and known as the cochlea. This spiral cavity is the most vital part of the organ of hearing and if uncoiled is found to be approximately 30 mm. long, and increases in diameter uniformly along its length to a maximum of 5 or 6 mm.; it is represented uncurled and diagrammatically in Fig. 10.4, together with the basilar membrane which divides the inner ear into two separate portions, except for a small gap at the end known as the helicotrema. When an incident sound wave causes the stirrup to exert a pressure on the liquid in the inner ear, the existence of the helicotrema enables this pressure to be relieved by a flow of liquid through the gap and the consequent *outward* movement of the round window. The basilar membrane varies in width from nearly 0·2 mm. at the window end to 0·5 mm. at the helicotrema end of the cochlea, and is stiffer along its breadth than along its length. It is covered transversely with a multitude of fibres, rather suggestive of a multi-stringed harp. These fibres are connected to nerves leading to the brain, and each fibre is maintained in tension, and its length corresponds to the width of the membrane at that point. It is reasonable, therefore, to presume that each part of the cochlea will respond to a different frequency, and only those fibres having the same frequency of vibration as the incident sound wave will be excited and so transmit nerve impulses of that frequency to the brain, which thus becomes conscious of pitch.

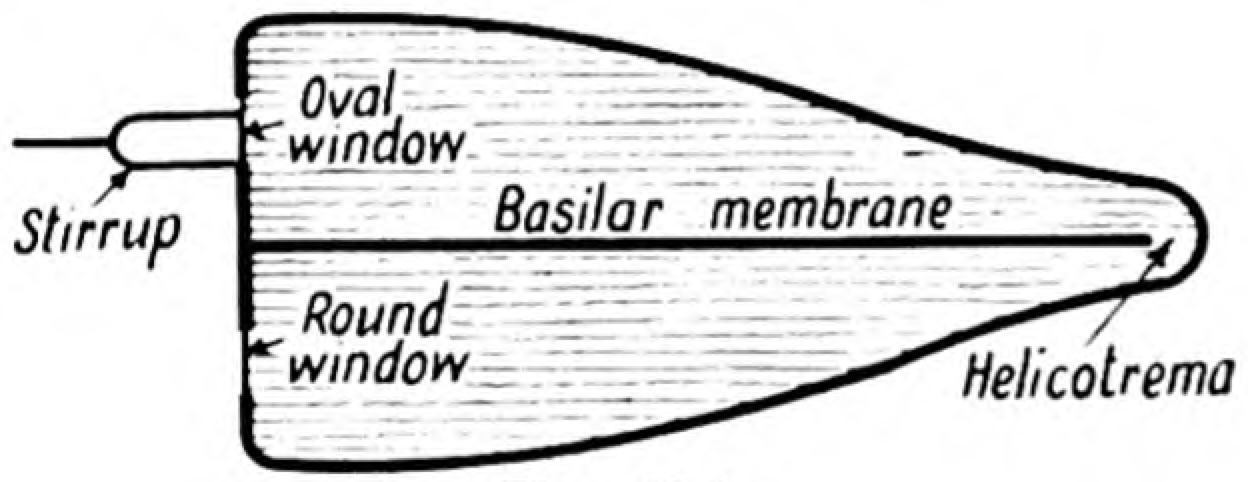


Fig. 10.4.

Hearing

This *resonance theory of hearing*, which assumes uniform tension in all fibres, is chiefly due to Helmholtz; it has been criticised on account of the fact that it does not permit of so large a range of frequency response as the human ear is known to possess. This objection has been partly met by a slight modification of the theory, in which the tension of the longer fibres is now assumed to be less than that of the shorter fibres; it then follows from eqn. (1), p. 45, that the frequency range will in consequence be extended. In support of the theory it has been noted, although disputed by some observers, that workmen who are continuously subjected to loud noises of a particular type, often become deafened to these sounds as if certain fibres had become overstrained. This aural effect has been reproduced in experiments with the ears of guinea pigs, using sounds of known frequencies, and the observed *positions* of comparative inactivity on the membrane were in accordance

with theoretical predictions. The human eye also shows a temporary fatigue effect of this nature, for if after viewing, for example, a red screen for an appreciable time the eye is turned towards a white screen, the latter will appear as the complementary colour green. As an alternative to the resonance theory, it has been suggested that the basilar membrane vibrates as a whole, like a telephone diaphragm, giving rise to nerve currents of the frequency of the incident sound. In this way the brain was supposed to be able to distinguish between the pitches of different notes, but later research has shown that the nerve fibres conduct currents, not as in electrical circuits, but rather as separate impulses. Since it has been found that the rate of these impulses cannot possibly exceed 1000 per sec., it means that this theory fails to account for the perception of frequencies in the audio-range from 1000 c.p.s. to 20,000 c.p.s.

As yet there is no precise explanation of how the brain interprets the information conveyed by the auditory nerve from the inner ear. Although this information may be very complex, as, for instance, when listening to the variety of noises in a city street, yet the brain is able to recognise the separate components of the babel of sound. That the resultant motion of the air at the ear-drum in such a case must be very complicated, is indicated in Chapter 2, and it can also be made evident by a microscopic examination of the groove of a gramophone record or film sound-track, which in itself is a representation of the complex sound wave incident upon the diaphragm of the instrument recording the sound. A "trained" ear can easily detect which member of an orchestra is not playing at any instant. It should be noted that in the above respect the ear is a more efficient instrument than the eye, for if two colours, say blue and yellow, enter the eye, only one colour, in this case green, is perceived.

There are two fundamental qualities of hearing which are important, namely the ability to detect a small change in sound intensity under specified conditions of loudness, pitch and background noise, and secondly the ability of an individual to recognise a small defect in a sound with which he is familiar.

As regards the perception of sound intensity, the ear is wonderfully sensitive, and Rayleigh has computed that the minimum intensity of sound (10^{-10} microwatt per sq. cm. approx.) capable of being perceived by the cochlea, is slightly less than the minimum value of light intensity required to excite the retina of the eye in the sensitive green part of the light spectrum. The sensitivity of the ear is greatest in the frequency range 500 to 6000 vibrations per second, and this range comprises in human speech the higher partials which serve to distinguish between vowels and consonants.

The *law of intensity perception* is in accordance with Weber's psychological law of sensation, which is applicable to the nervous system, and states that the change δS in any stimulus (vision, hearing or feeling) to produce a just perceptible increase δE of the sensation, bears a constant ratio to the total stimulus S, *i.e.* $\delta E = K\frac{\delta S}{S}$. The *integrated* form of Weber's law, viz. $E = K \log S$, is known as Fechner's

law, in which K has been assumed to be a constant. It should be carefully noted that it is the *fractional* change in stimulus, in this case intensity (I), which is the determining factor. It has been found that the ratio $\frac{\delta I}{I}=0{\cdot}1$ approximately, but there is an appreciable increase in this value at low intensities. From the above relation it is readily deducible that δI is less for smaller values of I, *i.e.* the ear detects much smaller differences in loudness in weak than in loud sounds.

The Weber-Fechner law has led to the definition of a unit for the *comparison* of the intensities of two sounds. If I_1 and I_2 are the respective intensities of the two sources then $N=\log_{10}\frac{I_1}{I_2}$ gives the ratio of the two intensities in *bels*. In practice the bel is rather a large unit and a smaller unit, the *decibel*, written *db*, is usually employed and is

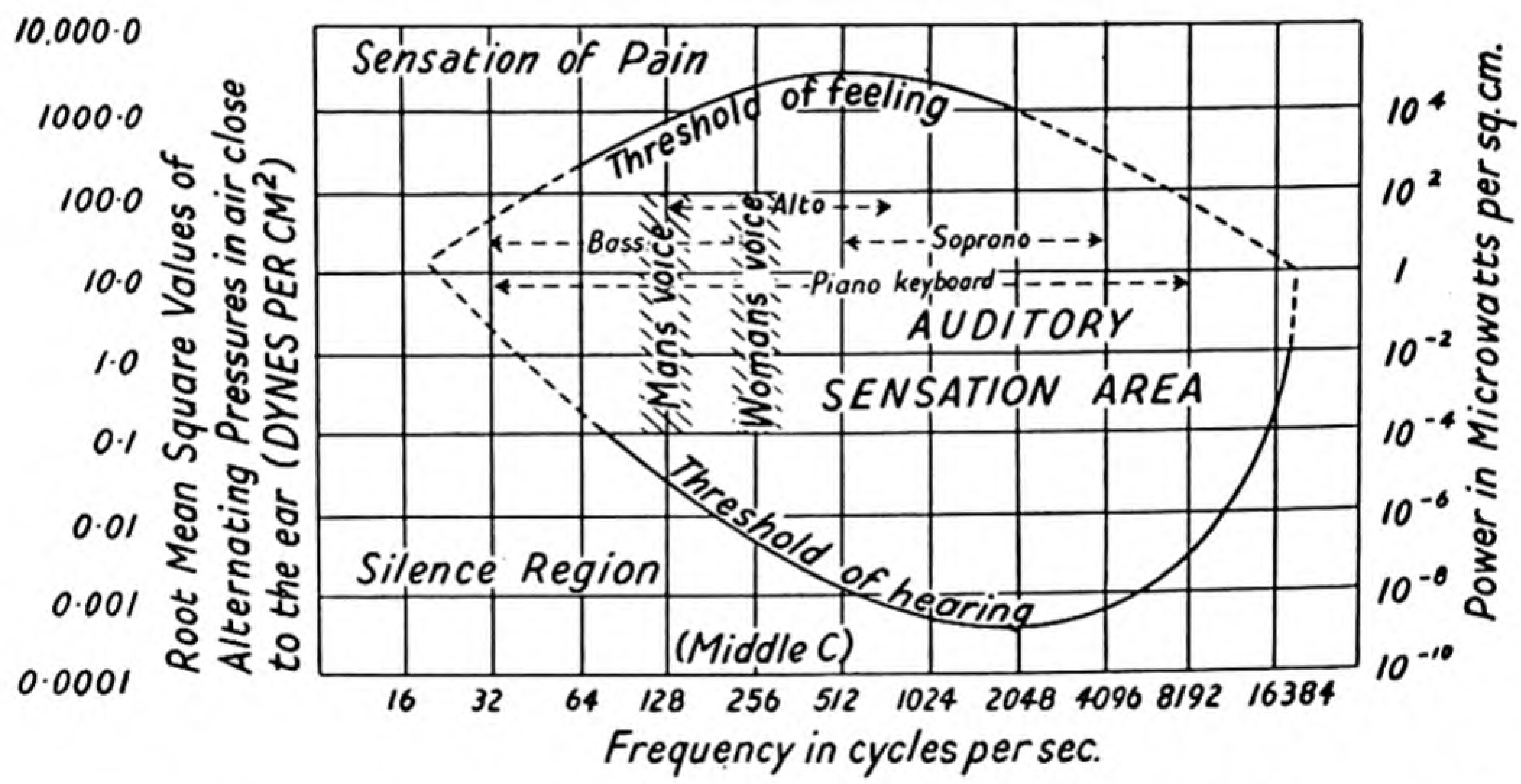


Fig. 10.5. Working limits of the ear for any pure tone as regards frequency and maximum change in air pressure close to the ear (Wegel).

defined by $N=10\log_{10}\frac{I_1}{I_2}$ db. The decibel is a convenient unit for it happens to be the smallest change in intensity detectable by the human ear.

The curve shown in Fig. 10.5 represents the average response of a large number of auditors of normal hearing. In order to obtain such a curve it is necessary to remember that the change of pressure due to the passage of a sound wave is alternating in character, hence the recording instrument must respond to alternating pressure changes, and should be situated close to the ear-drum of the auditor. The frequency of the generated note being known, its intensity is varied until the note is just audible and the corresponding value of the pressure is noted. This procedure is repeated for the range of audible frequencies, approximately 20 to 20,000 c.p.s., and in this manner the

lower curve of Fig. 10.5 is obtained. It is usually known as the curve of the *threshold of hearing*. The unit of pressure difference employed is the *micro-bar*, which is equal to 1 dyne per square centimetre, or to approximately one-millionth of the normal atmospheric pressure. It is seen from the threshold curve that the ear is most sensitive at a frequency of about 2000 c.p.s., where a change of pressure less than 10^{-3} micro-bar can be detected. If the intensity of a sound is gradually increased, a stage is reached when the ear experiences a tickling or painful sensation, and the value of the corresponding critical pressure excess is also a function of frequency, as shown by the upper curve of Fig. 10.5. It is indeed remarkable that at medium frequencies the ear can adjust itself to pressure changes varying from 10^{-3} micro-bar to 10^3 micro-bars, *i.e.* a million to one change. The dotted portions of the curves in Fig. 10.5 are not obtained experimentally but are extrapolations of the threshold curves drawn to meet at the upper and lower limits of audible frequency. It follows that any pure tone is audible if its intensity *and* frequency can be represented by the

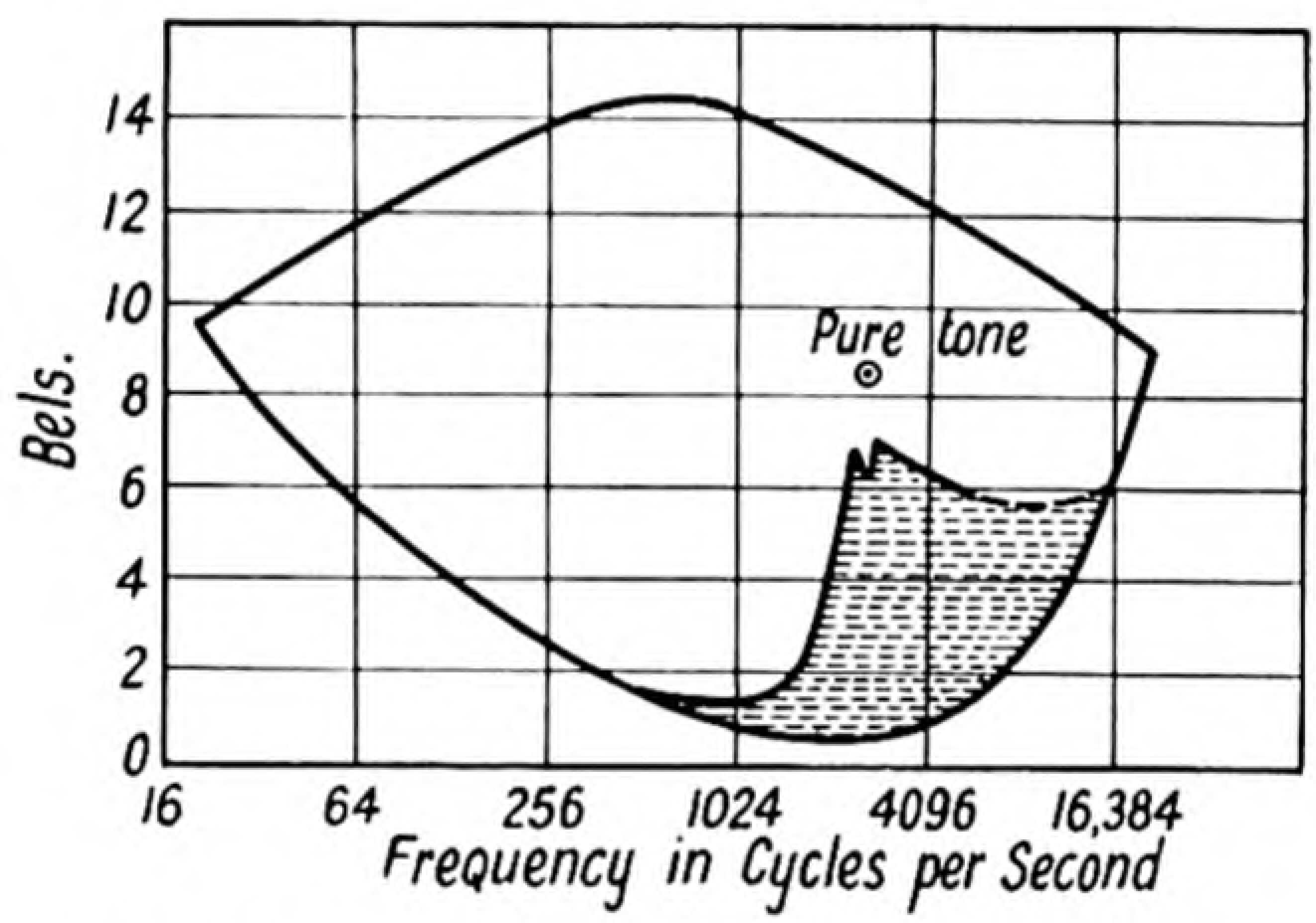


Fig. 10.6. Deafening effect of loud note.

coordinates of a point lying within the area bounded by the threshold curves. Should the point fall above the upper curve it means that the listener becomes liable to temporary, or even permanent deafness, if he is subjected to the noise. Emphasis should be drawn to the fact that the diagram was obtained when the various notes were sounded alone without any background noise, and the existence of any such noise will raise and distort the threshold of hearing curve (Fig. 10.6). Tones whose coordinates lie within the shaded area are inaudible. The effect of the presence of additional noises is usually referred to as *masking*, and it is the reason, for example, why the tone of a musical instrument heard in an orchestra differs from that heard when the instrument is played alone. Fig. 10.7 shows the auditory chart of an individual who is deaf to the higher audio-frequencies and who consequently would benefit by the use of a microphone suitably amplified in these higher frequencies.

In the measurement of the loudness of a tone the reference level of the intensity is taken as that corresponding to the threshold value

at *that* pitch. Considering Fig. 10.8, it is evident that by definition (p. 172) the loudness of the note corresponding to the intensity I_1' of P is given by $10 \log_{10} \frac{I_1'}{I_1} db$, where I_1 is intensity value of the threshold

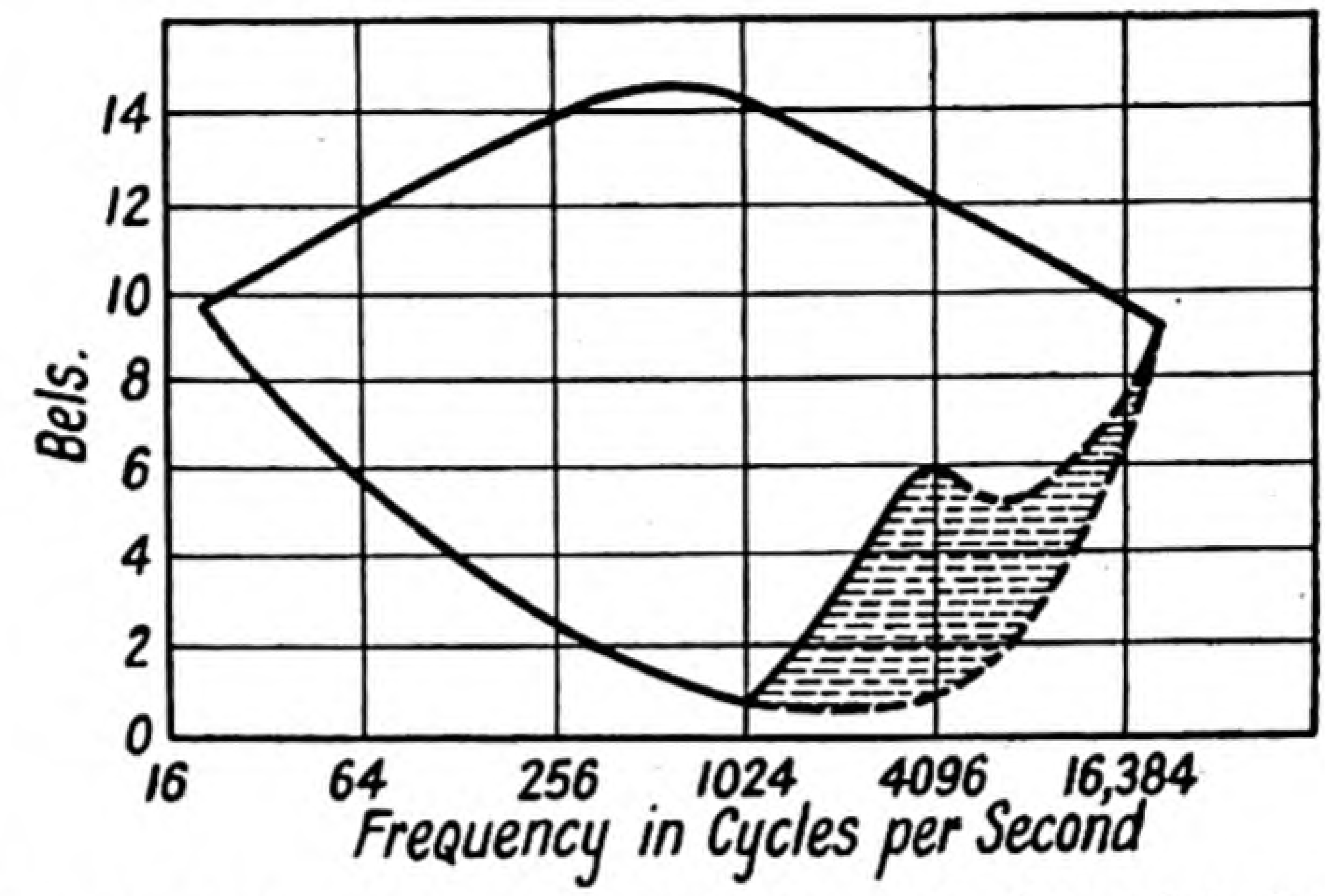


Fig. 10.7. Auditory chart of an individual deaf to high frequencies.

at that pitch. Now in order to make the reference intensity I_1 quite definite it is taken as corresponding to the lower threshold value of audibility (I_0) for a 1000 c.p.s. note. This value, after a large number of experiments here and in America, has been taken as the intensity due to a sound pressure (R.M.S.) of 0·0002 dyne per square centimetre. This new unit of loudness is called the *phon*, and one phon represents

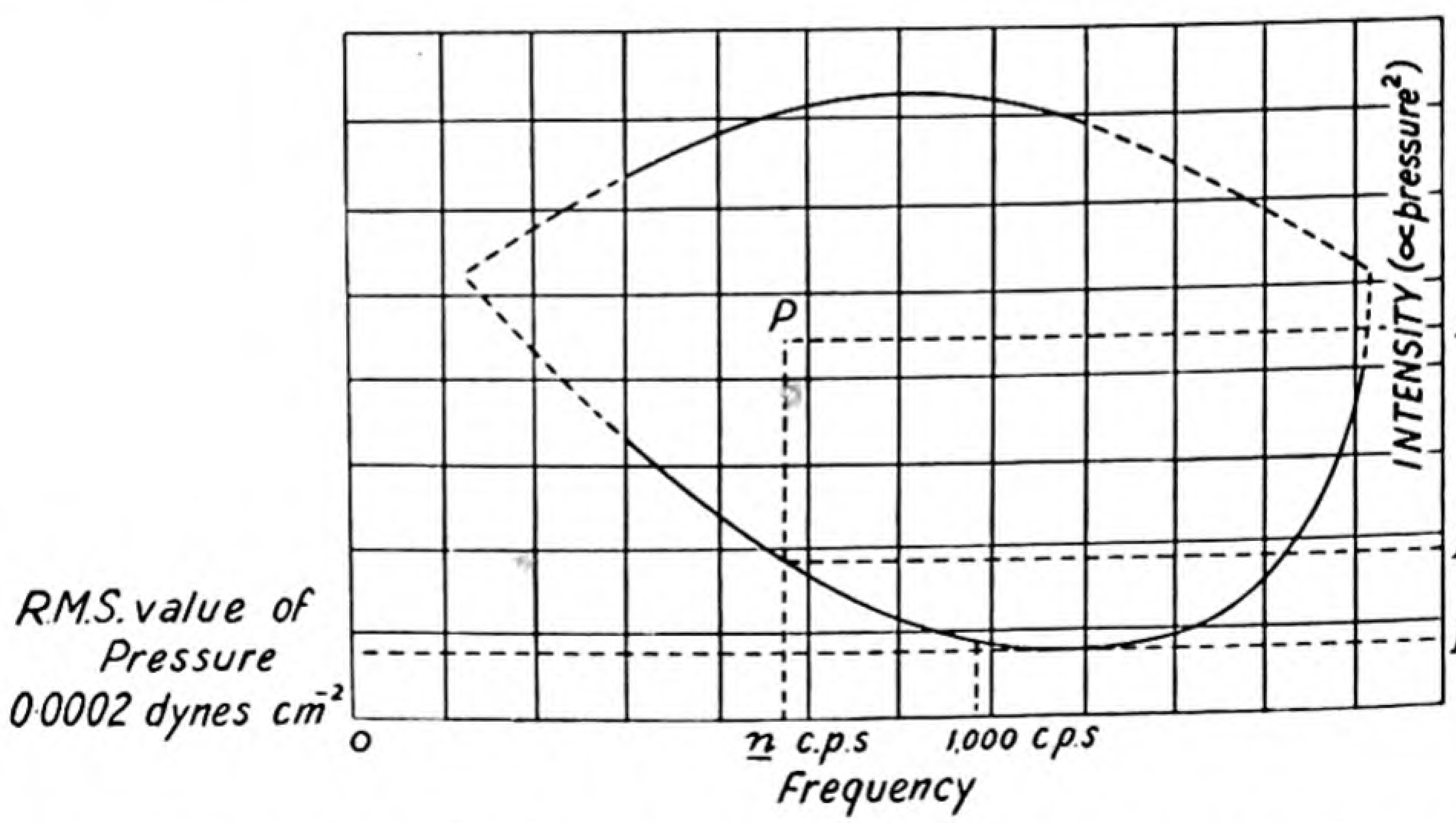


Fig. 10.8. I_1' is given intensity of note corresponding to P of frequency n c.p.s. I_1 is threshold intensity at frequency of n c.p.s. I_0 is standard reference intensity.

about the smallest change of loudness that can be detected by the human ear under average conditions. The loudness of the note P can now be expressed as $10 \log_{10} \frac{I_1'}{I_0}$ phons.

In the perception of frequency a *finite* change of frequency is necessary for the ear to detect a difference, just as in the case of intensity perception a finite change of intensity was required. The ear, however, is more sensitive to changes of pitch than to intensity; for it is able to detect a *percentage* change as small as 0·2 in frequency compared with the most favourable value of 5·0 in intensity. The ratio of the minimum perceptible change δf in frequency to the frequency f is

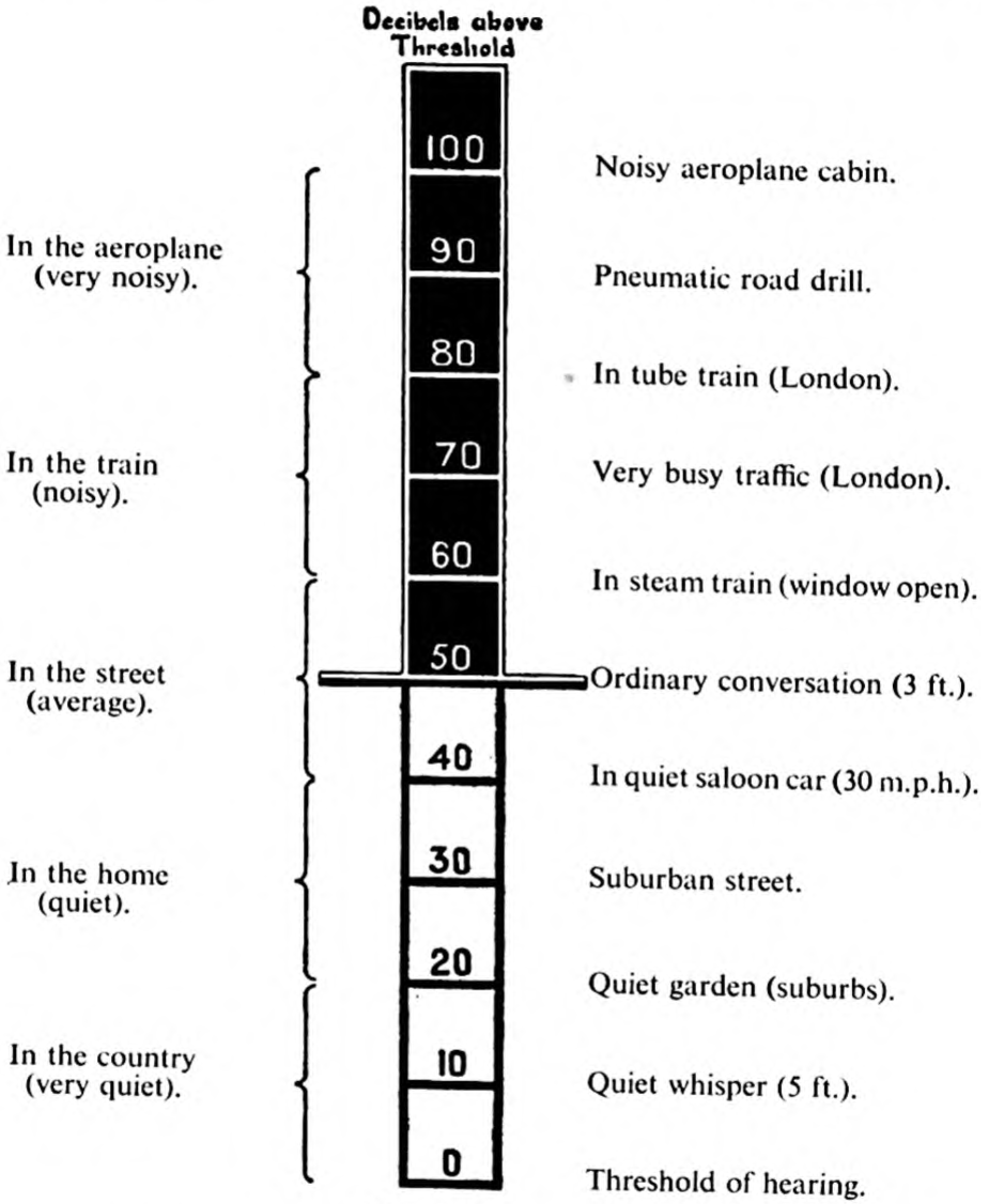


Fig. 10.9. [*After Kaye.*

approximately constant over the frequency range, 500 to 4000 c.p.s., which is to be expected if frequency perception like intensity perception is to follow Weber's law of sensation. Fig. 10.9 gives a table of loudness levels of some common noises.

Combination tones

If two simple tones are sounded together, the ear sometimes detects along with them a third tone, often spoken of as a Tartini tone, after

its accredited discoverer, an Italian violinist of the eighteenth century. The frequency of this combination tone as originally discovered was equal to the *difference* (f_1-f_2) between the frequencies f_1 and f_2 of the generating tones, but Helmholtz later revealed the existence of *summation* (f_1+f_2), as well as difference tones. These *first* summation and difference tones can combine with the original tones and give rise to higher order combination tones, e.g. (f_1-2f_2), (f_1+2f_2), etc. The explanation of the production of these tones, as put forward by Helmholtz, is dependent upon the asymmetric vibrational characteristics of the ear-drum, and is a direct consequence of the theory of such vibrations (App. 8). Waetzmann has also verified the existence of combination tones by experiments with a light rubber membrane mounted vertically and loaded asymmetrically with a piece of wax.

An interesting example of the production of combination tones occurs with the playing of stringed instruments, where the fundamental tones are much *weaker* than the other partial tones, owing to the fact that the linear dimensions of the "resonating bodies" of these instruments are small compared with the wave-lengths of their fundamental frequencies. As a result of the non-linear response of the ear, however, the higher partial tones are balanced in the act of *hearing* by the formation of the fundamental tones by the combination of the higher frequencies.

Deafness

To complete this brief survey of the ear, a short mention will be made of the help that can be given by the applications of modern science to sufferers from deafness, although progress in this direction has been tardy compared with that rendered to those with defective sight. It is hoped, however, that the day is not far distant when the counterpart of the optician in the aural world, the *otician*, will be readily available to the mass of the public, and that he will be a man fully trained, both from the medical and the electro-acoustic aspect.

The general method of determining the nature and degree of deafness of an individual is to perform what is termed an *audiometer* test. This test involves the estimation of the sound intensity which is just audible to the patient at a representative number of frequencies, and is carried out for both air and bone conduction. The graph obtained by plotting the estimated number of decibels below normal hearing intensity against the frequency is known as an audiogram. Three main types of deafness are classified, viz. conductive, nerve and cortical. Conductive deafness refers to a defect in the parts of the ear which "conduct" the vibrations to the cochlea, and may be caused, for example, by the stiffening of the joints of the ossicles or a thickening of the ear-drum. It is made evident by a feeling of "stuffiness" in the ears, such as is felt when suffering from a common cold. Nerve deafness accrues from defects in the cochlea or inner ear, and the characteristic feature is that the ear may be insensitive to low intensities, but at higher levels have a sensitivity approaching that of a normal ear. This fact is borne out by the phenomena of some deaf individuals appearing to hear better in a background of noise, for it is usual in such circumstances for speakers, unconsciously, to raise the intensity levels of their voices

to overcome the loss of intelligibility. It has been found, however, that some individuals actually do hear better when a noise is present, as if there is an initial stickiness to be overcome in the hearing mechanism before it is stimulated, and the effect has been likened to back-lash in a mechanical instrument. Cortical deafness, which chiefly affects old people, is characterised by an inability to *interpret* sounds correctly, even although the organs of hearing may be unimpaired, and in effect it means that the brain centres are suffering from a loss of "sound" memory. In such cases this loss of "language factor," as it is termed, may usually be overcome if the individual is spoken to deliberately and slowly. Fletcher and Wegel have found that up to frequencies of 4000 c.p.s., persons requiring a pressure variation of 1 dyne per $cm.^2$ or less for audibility can usually manage to follow ordinary conversation, but artificial aids are definitely needed by persons insensitive to changes less than 10 dynes per $cm.^2$.

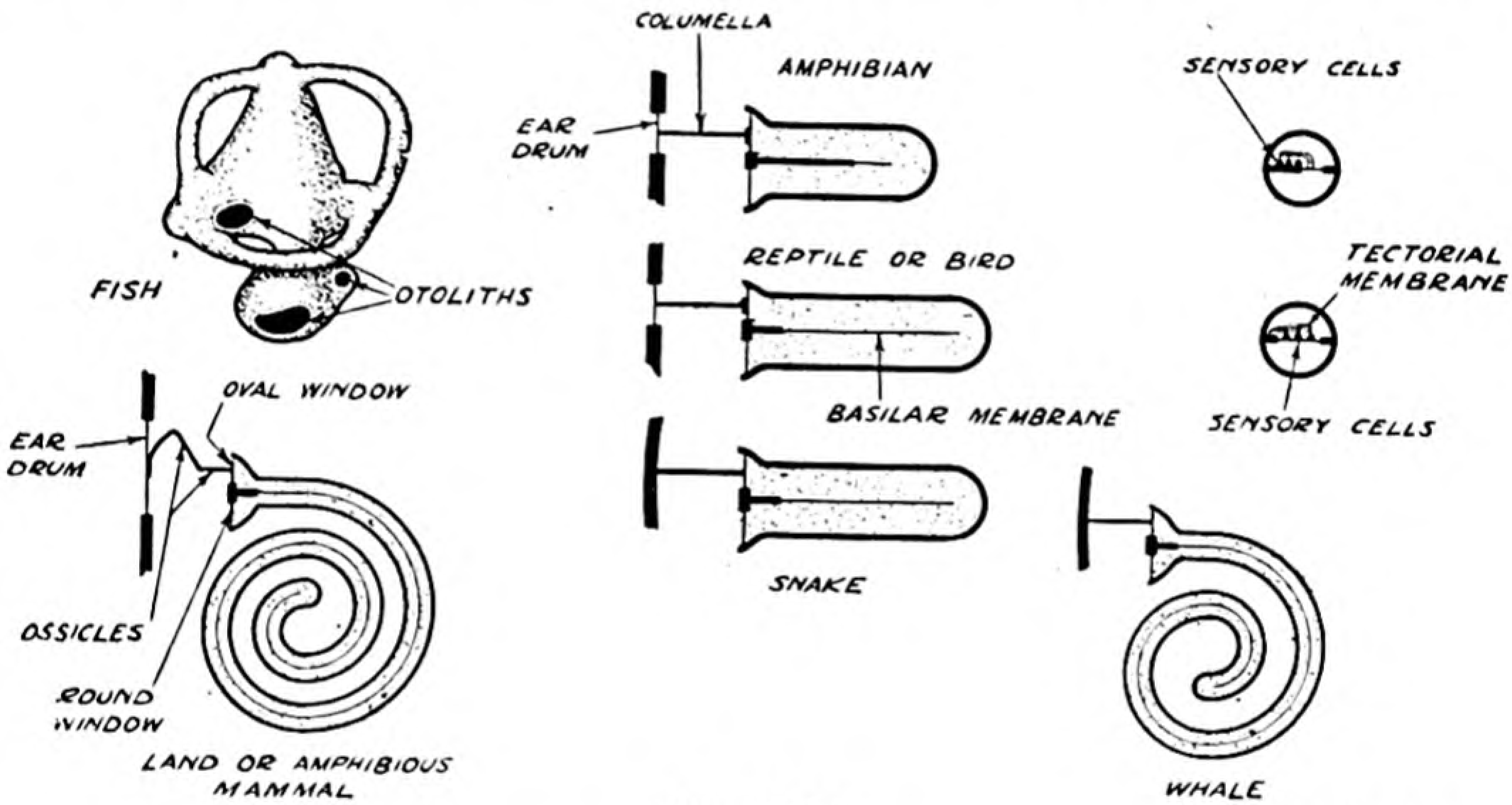


Fig. 10.10. [*After R. T. Beatty.*

It is evident that there is a natural limit to the degree of amplification of the sound which can be used owing to the existence of the threshold of feeling.

Hearing-aids

A hearing-aid comprises essentially three parts, a microphone or sound receiver, an amplifier, and a reproducer whose function is to transfer the amplified signal to the inner ear. This reproducer may be either an earphone or, as in the case of bad conductive deafness, a bone conductor, which is the means of transferring the incident vibrations direct to the inner ear by conduction through the mastoid bone. By suitably designing the component parts of a hearing-aid unit it should be possible to attain a close approximation to the desired characteristics for the correction of the hearing defects of the patient. For an example, in the case of a person suffering from nerve deafness, it is necessary to employ an electrical amplifying circuit which gives a

greater amplification for weak than for loud signals, otherwise the latter are liable to attain the threshold of pain.

Hearing in animals

The evolution of the hearing mechanism in various forms of animal life, ranging from a fish to a mammal, is shown diagrammatically in Fig. 10.10, and it is seen that certain features are common in the auditory organs of all *animals*, such as the stretched membrane to receive the incident sound waves. In the fish the vague beginning of an organ of *true* hearing is to be noted in the otolith organs, which consist of several fine and delicately balanced hairs, and these sway under the action of gravity when the fish changes its position. This mechanism which enables the fish to swim on an "even keel," is not suitable for a *sensitive* analysis of sounds, but such a quality is not really required, as will be made evident after considering the nature of ocean sounds. These noises will be chiefly localised at the surface of the ocean, for the fishes themselves are streamlined and move about very smoothly and silently in the interior and, being supported by the medium, they do not require to engage in a rapid "beating" of the fluid medium as birds do in the air. Furthermore, in the depths of the ocean the actual "flow" movement of water is small. Near the shore, however, the surface noises will be appreciable, and here the roar of the breakers comes from the exploding of innumerable small air bubbles. In the foam, etc., these bubbles become distorted but tend to regain their spherical shape, and in so doing are set into vibration. Owing to the smallness of the bubbles, however, these vibrations will be mainly of fairly high frequency, and in some cases the pitch is ultrasonic and undetectable by the human ear.

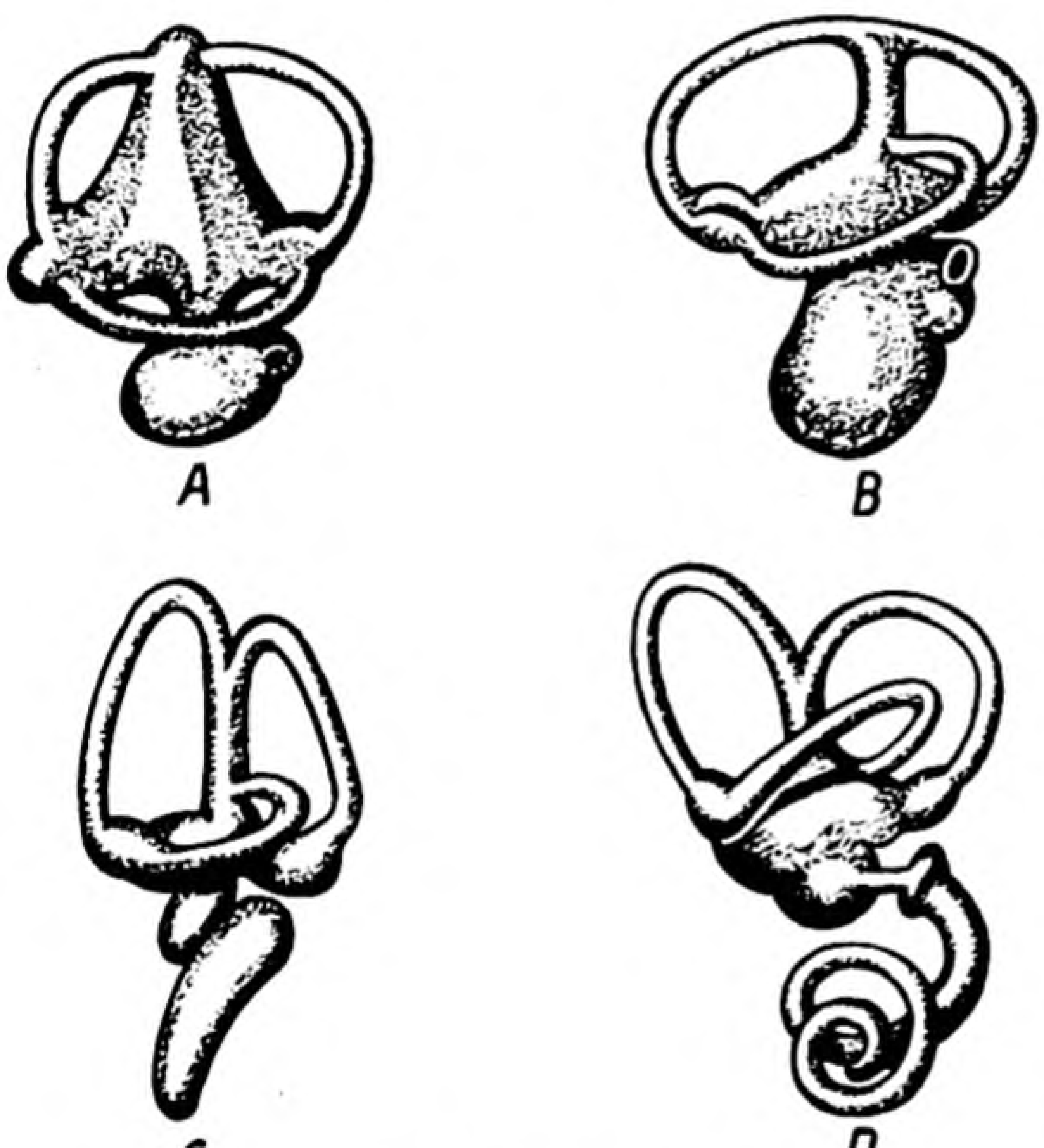


Fig. 10.11. [*After Beatty*.

The various stages of development of the cochlea in the animal world is shown in Fig. 10.11. In the case of a fish (*A*) the cochlea is non-existent, but it appears as a small swelling on the right-hand side in case of the frog (*B*). Diagram *C* indicates that a bird has a slightly curved cochlea, and finally in a mammal (*D*) it has become a spiral of several turns. It is interesting to note (Fig. 10.10) the absence of an ear-drum in the snake, which has the outer end of its columella connected to the quadrate bone of the skull, so that it is able to detect sounds conveyed through its body and the ground but is not

conscious of air-borne vibrations. Consequently, it is only because the cobra is able to follow the motion of the snake-charmer's body that it sways in rhythm to the music, for the snake is quite unaware of any tunes that may be forthcoming if the performer remains hidden.

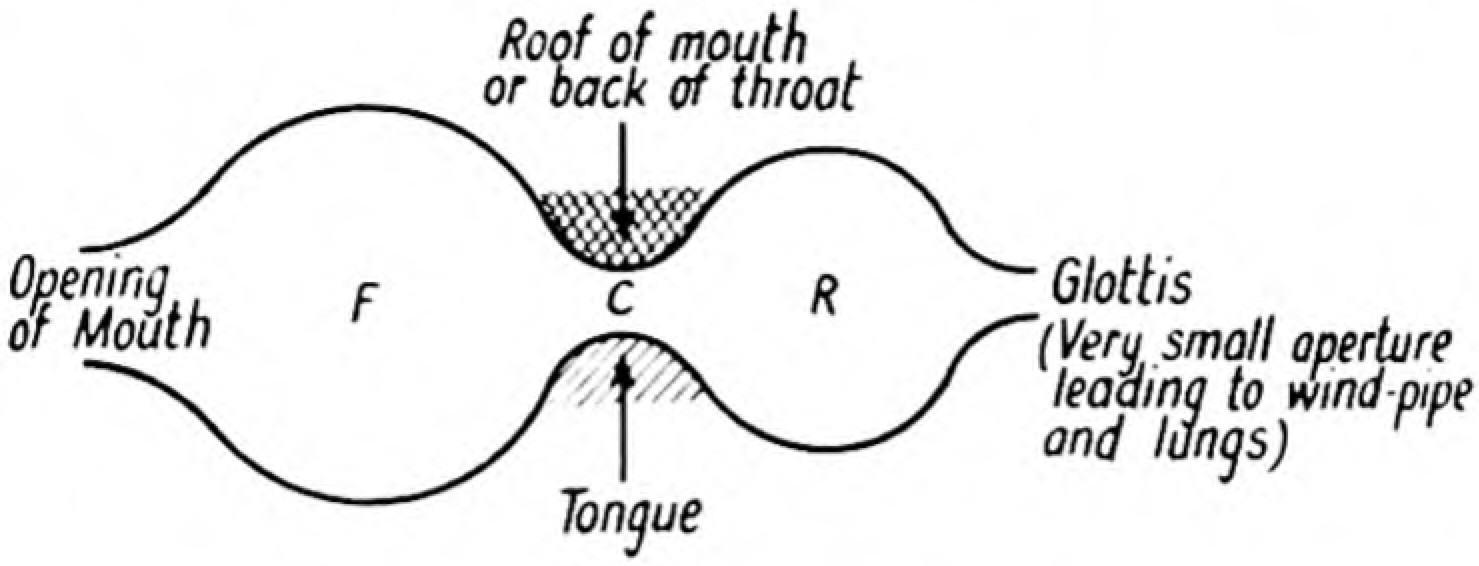


Fig. 10.12.

Apart from the vertebrates, the power of hearing air-borne sounds has only been noted in certain insects, and in these the auditory organs are not situated on the head. Cicadas, for instance, have their hearing mechanism situated on the body, while in crickets it is located on the front legs. Gnats, which are sensitive to high-pitched sounds, are supposed to recognise these noises by means of hairs on their bodies, the hairs being set into vibration at select frequencies.

Speech

The sound generating system of the human voice may be conveniently likened to a sound producer in which the bellows, the source of vibrational motion, and the resonators are replaced respectively by the lungs, the larynx, and the cavities of the throat, mouth and nose (Fig. 10.12). The cavity at the back of the mouth leading to the nose is called the pharynx. Actually, the system bears a close resemblance to the trombone type of instrument, for the length and tension of the vocal cords contained within the larynx (popularly known as "Adam's apple") are variable, and so are the sizes of the resonating cavities. Each vocal cord is a piece of flesh of triangular shape, and there are two pairs, F_1F_2 and T_1T_2 (Fig. 10.13), the lower pair (T_1T_2) being known as the tone cords, and the upper (F_1F_2) as the false cords. These cords are in contact at the front, but at the back their relative positions are adjustable; when taut, as for song or speech, they may be separated by a few tenths of a millimetre only. The separating gap between them is known as the glottis.

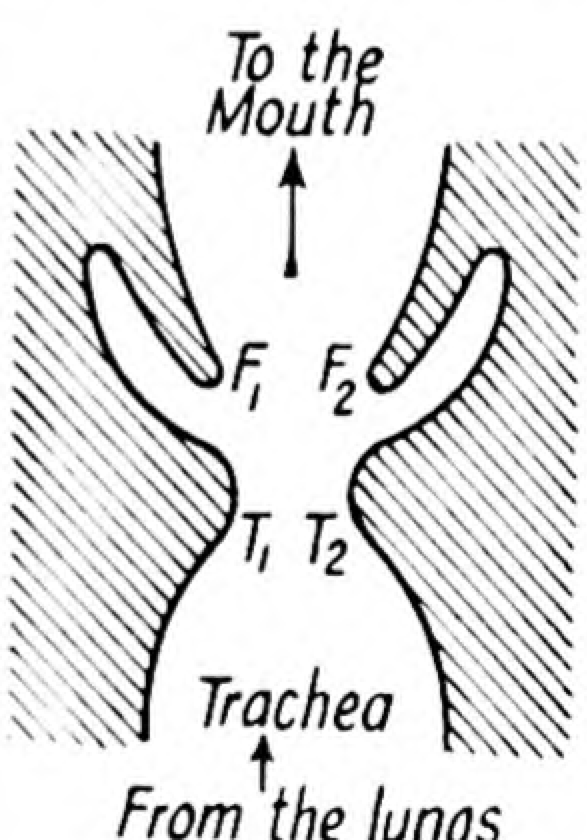


Fig. 10.13.

A continuous supply of air from the lungs passes up the wind-pipe or trachea which is nearly one inch in diameter. When the voice is operative the pressure of air in the trachea gradually increases until it is sufficient to displace the vocal cords. This displacement, however, will tend to increase the tension in the cords and so they are restored to their initial

positions. Again the pressure of the air in the trachea builds up sufficiently to force the cords apart and again they are restored, and so long as this sequence is repeated the cords are maintained in vibration. The maintenance of the motion is, therefore, rather similar to that of a violin string, for the air, like the violin bow, moves only in one direction; the efficiency of maintenance will depend upon the volume of breath expelled per second, and a poor efficiency, *i.e.* a large volume, will cause a singer's voice to sound "throaty." Stanley and Sheldon, two American scientific workers, have found that a skilled singer uses less breath when singing loudly than when singing softly, and that in the latter case it could amount to 400 cc. per sec.

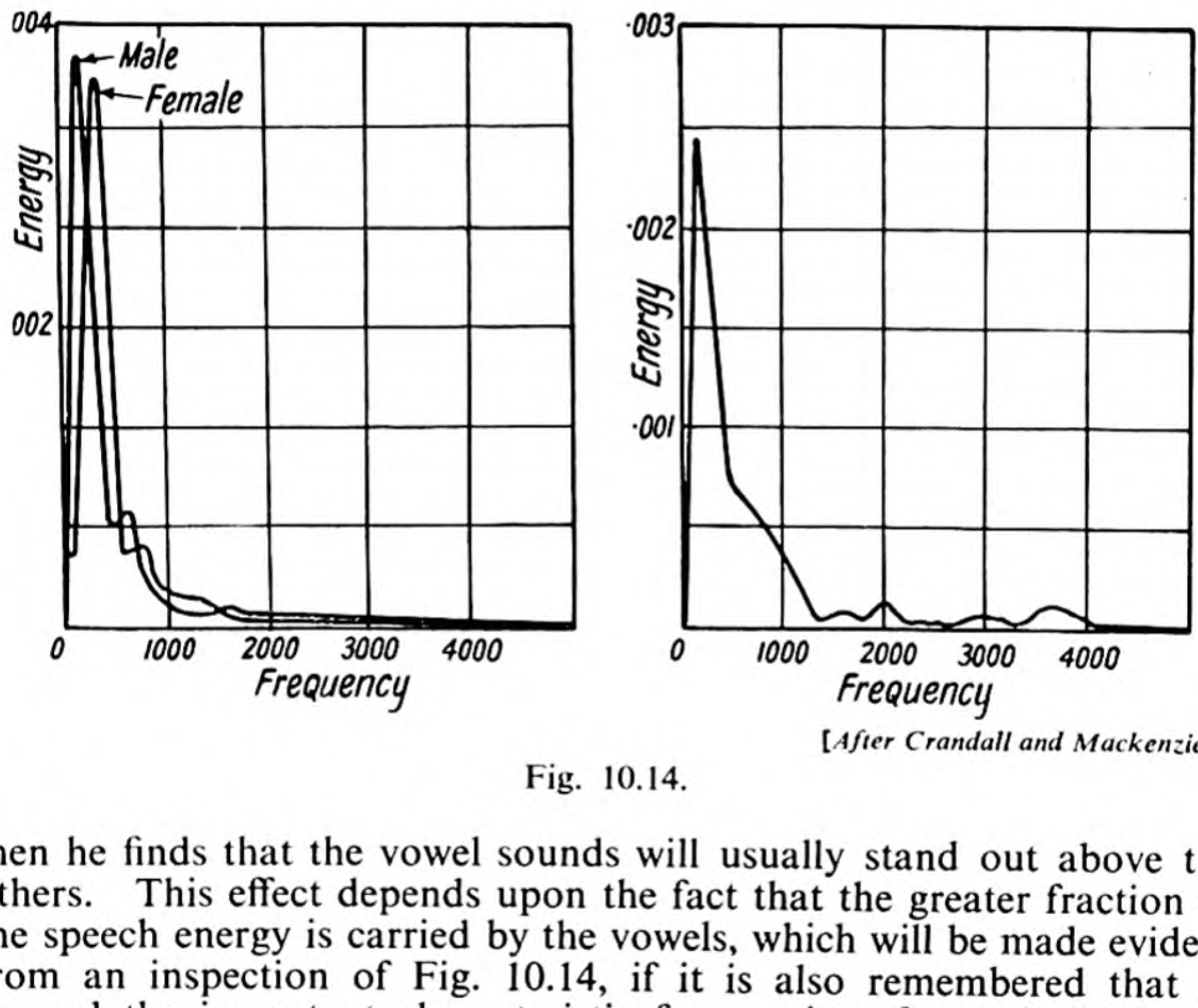
The rapidity of the vibrations will be determined by the length and tension of the vocal cords, the shorter the cord the higher the frequency. Hence it follows that since the vocal cords of women (and children) are thinner and about two-thirds of the length of those of men, they produce notes of a higher pitch. When a boy's voice "breaks," its pitch falls by almost an octave owing to the rapid growth of the larynx at adolescence. The range of the fundamental frequencies of the human voice is from about 80 to 770 vibrations per second, *i.e.* approximately 3 octaves.

The note produced by the vocal cords is known as the laryngal tone, but it would be quite unmusical without the modulating effects of the resonator cavities, and it is this feature which distinguishes human speech from that of the lower animals. The parrot and the ape, however, are exceptions amongst the latter as they are able to perform a limited degree of modulation. If the vocal cords are relaxed, *i.e.* drawn aside, the stream of air will suffer little disturbance in its passage through the larynx, and any sound heard will correspond to the natural vibrations of the resonant cavities, *e.g.* mouth, etc. Such an effect occurs in whistling and in whispering, and is referred to as "unvoiced" speech. The majority of consonants, *e.g.* k, p, s, are generated in this manner without any help from the vocal cords.

The larger fraction of the walls of the voice resonators is formed of soft flesh which possesses a pronounced sound absorbing quality, and it follows that the vibrations of the air stream will become strongly damped. In consequence, the resonances of the voice are very broad when compared, for example, with those of a brass wind instrument whose walls are rigid. The mouth, throat, nose and pharynx, however, can be altered both in size and shape to enable the system to resound now to one note, now to another; in fact, Helmholtz says that they permit "of much variety of form, so that many more qualities of tone can thus be produced than in any instrument of artificial construction." In contrast, the resonance effect of the lungs in response to the laryngal tone will be very small, since their communication to the lower end of the trachea is by way of passages which are both narrow and contained by walls which have good sound-absorbing qualities. If the mouth is kept nearly shut when speaking, it means that the pharynx must undergo large changes of volume in order to produce the various vowel sounds, and this is the basis of the art cultivated by ventriloquists.

In direct contrast, a singer uses his pharynx muscles sparingly and must therefore adopt a wider mouth opening. If the lips are actually closed, the sound can only pass through the nose, as occurs in humming.

The separate sounds that constitute speech and song are conveniently divided into vowels, semi-vowels (*e.g.* r, l, m and n) and consonants, and it is upon the existence of the latter class that the intelligibility of speech is primarily dependent. This fact may be verified by hearing a passage read aloud, firstly as written, and then repeated with all the vowels replaced by a *single* vowel tone. In general, the hearer will find that the gist of the passage is still understandable during this second modified reading. If, however, he increases his distance from the speaker, so that the speech becomes unintelligible by the general lowering of the perceived sound intensity,



[*After Crandall and Mackenzie.*

Fig. 10.14.

then he finds that the vowel sounds will usually stand out above the others. This effect depends upon the fact that the greater fraction of the speech energy is carried by the vowels, which will be made evident from an inspection of Fig. 10.14, if it is also remembered that in general the important characteristic frequencies of vowels lie below 1000 c.p.s., while those of the consonants are at higher frequencies.

The curves on the left-hand side of Fig. 10.14, obtained by Crandall and Mackenzie, show the mean energy distribution in the speech of men and women speakers, repeating a given sentence of connected speech and a given set of disconnected syllables. The right-hand curve is an average of all the observations made; the ordinate scale is quite arbitrary, being proportional to the energy flow.

Speech-aids

A person who has been deprived, *e.g.* through paralysis, of the use of the vocal cords, can now receive aid from an artificial larynx developed by the Western Electric Company of America, which

is shown in detail in Fig. 10.15. This larynx fulfils the double purpose of simulating the action of the vocal cords, by means of a vibrating reed, and also of supplying the connection between the lungs and the mouth, for as a result of the operation performed, the trachea is terminated at a small opening in the lower end of the throat. In

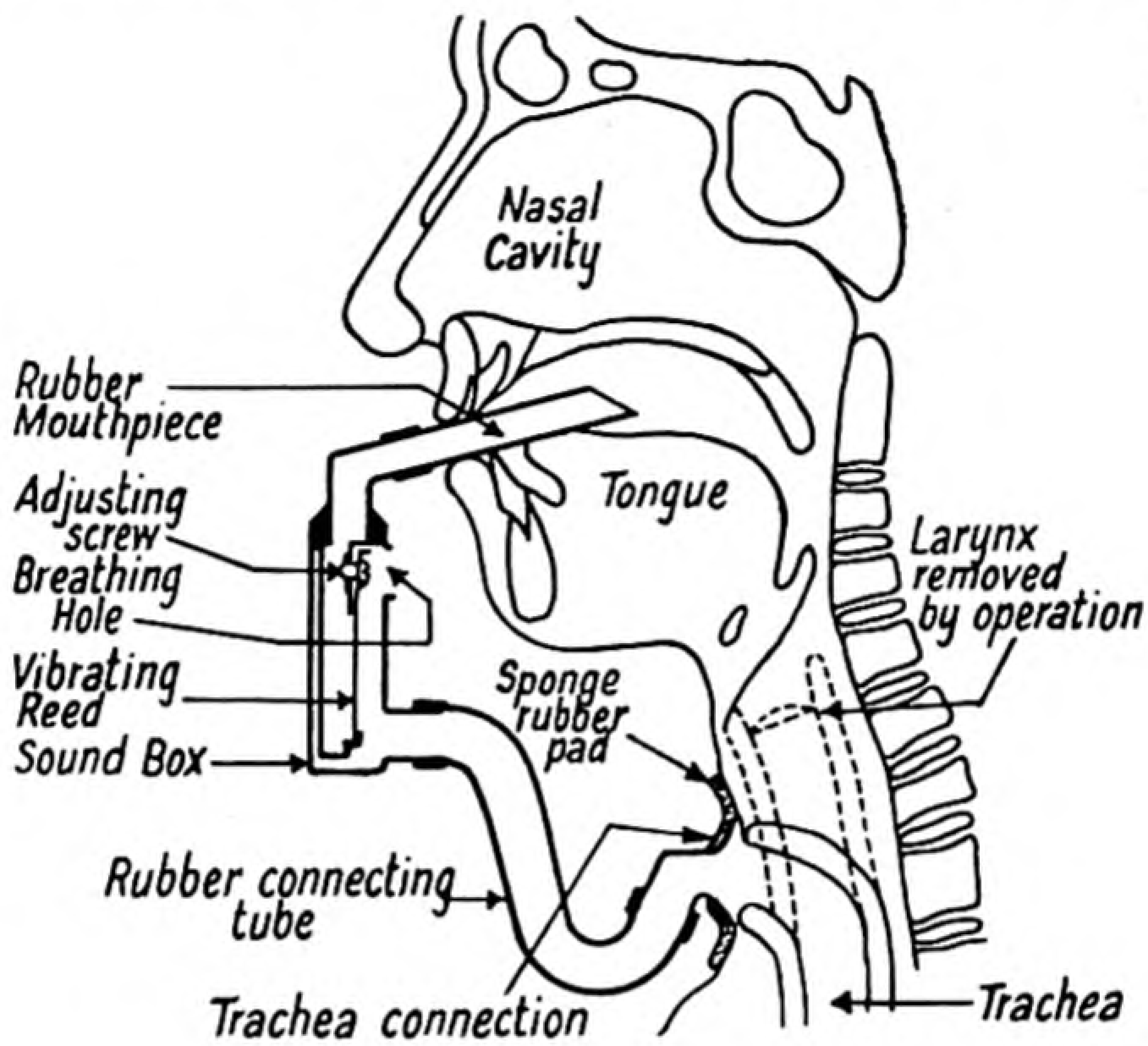


Fig. 10.15. [*Courtesy of Western Electric Co.*

the act of speech the user sets the reed into vibration at will by placing a finger over the breathing hole shown in the diagram, and the modulated air-stream then enters the resonating cavities of the mouth, throat and nose, which are adjustable as for the ordinary individual.

In places where the general noise level is extremely high, *e.g.* in the cabin of an aeroplane, the sound of the human voice is inaudible and

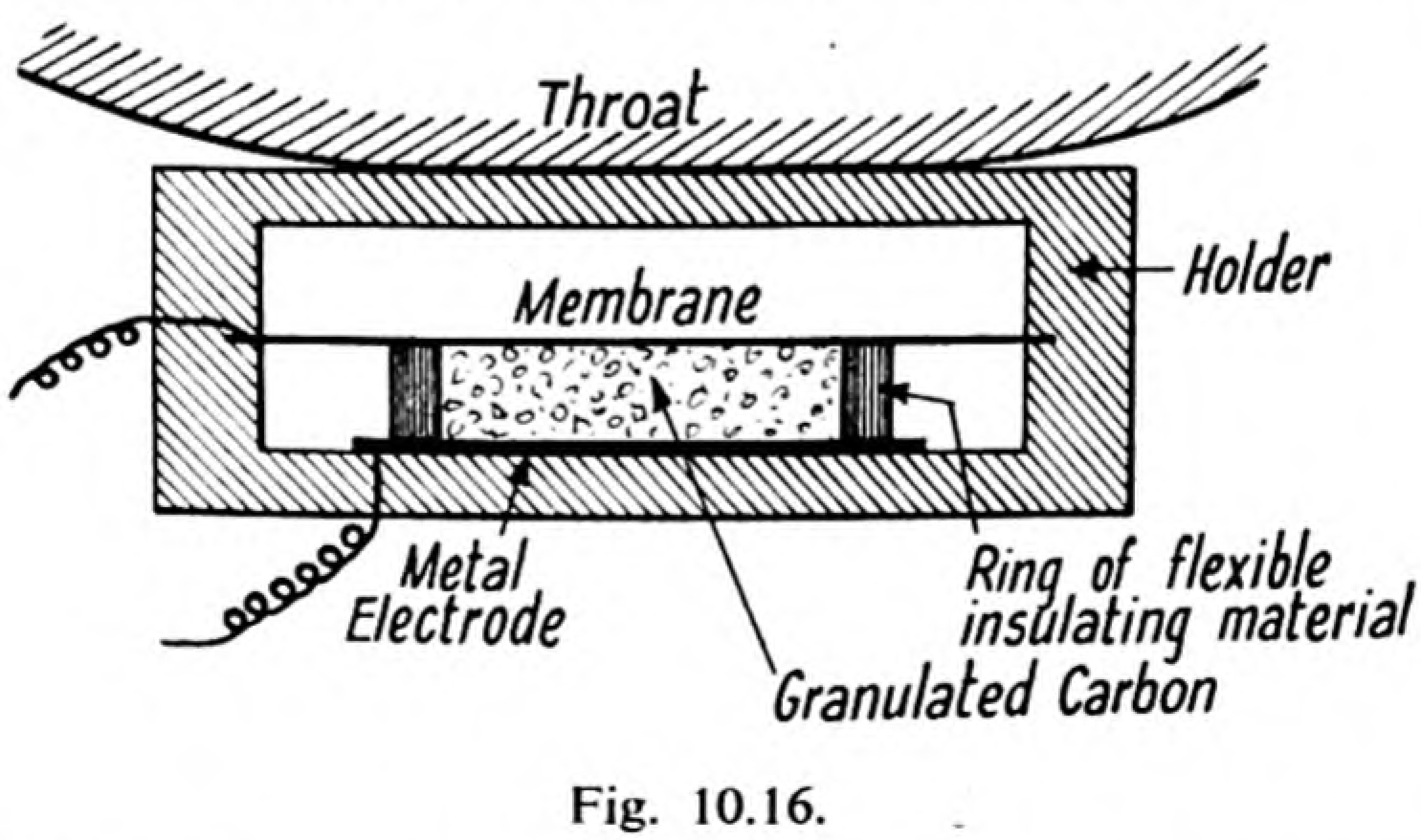


Fig. 10.16.

the use of the ordinary telephone transmitter becomes impracticable. However, this difficulty may be overcome by utilising, not the vibrations of the air in front of the speaker's mouth, but the mechanical vibrations of his throat. The instrument developed for this purpose, known as the *laryngophone*, depends for its action upon the fact that

the vibrations of the air in the mouth are also communicated to the flesh and bones of the neck and face. The damping layer between the outer skin and the cavity is thinnest at the throat, which therefore affords the most suitable place for "picking-up" these vibrations. The throat vibrations, as is to be expected, will not represent all speech

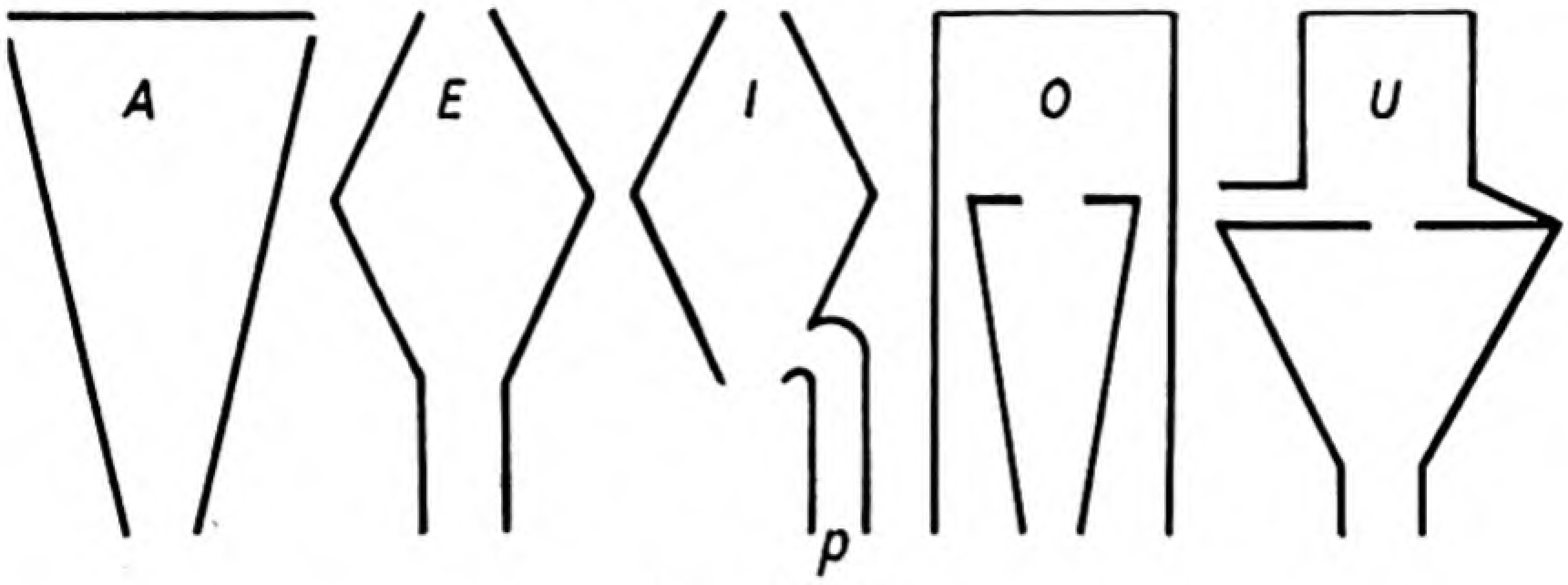


Fig. 10.17.

with the same fidelity, *e.g.* the "s" sounds and the explosive consonants p, k and t are poorly represented for they are excited in the forward parts of the mouth. The vowels and other consonants, however, are sufficiently well reproduced by the instrument for the intelligibility to be reasonable. In order to obtain the desired frequency characteristics, J. and K. de Boer have shown that it is preferable to excite the microphone (crystal or carbon) indirectly, so that it is the holder itself which becomes directly subjected to the mechanical vibrations. This procedure also possesses the advantage that no moving parts project outside of the holder, as shown by Fig. 10.16, where the circular membrane of the carbon microphone is stretched across the inside of its holder.

Nature of speech

So far only the general mechanism of the human voice has been considered, and a brief survey of the work that has been done on the nature of the human speech will now be given. This subject aroused interest in the middle of the seventeenth century, at about the time that J. Wilkins, a founder of the Royal Society, invented a phonetic alphabet. Towards the end of the next century the Russian Imperial Academy offered their annual prize to the constructor of an instrument which would produce the sound of the vowels *a*, *e*, *i*, *o*, *u*. The award went to a Professor Kratzenstein who, from observations on the form and dimensions of the human mouth when uttering the various vowels, devised the series of tubes shown sectionally in Fig. 10.17. The pipes were excited by blowing air from a bellows on to a reed fitted to each, although *i* could be sounded merely by blowing into the pipe *p* without the use of a reed. This apparatus was improved by an Austrian, de Kempelen, who was able to use it to pronounce several consonants in addition to the vowels. In his first design (Fig. 10.18) the position

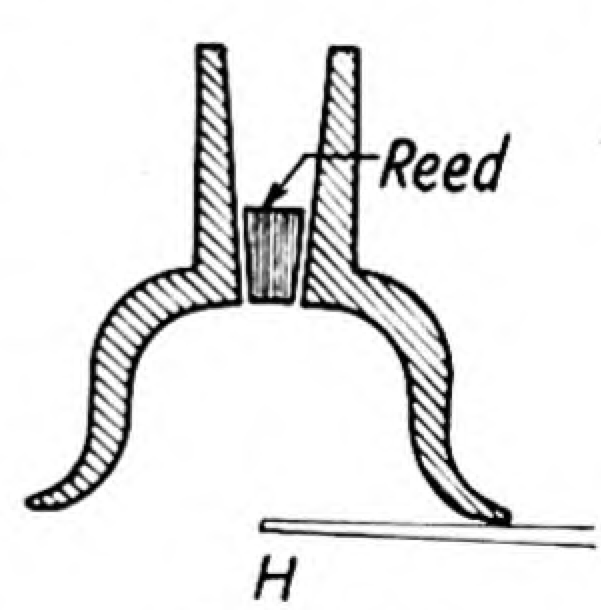


Fig. 10.18.

of the hand *H* was employed to vary the tone elicited on exciting the reed. About this time (1830) Willis, of Cambridge University, simulated the speech organs by using a free reed placed near the

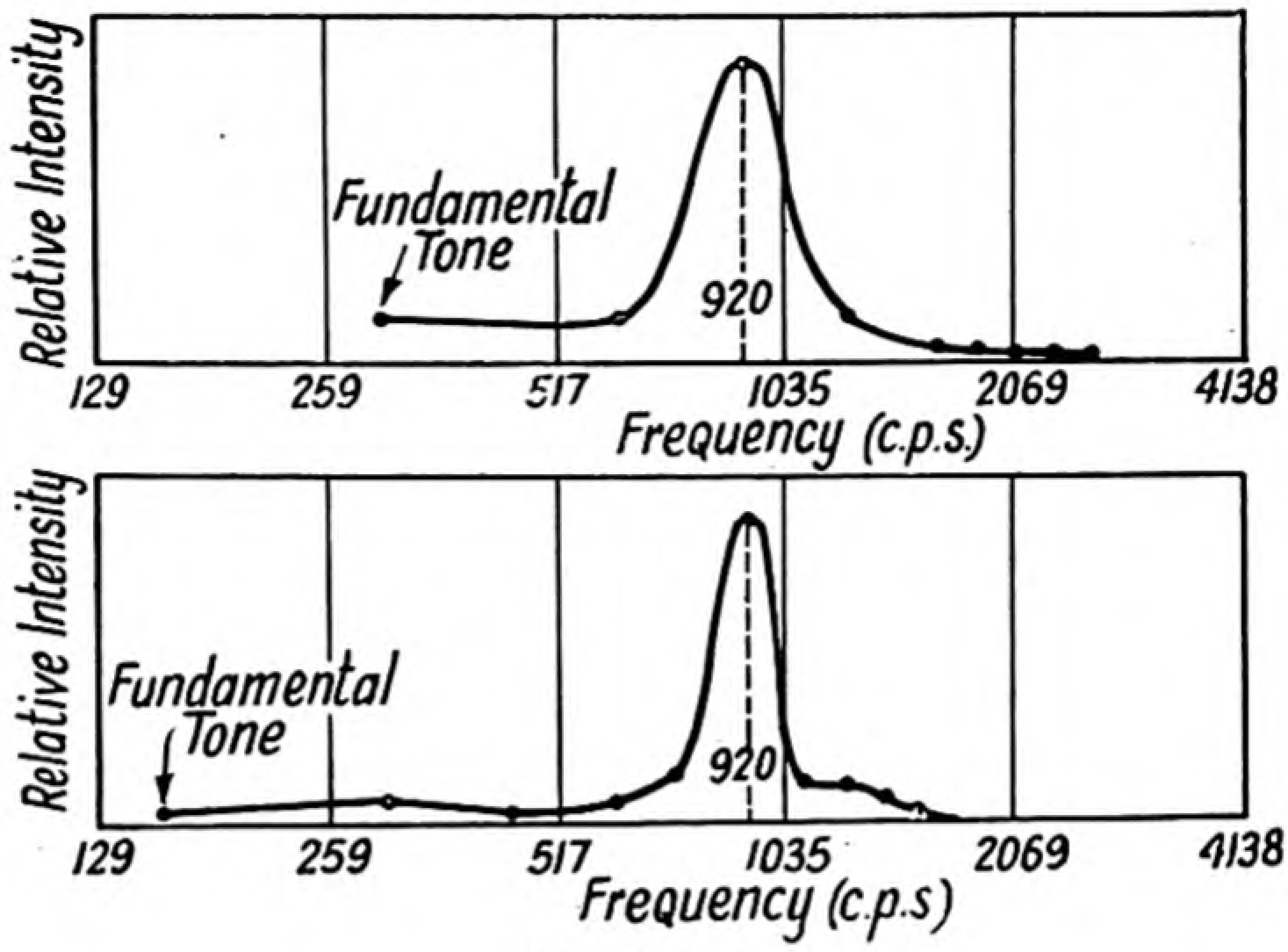


Fig. 10.19.

closed end of a tube, whose natural frequency was *greater* than that of the reed. These experiments showed that the characteristic pitch (or *formant*, as it is now termed) of a vowel sound was given by the reed (*i.e.* the vocal cords), but that the quality was dependent upon the heavily damped vibrations of the air in the tube (*i.e.* the vocal

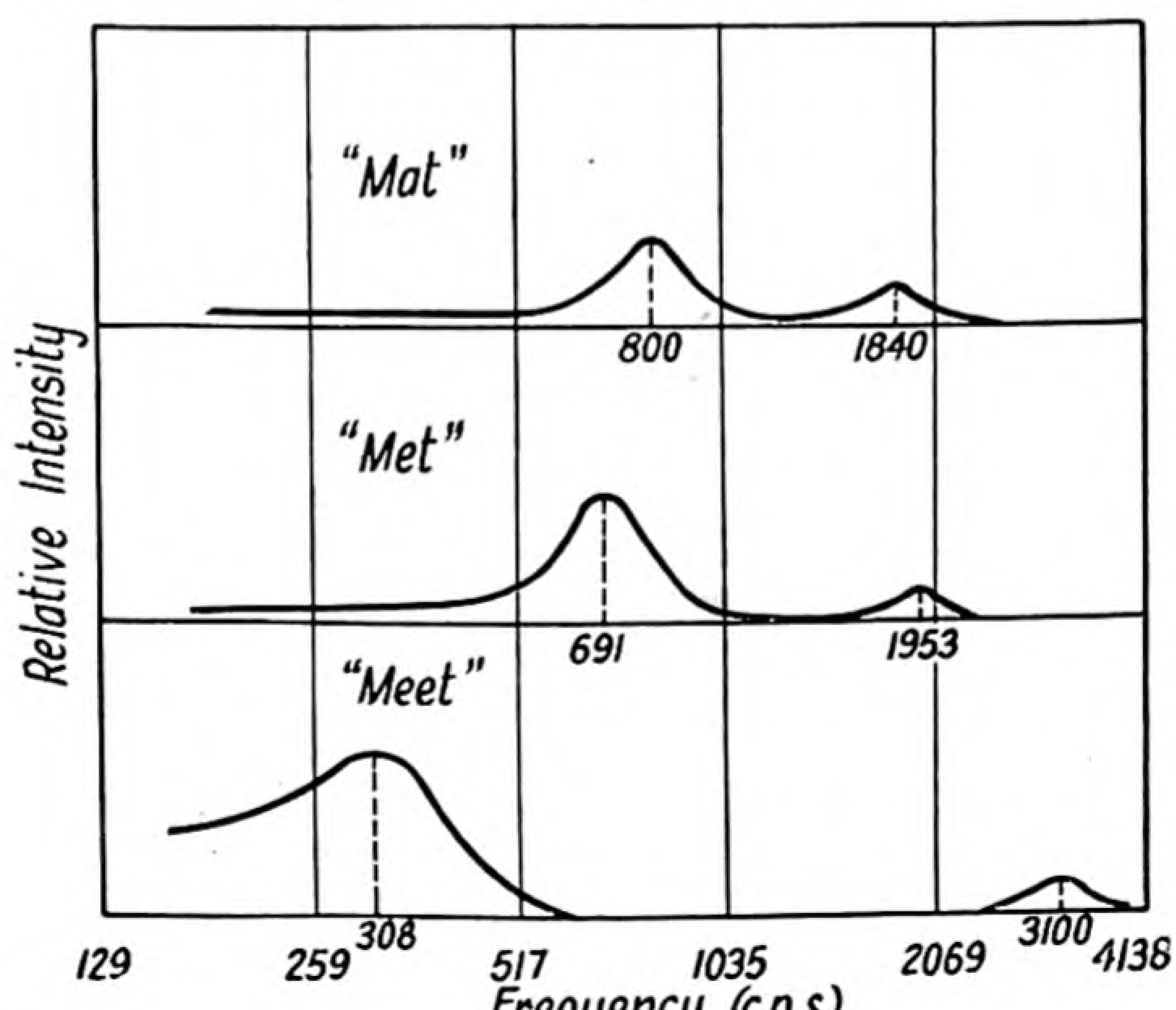


Fig. 10.20.

resonators). More recently D. C. Miller has investigated the composition of *sustained* vowel sounds, and has found that if any member of a certain group of vowel sounds is produced at two different

frequencies, the distribution of energy in the frequency spectrum (Fig. 10.19) is different, but both distributions show the greatest loudness in the same limited frequency range. In a second class of vowel sounds two characteristic frequency ranges are exhibited (Fig. 10.20), which is to be expected if the resonating system of the voice may be regarded as approximating to two Helmholtz resonators coupled together (Fig. 10.12). This result suggests that in the first class of vowels the energy in the upper frequency range is extremely small and therefore less easily observed. It should be emphasised here that since resonance in a system requires a finite time to become established, it follows that the experimental observations of Miller will not strictly apply to the vowel sounds of ordinary speech.

Fig. 10.21 represents observations made by Sir Richard Paget which confirmed that all vowel sounds seem to depend on at least two resonances, which are not sharp and extend over several semitones. Even

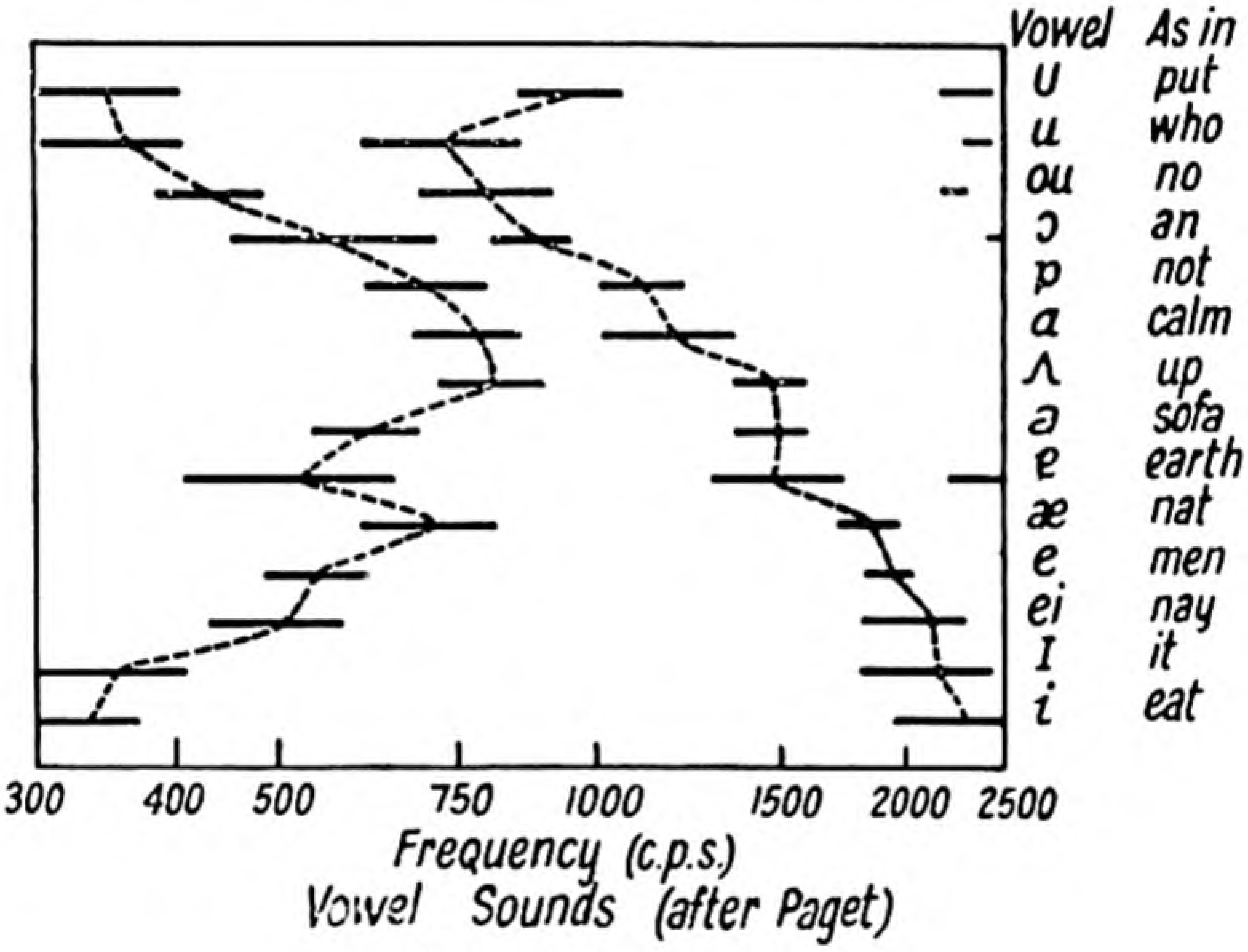


Fig. 10.21.

if two vowels occupy approximately the same particular frequency range, the other resonance is always different. Paget confirmed the earlier observation of Lloyd that the upper resonance is pharyngeal and its pitch may be varied by compressing the pharynx.

In the double-resonator theory the tongue is regarded as dividing the volume of the mouth into two parts, known respectively as the front and rear cavities (*F* and *R* in Fig. 10.12), communication between the two being provided by the channel *C*. If the size of the mouth opening has been correctly chosen, it has been found that all the vowels can be articulated without changing the opening, which is illustrated in Fig. 10.22 by the humping of the tongue of the experimenter, Sir Richard Paget, to produce the various vowels indicated. Using this result and assuming that the total volume of *F* and *R* is constant for all tongue positions, a theory has been developed to account for certain vowel sounds. This theory follows that of the

ideal Helmholtz double resonator with an essential modification to include the possibility of standing vibrations (*i.e.* pipe-resonance).

Summarising, therefore, the simple theory of vowel production suggests that if *different* vowel sounds are spoken at the *same* pitch, it is only the form and size of the vocal resonators, *e.g.* shape of the mouth and tongue position, which require to be altered, the role played by the larynx being to add "carrying power" to the voice. The different shapes of resonant chambers required for the various vowels have been moulded in plasticine by Paget, and by means of these models (Fig. 10.23) he has been able to imitate the vowel sounds with much success.

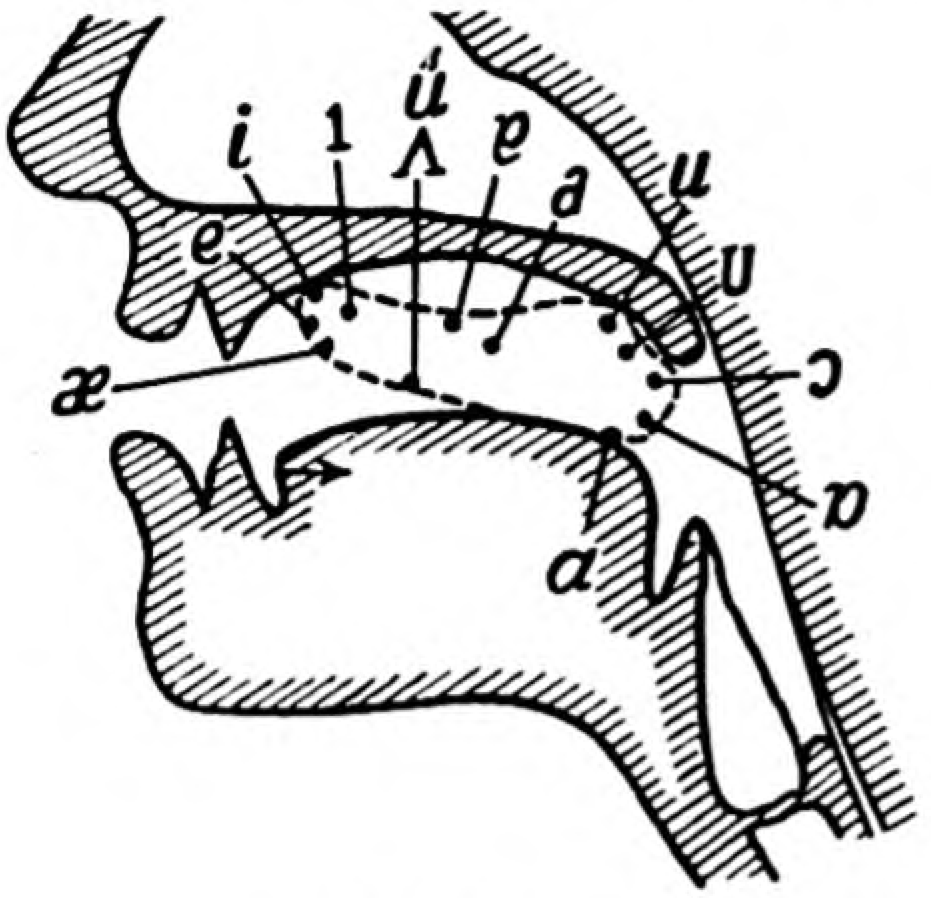


Fig. 10.22. [*After Paget.*

The electro-mechanical analysis of speech, and in particular of words of command, promises to play a prominent part in future developments of acoustics, as, for example, in the application of the human voice to the control of machinery, etc. The Western Electric Company of America have devised an automatic telephone exchange in which the mechanical dialling action is replaced by spoken words, the frequency composition of these being used to generate impulses for operating selector switches. Another idea which is being developed is the operation of a typewriter entirely by the spoken word. The chief difficulty here lies in the fact that it is much more difficult to simulate the frequency spectrum of a consonant than of a vowel. In actual fact, the rate of change of frequency with time is of greater importance for the enunciation of the consonant than the absolute value of the frequency, so that most complex electrical circuits are required for consonants and consonant-vowel combinations. A system of lift control by the use of the human voice has recently been worked out by a British experimenter, Telcs.

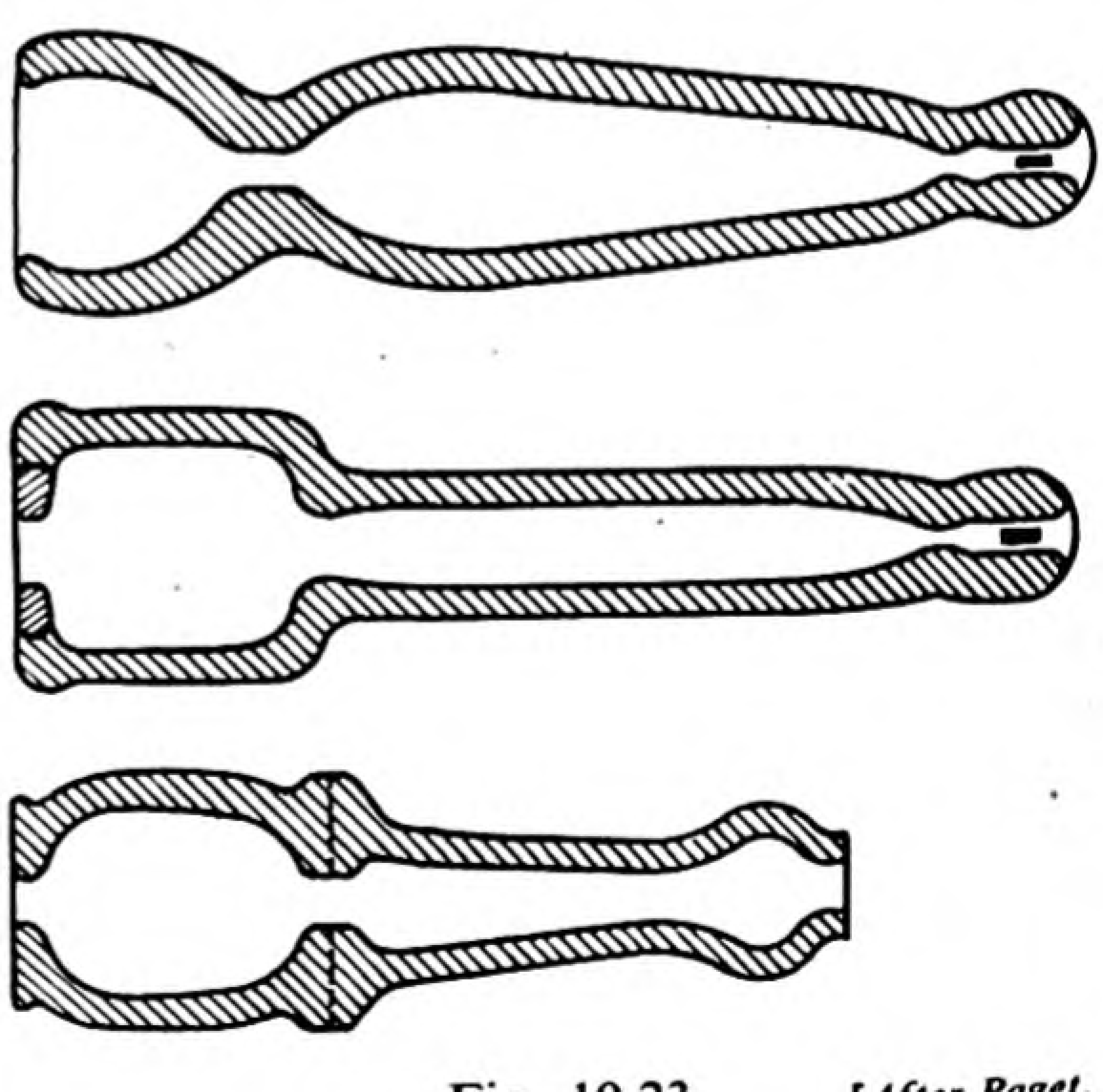


Fig. 10.23. [*After Paget.*

This section on voice production will be concluded with a brief note on the physical aspect of a bird's production of song. A. L. Turnbull has defined the song of a bird as its highest form of vocal expression, which is couched in the form of a modulated theme, with traceable movements and motifs.

The ability of expression in this particular manner is almost solely possessed by what are known as the Passerine group, or "Perching Birds," so that it would seem as if posture and stance have a direct bearing on song production. The vocal organs of the lungs, larynx and lips in a human being have the more or less corresponding counterparts in a bird of the lungs, the syrinx or "sound-box," and the beak. The analogy is not rigid for, as already pointed out, the human tongue is used in conjunction with the lips.

The syrinx of a bird is the organ equivalent to the larynx, but unlike the latter, it is not situated in the throat but in the breast. There are membranes corresponding to the vocal cords within the syrinx, and they are energised by the current of air from the lungs. The main function of the beak seems to be for sound projection, and this property, it is suggested, is used by some birds, *e.g.* the nightingale, to obtain ventriloquial power. On the other hand, the wren uses its beak to obtain expression, while the warbler makes no apparent use of it.

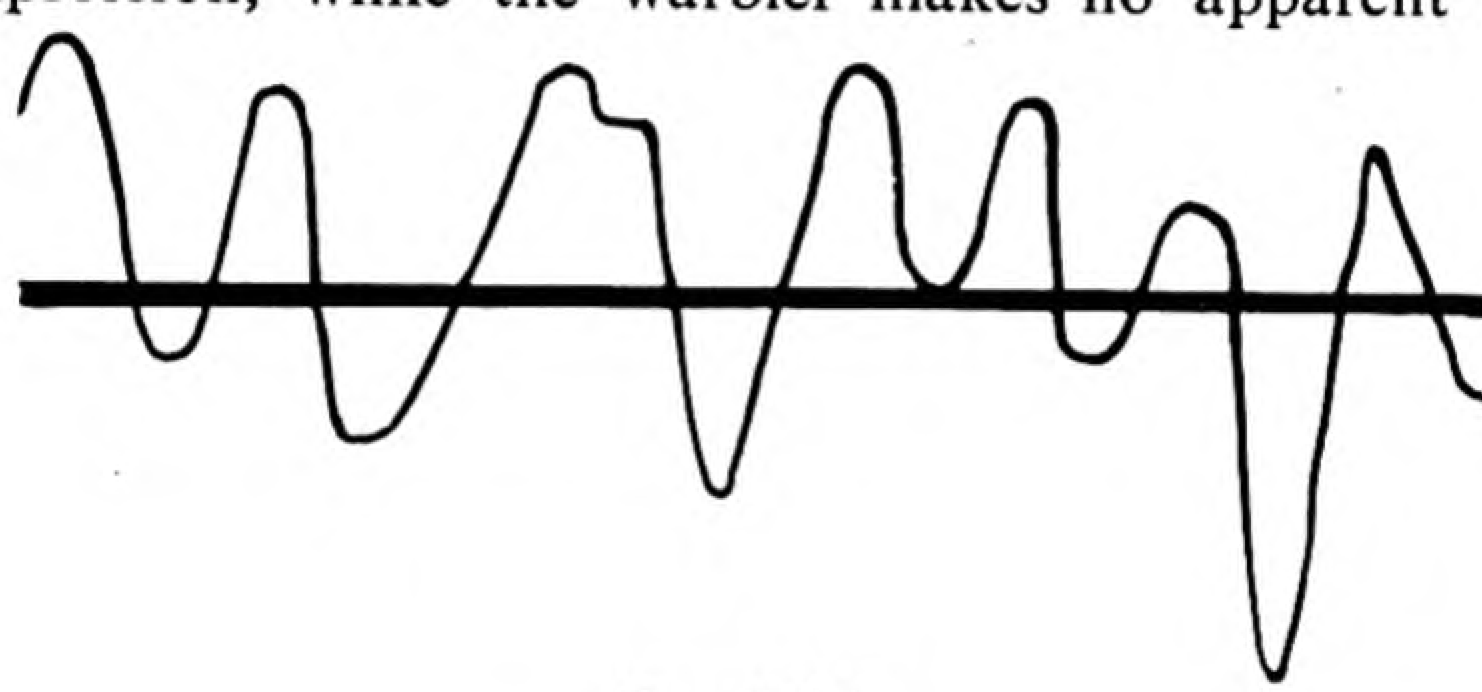


Fig. 10.24.

The mechanism of production of sound by insects is very varied, but that of the grasshopper is of special interest. One of its wing-cases has a serrated edge and on the other is set a ridge, so that when one wing-case is moved over the other the serrations or teeth passing across the ridge cause the wing to quiver. An air-pulse is therefore generated after the manner of Savart's disc and wheel, and the intensity is increased by the provision of a broad, flat surface in the form of a small tambourine set in the wing.

By the use of a "water-proofed" microphone and a cathode-ray oscillograph Dr. Coates, of New York, has recently demonstrated the existence of some form of inter-communication between fishes. Usually any sound forthcoming is attributed to the grinding of teeth, but some fish produce a form of croaking by blowing air from a swim-bladder. Coates was able to make a rough classification of his records according to whether a fish was feeding, excited or at ease; a typical recording on the C.R.O. screen is shown in Fig. 10.24.

For further reading

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CHAPTER 11

WAVE ANALYSIS AND SYNTHESIS

Wave analysis

Wave analysis is the expression of any arbitrary wave-form in terms of its simple harmonic components, which themselves should be fully defined as regards frequency, amplitude and relative phase. Nowadays this wave analysis usually means, ultimately, the analysis of an electrical wave-form, since the time and space variations of essentially non-electrical phenomena, *e.g.* acoustical or physiological, may be easily transformed into corresponding variations of electric current.

The main types of wave-form occurring in practice can be conveniently classified as (i) periodic, (ii) non-periodic but continuous, and (iii) transients. The analysis of these wave-forms are exhibited very conveniently by means of spectrum diagrams (Figs. 11.1 and 11.2), similar to those adopted for specifying the constitution of light sources.

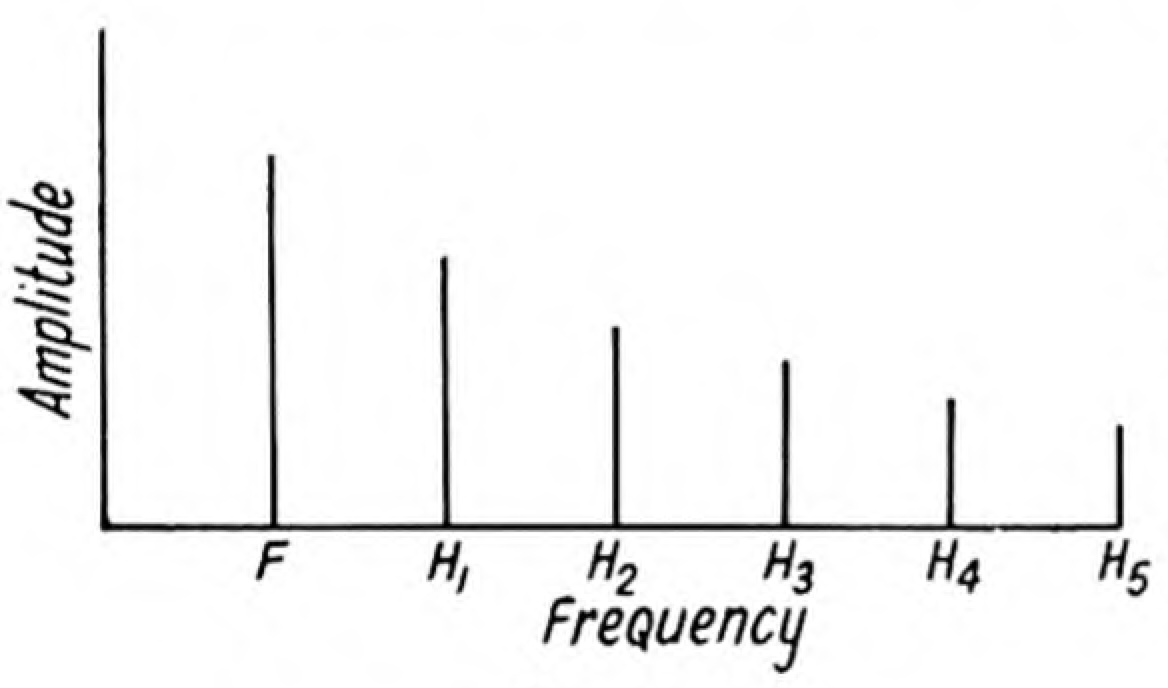

Fig. 11.1.

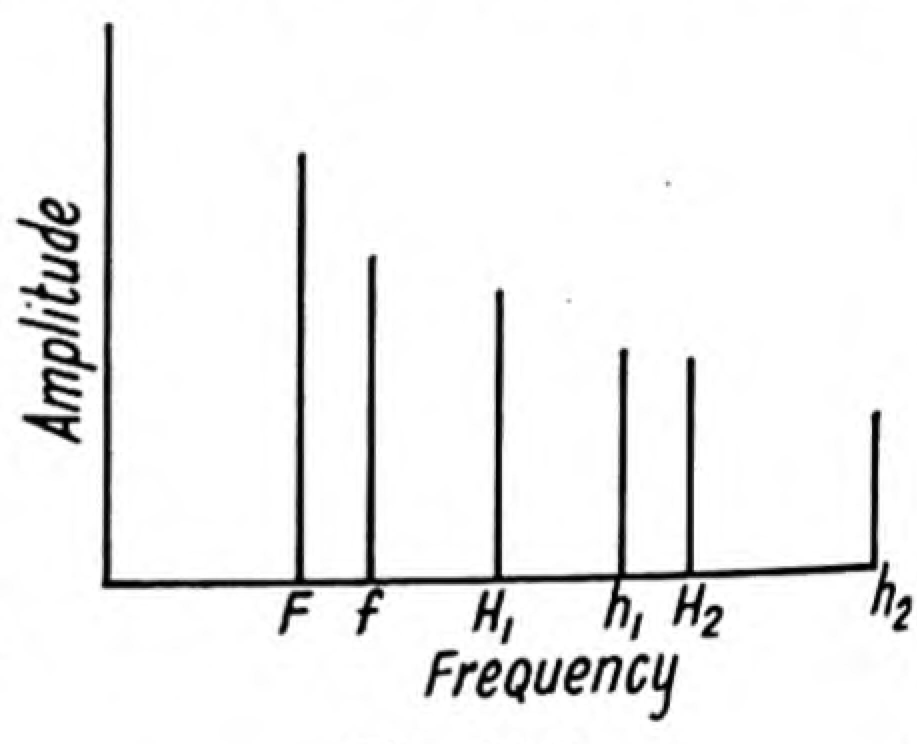

Fig. 11.2.

It must be emphasised, however, that these sound spectra do not provide any information regarding the relative phases of the harmonic constituents. This knowledge, although essential from the point of view of the graphical or mathematical reconstruction of the original wave-form, is not required when considering certain aspects of wave motion and, furthermore, according to Helmholtz and other workers, the *quality* of a musical sound is almost independent of the relative phases of the component harmonic vibrations. The first type of wave-form, the periodic, may be further sub-divided into two classes, (*a*) single, and (*b*) multiple period waves.

Type [i (*a*)] shows *one* well-defined fundamental tone (F in Fig. 11.1) with harmonic overtones H_1, H_2, etc., and is to be found in the notes of musical instruments, in the output of a valve oscillator, etc. The characteristic feature of the multiple period wave, type [i (*b*)], is the existence of two or more periodic functions whose fundamental frequencies do not bear any integral relation to one another. A wave-form of this type would result from the combination of a fundamental

note and an interfering note, and a typical line spectrum would be that shown in Fig. 11.2, where F and f are the fundamental tones of the two notes, and H_1, H_2, etc., and h_1, h_2, etc., are the corresponding overtones.

The wave-forms represented by type (ii) are continuous but non-periodic, and essentially comprise a fundamental tone and its harmonic

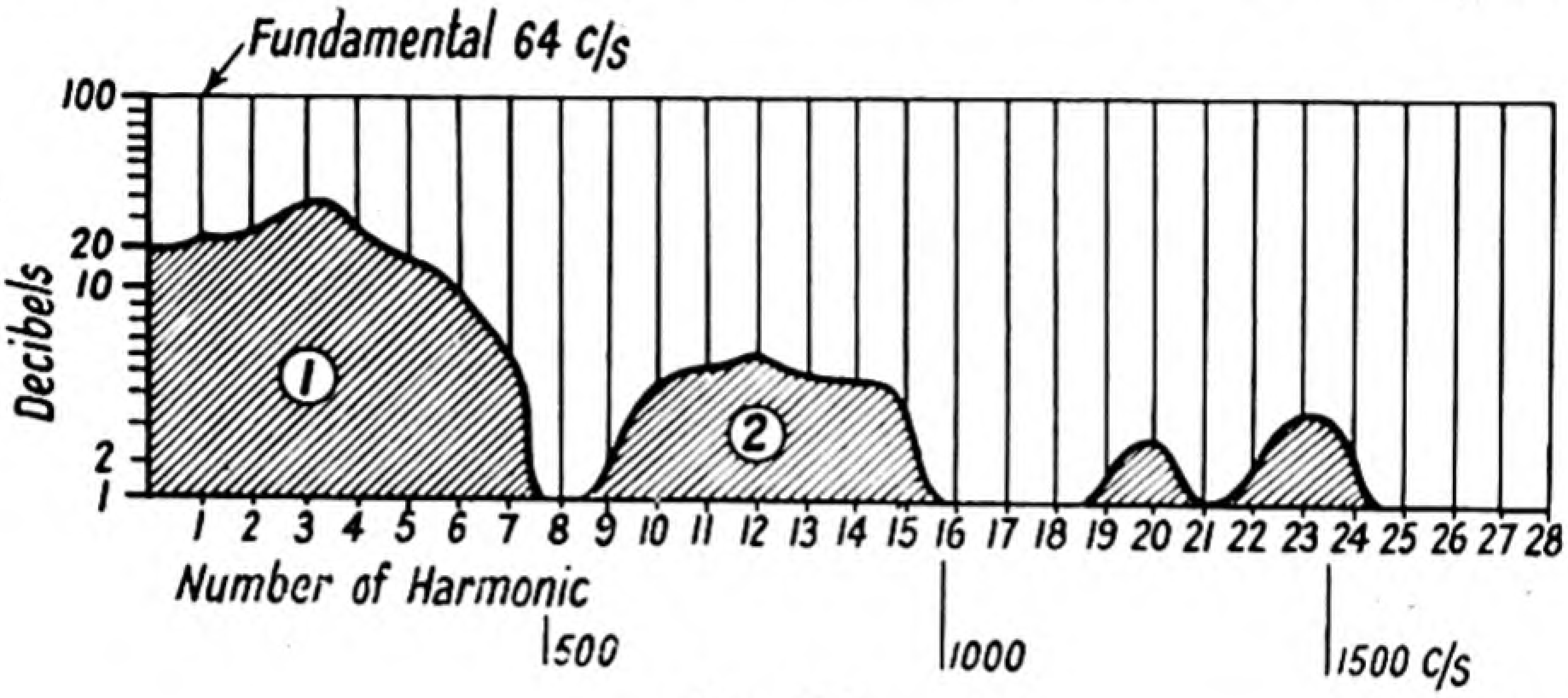

Fig. 11.3. [*After* McLachlan.

overtones *together* with a background spectrum. An example of this type is shown in Fig. 11.3, which shows the spectrum of a note struck on a pianoforte as comprising a line spectrum,* similar to that of the corresponding *bowed* violin string, together with a *band* spectrum due to the impulsive blow of the hammer. Transients [type (iii)] are the result of impulses or of notes of short duration, or may be brought about by a sudden change in the frequency, amplitude or phase of a continuous note. The effect of a *single* excitation of a damped vibrating system would be to produce a *single train* of waves of continuously decreasing amplitude (Fig. 11.4). The spectrum of such a transient will be continuous and usually *infinite* in extent (Fig. 11.5).

The experimental analysis of periodic wave-forms may be approached in two distinct ways: either the sound is recorded graphically at the time it is generated and the analysis is performed at leisure afterwards, or, alternatively, the note is analysed at the time of its production. The latter procedure is particularly advantageous if the wave-form is rapidly changing and is of short duration, as in speech, so that speed of analysis is necessary in order to obtain even an approximation to the instantaneous sound spectrum. It must be remembered, however,

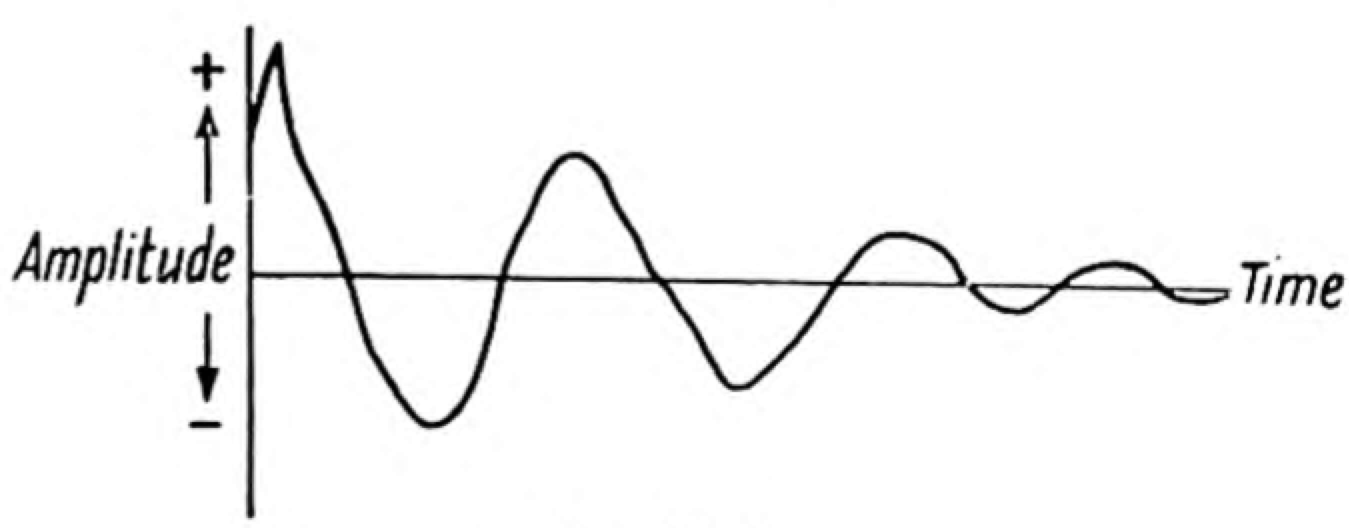

Fig. 11.4.

* The vertical lines represent the line spectrum (without regard to relative amplitudes) of a 64 c.p.s. pianoforte note with its harmonic overtones. 1 and 2 represent the band spectrum.

that the more selective the frequency discrimination of the sound receiving element, the longer the time taken for such a resonating element to build up an appreciable response. As a consequence, in this type of analyser there must be an inevitable compromise between the efficiency of resolution of the component frequencies of the source and the rapidity of analysis. The speed may be increased by multiplying the number of selective receiving elements, and the ultimate limit in this direction is attained in the acoustic diffraction grating which, as constructed by Meyer, consisted of some 300 steel needles, 3 mm. diameter, fixed between parallel iron plates 12 cm. apart to form a concave diffracting system. The distance between adjacent grating elements was about 1 cm. The current to be analysed was amplified and used to modulate an ultrasonic carrier frequency—in Meyer's case it was 45,000 c.p.s. A ribbon loud-speaker was employed so as to obtain approximately cylindrical waves. The ultrasonic sound waves generated, after diffraction by the grating, were received by a condenser microphone moving along a calculated path in order to examine the acoustic spectrum.

Other selective receiving devices employed are vibrating reeds and band-filters, the latter consisting of special electrical circuits which offer a low impedance to alternating electrical currents of a certain range of frequencies only. The acoustic spectrometer of Freystedt (Fig. 11.6) employs a number of band-filters, the rectified output of each one being used to charge up an electrical condenser C; a typical circuit, in which R is the rectifier, being shown in the lower left-hand portion of Fig. 11.6. By means of a rotating switch Sy these electric charges, in turn, modulate a carrier frequency, and after suitable amplification and rectification, produce the ordinate deflections, *i.e.* those in the direction Oy, on the screen of a cathode-ray oscillograph. Since the switches Sy and Sx are coupled together the abscissa deflection at any stud of the deflection-potentiometer circuit, will always correspond to the same band-pass filter, *i.e.* it will represent the frequency appropriate to the ordinate deflection. Although historically out of sequence, mention should be made here of Helmholtz's experiment on wave analysis which was the first with any pretence at quantitative measurement. He determined the magnitude of the overtone pressures in a given complex sound by the combined use of his double-resonator and the Rayleigh disc (Fig. 11.7).

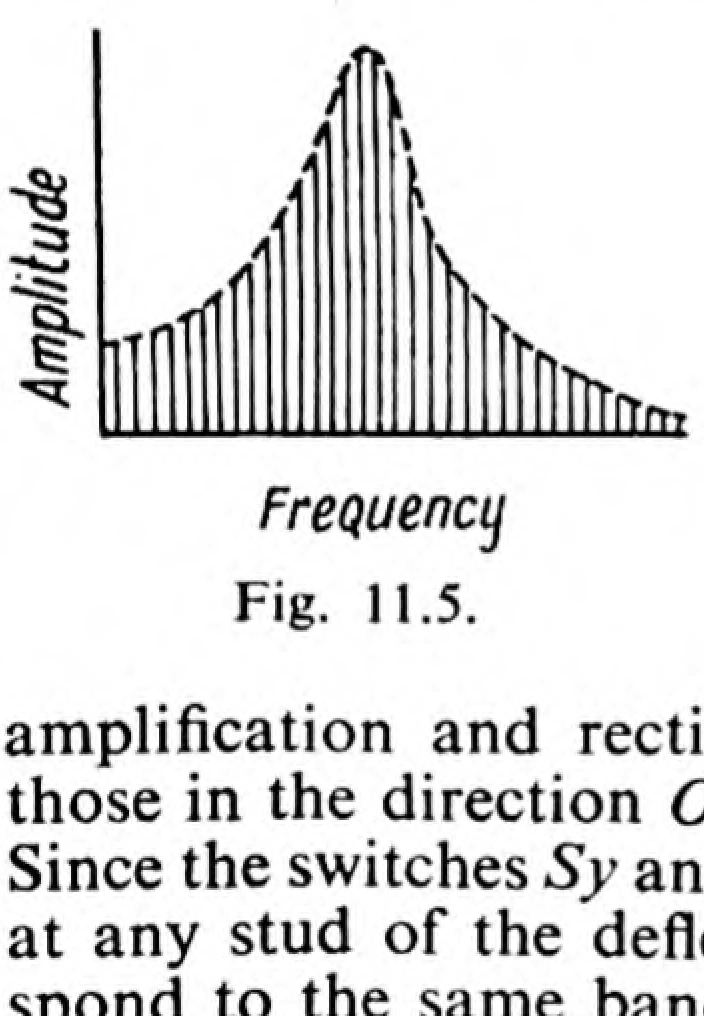

Fig. 11.5.

In the other method of wave analysis, viz. that in which a record of the sound is made at the time of generation for subsequent analysis at a time convenient to the observer, the considerable development in the sound-film industry of recent years has led to such a high standard in film-recording technique that a very close copy of the original waveform may now be obtained very rapidly. Hence, from the analysis of this record, *precise* information may be deduced at will regarding the frequency, amplitude and phase of the harmonic components of the sound.

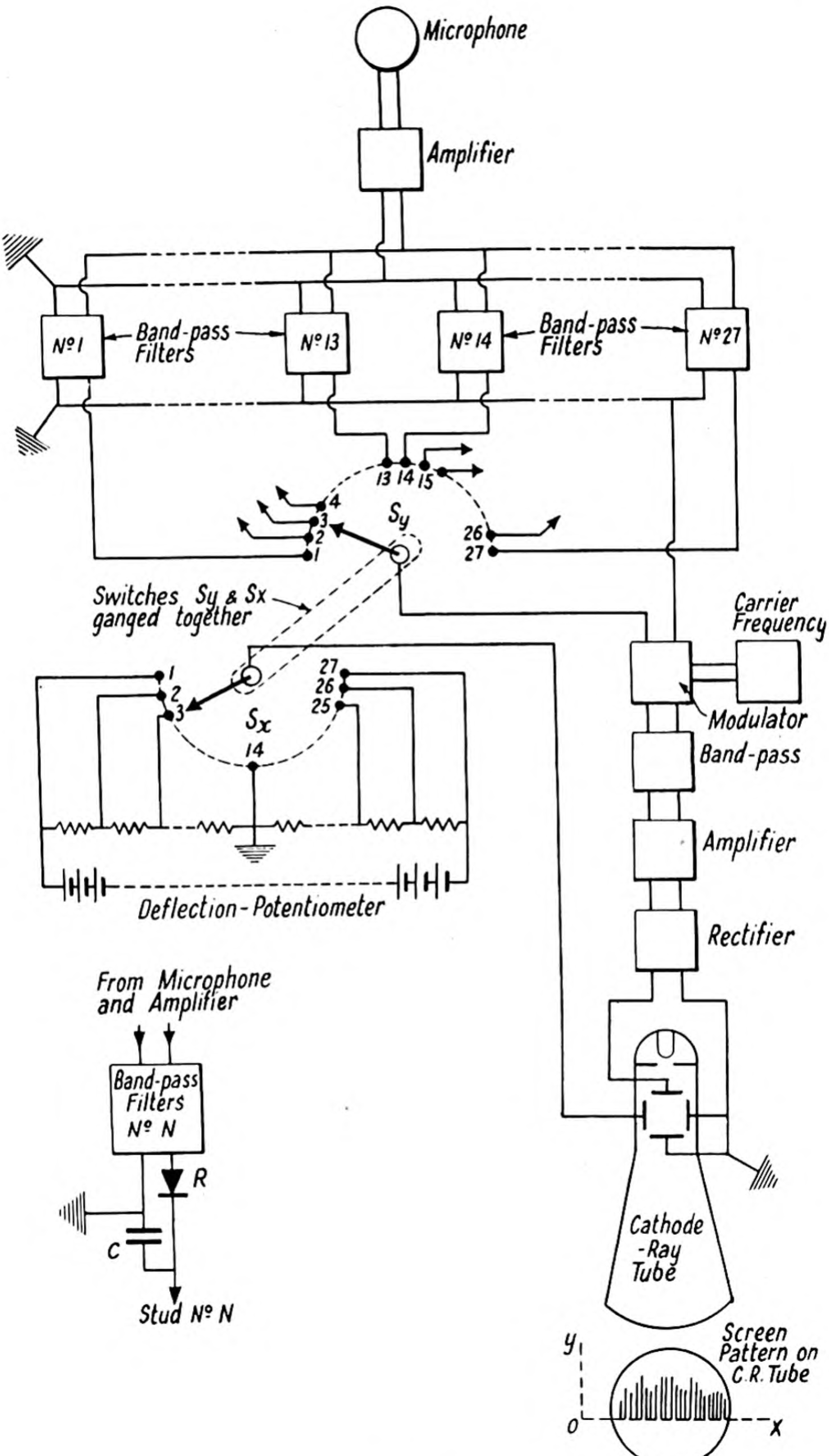
Microphone
Amplifier
Nº 1
Band-pass Filters
Nº 13
Nº 14
Band-pass Filters
Nº 27
13 14 15
4
3
2
1
Sy
26
27
Switches Sy & Sx ganged together
Carrier Frequency
Modulator
Band-pass
Amplifier
Rectifier
1
2
3
27
26
25
Sx
14
Deflection-Potentiometer
From Microphone and Amplifier
Band-pass Filters Nº N
R
C
Stud Nº N
Cathode -Ray Tube
Screen Pattern on C.R. Tube
y
0
x

Fig. 11.6.

Before dealing with the various methods of analysing these sound records, a brief consideration will be given to the classical work of D. C. Miller on the tone qualities of musical instruments as being representative of a solely mechanical method of recording. His apparatus, known as the phonodeik (Fig. 11.8), consisted of a horn *H*

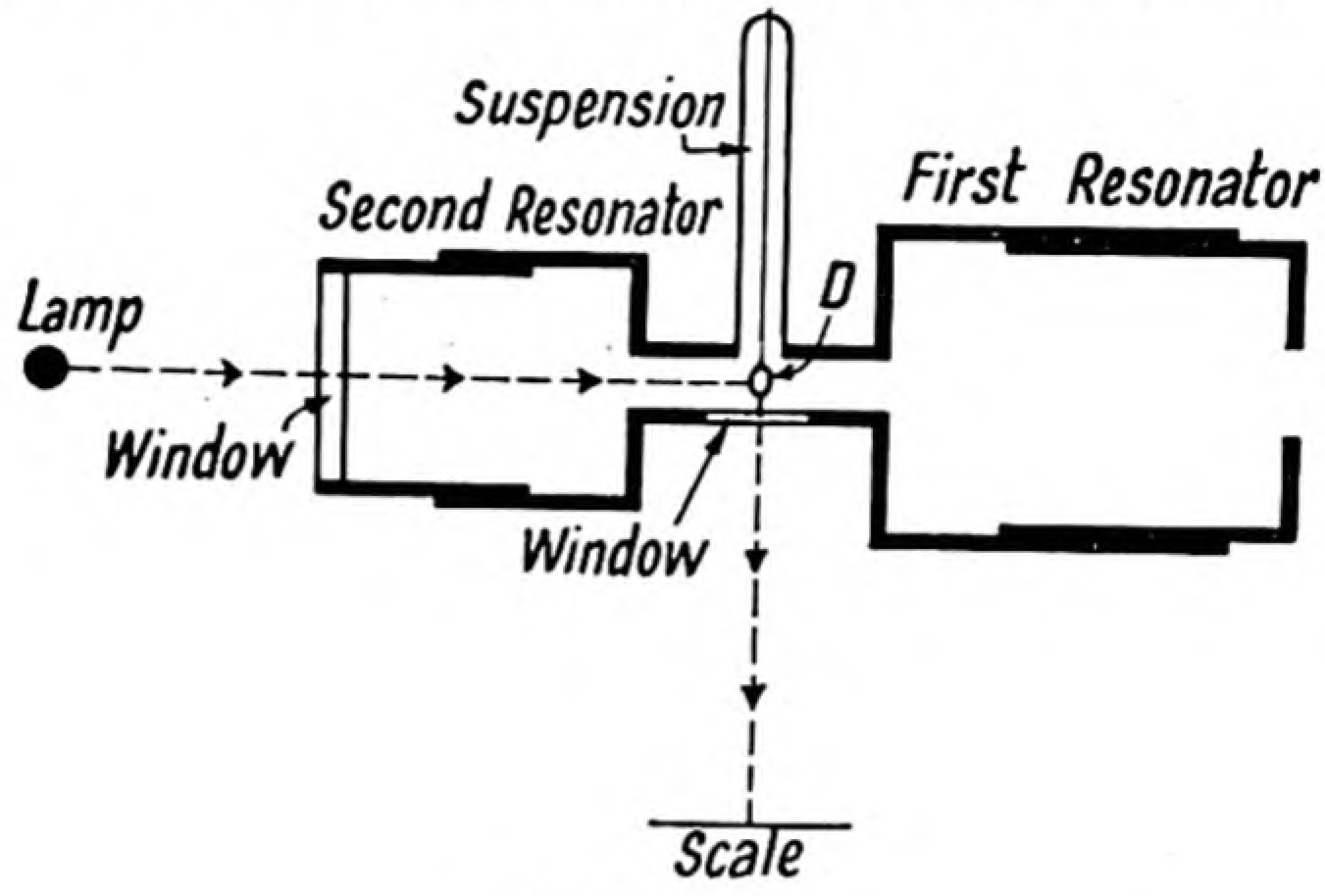

Fig. 11.7.

which was closed at the narrow end by a thin glass diaphragm *D*, upon which were received the sound waves to be recorded. A very fine wire (or thread) *T* is fixed to the centre of the diaphragm, and after passing once around the small cylinder *C*, is maintained taut by attachment to a light spring. This cylinder is fixed to a thin steel spindle which is mounted in jewelled bearings at each extremity, and to the upper portion of the shaft is fixed a small mirror *M*, about 1 mm. square. A beam of light from a small source *S* is reflected from the mirror and focused on a film *F*, which is moved in a vertical direction at any speed up to 40 ft. per sec. Now any rotation of the spindle due to the movement of the diaphragm under the action of a sound wave will result in a horizontal movement of the light image focused on the film. A magnification of 2000, or greater, is easily obtainable so that a diaphragm displacement of 0·001 cm. as produced by sounds of normal intensity will be recorded as a movement of 2·5 cm.

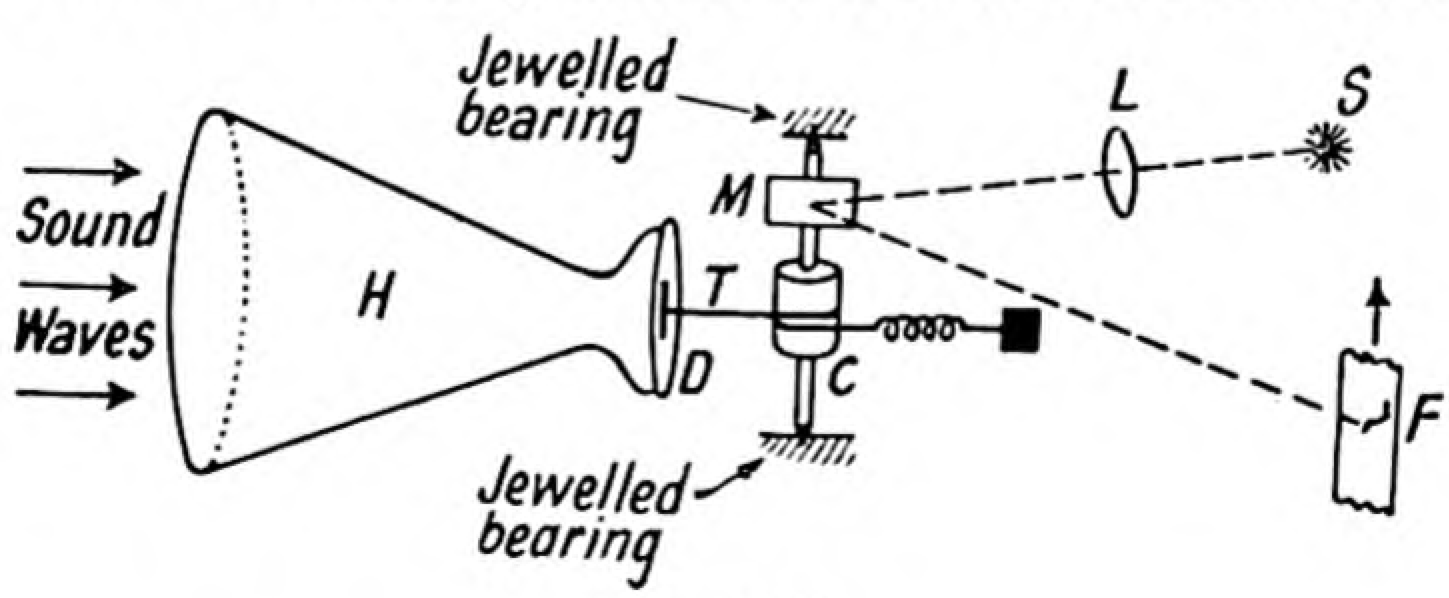

Fig. 11.8.

or more. It is evident that with this large magnification a very high standard of workmanship was required in the construction of the apparatus, to avoid the recording of any small spurious motion of the indicating system. The sound-film method of sound recording is described later in this chapter.

The analysis of a film wave-record may be carried out mechanically (or graphically), electro-mechanically, or by optical diffraction means. The first method, which can be tedious and long if there are numerous harmonics, would involve the Fourier analysis of an enlarged image of the sound record, either by calculation or automatically by the use of an harmonic analyser. In the electro-mechanical method the experimental arrangement is similar to that of the sound-film projection apparatus (p. 202), except for the omission of the loud-speaker system. The electrical output from the amplifier is examined by a standard procedure; a simple form of apparatus, typical of variable-frequency resonator methods, is shown in Fig. 11.9. The E.M.F. (E) under examination is fed into a circuit containing a variable capacitance C, and inductance L (R_L being its effective resistance), and the potential difference across the latter is measured by a valve-voltmeter (*V.V.*), or other suitable instrument of high impedance. This p.d. will be a maximum when the circuit is resonant to one of the sinusoidal components of the complex wave-form under investigation, and the meter reading will be a measure of the magnitude of this component. The particular setting of C for resonance will indicate the resonant frequency from a previous calibration.

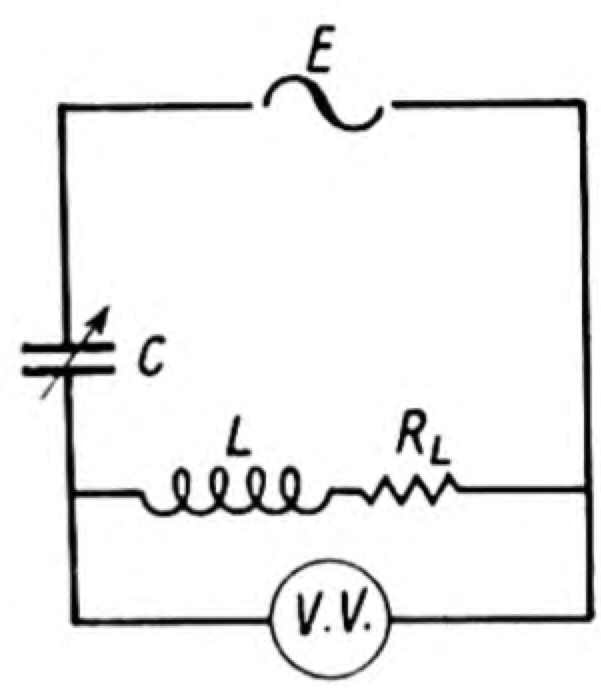

Fig. 11.9.

As regards the optical method of analysis, the underlying principle is to use the film-record as a photographic optical grating, which is illuminated by a monochromatic source of light; but in this case the grating is irregular in spacing and so will give rise to a non-regular diffraction spectra. The theory has been fully worked out by Schouten, and the interpretation of the diffraction spectrum obtained from the sound-film is shown to be most easily interpreted when the film is of the variable-density type (p. 200).

Synthetic sound

The traditional forms of musical instruments are dependent on the limitations of mechanical resonating systems, and this restricts their degree of flexibility of control and the tone colours they are capable of producing. It is evident that from a theoretical point of view, it should be possible to generate electric currents of any desired wave-form, *i.e.* of any given fundamental frequency and range of harmonics, and the consequent transformation of electrical into acoustical energy is nowadays, of course, an accepted procedure. The great advantage of these electronic methods would appear to be their flexibility of control, both as regards frequency and intensity, without recourse to such large structures as pipe organs. Two distinct methods may be employed in the construction of any type of electronic musical instrument, either the desired fundamental and harmonic frequencies are generated separately, and by suitable "mixing" circuits they are combined to form the complex wave-form, or alternatively the complex note is generated directly.

Perhaps the first breakaway from the conventional type of sound generator was the singing arc of Duddell (Fig. 11.10), in which a carbon electric arc was shunted by an inductance L and a capacitance C, the values of which determined the frequency of oscillation of the arc. The earliest type of electronic musical generator was that due to Theremin; it was of a single note type, and depended upon the audio-beat frequency (f_1-f_2) between two ultrasonic valve oscillators of frequencies f_1 and f_2 respectively. This beat frequency could be varied at will, by varying the position of the hand with respect to a vertical conductor projecting upwards from one of the oscillators, *i.e.* by altering the effective capacity, and therefore the frequency, of one of the oscillating circuits. Starting and stopping of the oscillation was effected by means of a switch. Various forms of oscillating valve and neon-lamp circuits have also been designed to produce audio-frequency vibrations, the overtones in general, being generated separately and superposed on the fundamental notes to produce a limited range of tone colours.

In electronic instruments developed to simulate the tones of a grand piano, metal bars are located at various points along each piano string so that small gaps exist between the wire and the bars, and the latter are connected to various excitation potentials.

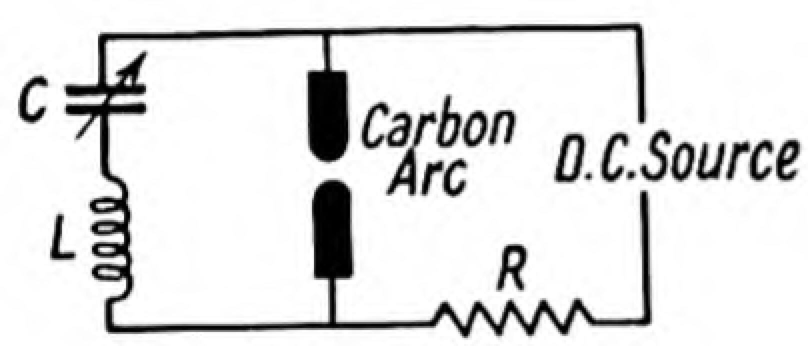

Fig. 11.10.

On setting the wire into vibration by a hammer-blow, the consequent variations of the electrostatic capacitances of these gaps endow the fundamental note with harmonics to an extent which depends upon the positions and potentials of the various "pick-ups." The wires are electrically connected to the grid-circuit of an amplifier and thence to the loud-speaker system. Electromagnetic "pick-ups" have also been used instead of the electrostatic variety.

The chief attention in the development of electronic instruments has been directed to the production of the pipeless organ, with its possibilities of cost and size reduction, compared with the traditional type. Of the various forms of generator suggested, only the multiple oscillator circuit, the wind-maintained reed generator, and the rotary form of generator have really emerged from the experimental stage. A major difficulty encountered in all types is the exact imitation of the requisite starting and stopping transient conditions of a musical tone. In the valve oscillator the stability of its frequency is a most essential consideration, as this is the factor upon which the constancy of tuning of the instrument depends, hence it is most important that the valves and circuit components employed should possess stable characteristics. Coupleux, a Frenchman, built a 3-manual and pedal organ of 76 stops which was used, before the last war, in a Paris broadcasting station. Over 400 valves were employed in its construction, and tuning-frequency adjustments could be made on each note-frequency circuit. A limited range of tone colours was provided by special wave-filter circuits actuated by a stop-key mechanism.

As regards the rotary form of generator there are three distinct

types, photo-electric, electromagnetic and electrostatic, and in the first of these are two principal groupings. In one group the modulation of the beam of light from the source is effected in the manner of the talkie film, previously described, *i.e.* a variable-density (or variable-area) mask of the desired wave-form is moved in front of an illuminated stationary slit, and the transmitted light beam is focused on a photo-cell and the photo-electric current after amplification is fed into the loud-speaker. In the second group the mask is kept stationary while a series of slits, S_1, S_2, S_3, etc., having a constant fundamental separation, move in front of a mask as indicated in Fig. 11.11. The mask shown is cut out according to the polar equation of a sine wave, so that as the disc turns about its centre O, the amount of light transmitted at each moment is proportional to the height of the slit cut off by the curve ABC. The separation of the slits is such that when one slit has traversed the stencil, the other is just about to enter at the other side. The frequency is regulated by the speed of the motor, so that if this is r rev. per sec., and n is the number of slits in a circle, the fundamental frequency is nr c.p.s. A complex wave-form may be cut out in a single mask, or built up by means of stencils of the various sinusoidal components and placing them one above the other. Alternatively, by displacing the stencils with respect to one another, a very convenient means has been provided for investigations of the effect of phase on sound perception.

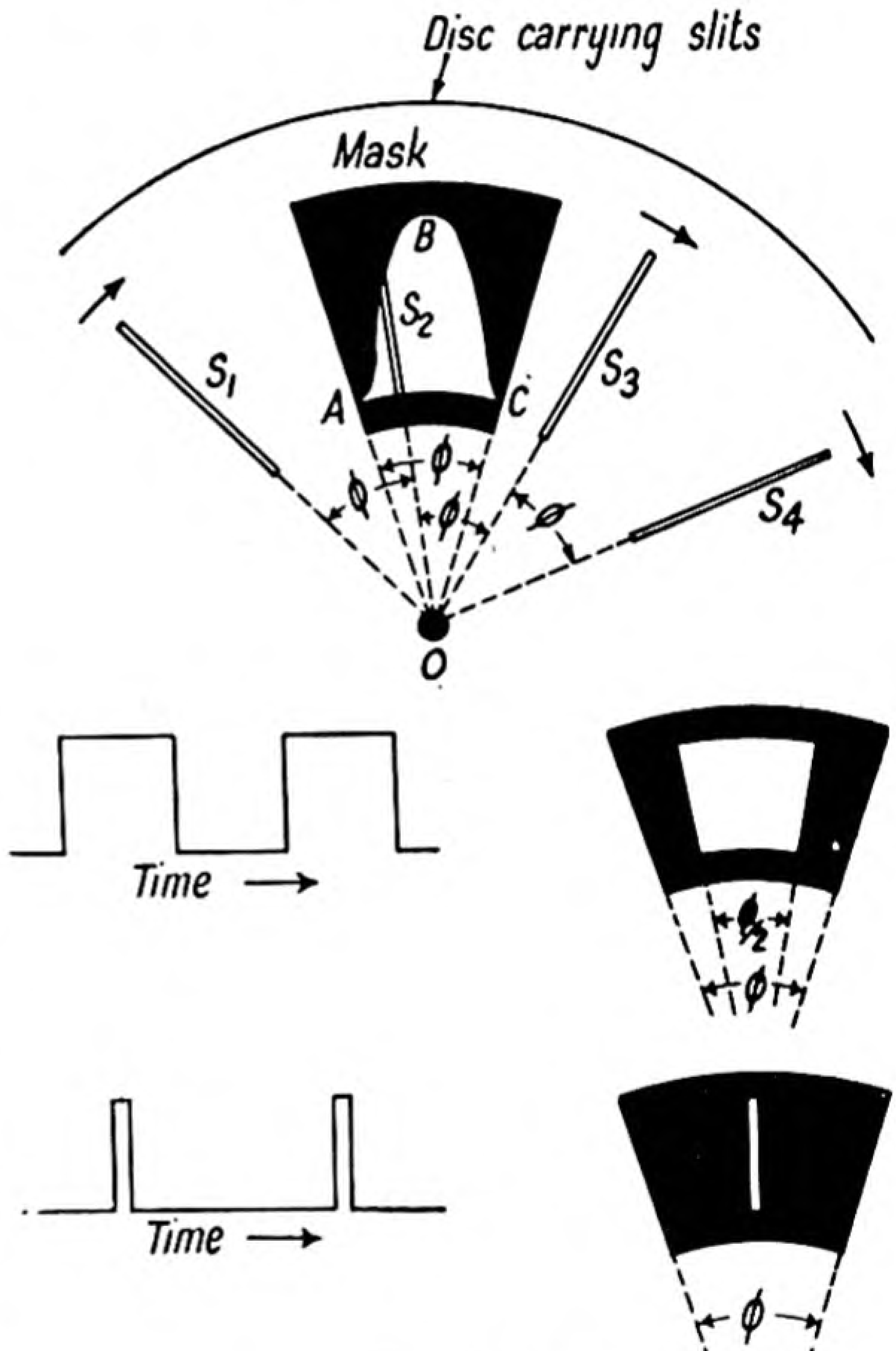

Fig. 11.11.

The most serious source of error in the above form of generator is inherent in the finite width of the moving slits, *i.e.* the apparatus has a limited resolving power (*cf.* optical instruments and the analysis of the fine structure of luminous spectra). The stencils required for a "square-wave" and a narrow rectangular "pulse" are shown respectively in the middle and lower parts of Fig. 11.11. The greatest divergence between a desired wave-form, and that actually obtained in practice has been found to occur with the "square-wave," and the error consists in the slight slant given to the horizontal portions of the wave, which is conveniently shown on the screen of a cathode-ray oscillograph.

This error may be traced to an inherent defect of amplifiers at zero frequency. In a photo-electric organ of this type, devised by Toulon, the rotating disc has a series of concentric rings, and each contains a number of slits and has its associated masks, the number of slits in each successive ring being made proportional to the frequencies of the chromatic scale.

The Hammond organ is probably the best known of the electromagnetic generators, and a typical unit of such an instrument is shown in Fig. 11.12. It consists of a small iron disc, about the size of half-a-crown, with a suitably shaped periphery. When the disc is rotated in front of an electromagnetic "pick-up," and a high point on the edge passes the pole of the permanent magnet, a momentary current is induced in the "pick-up" coil. The frequency of the alternating current produced will depend upon the number of "corners" on the wheel and the speed of rotation. In order to obtain the requisite number of partials for the attainment of the desired quality of reproduction, some ninety of the iron discs are mounted on a common shaft driven by a synchronous motor. By combining the "pick-up" currents of various frequencies in controllable proportions, it should be possible to produce any desired complex wave-form. The various currents are "mixed" in the harmonic controller of the organ, the *final* electrical current associated with the complex wave-form is then amplified to the desired extent, before conversion into sound waves via the loud-speaker system.

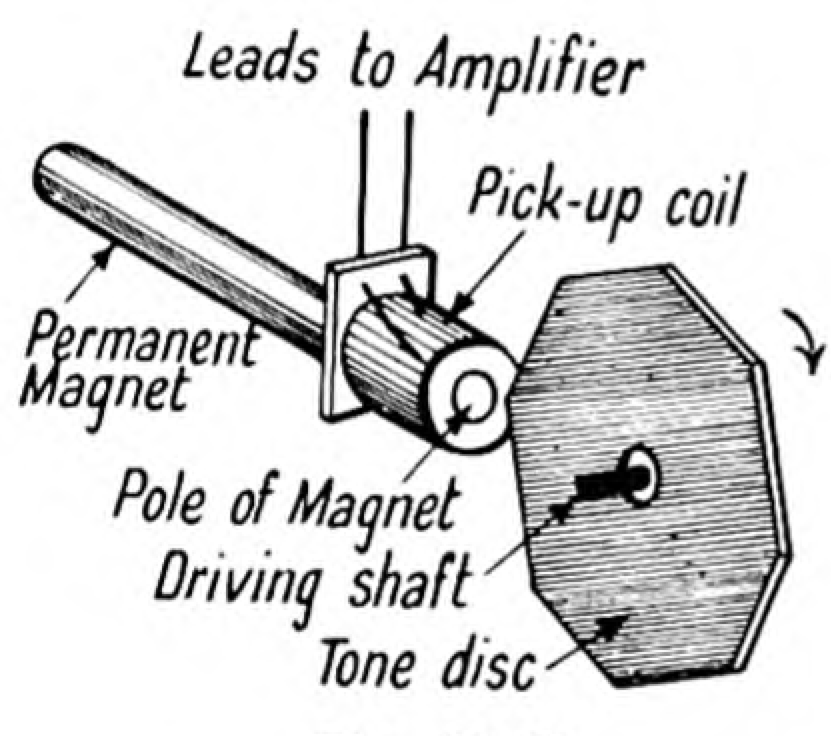

Fig. 11.12.

The electrostatic type of rotary generator is employed in the Compton and Midgley organ; the general principle of action is illustrated by the early apparatus of Bourn, in which there are a number of concentric rings on a stationary disc of insulating material. One edge of each ring is serrated in the form of a sine-wave and above it, a short distance away, moves a flat metal arm with its axis of rotation passing through the centre of the disc. The metal arm and ring therefore form a capacitance which varies sinusoidally as the arm is rotated. If there are a series of these rings, each connected to a suitable excitation potential, then it is evident that the means are available for generating any desired complex wave-form.

The type of rotary generator of those described above which offers the greatest promise is undoubtedly the electromagnetic type, for besides its simplicity and economy of construction, its electrical output is much higher than for the others, and so less amplification is required.

Requirements for perfect reproduction

The ideal objective to be attained is for the sounds heard and the original sounds to be indistinguishable. In other words, the complex sounds produced by the reproducer should contain all the source

tones at their original relative intensities, and this rendering should be independent of the general intensity level. These two conditions may be expressed concisely as follows. If I_0 is the original intensity of a particular tone, and I_r is its reproduced intensity, then the condition for no "frequency distortion" is that the ratio $\frac{I_r}{I_0}$ should be constant over a wide range of frequencies. Secondly, the ratio should be independent of the loudness or intensity I_0 of the sound, and if this condition is not fulfilled "amplitude distortion" will occur. This departure from linearity of response to intensity becomes more apparent at high energy levels, when an additional defect is brought about by the setting up of component frequencies not present in the original sounds. The intensity level of the reproduced sound will be partly governed, however, by the type of music and the size of enclosure in which it is to be reproduced. If the intensity is too small the lower frequency notes fall below the threshold of audibility and the quality of reproduction suffers from deficiency in bass. Such an effect is equivalent to the auditor moving farther away from the sound source. The most difficult sounds to reproduce faithfully are those notes in which the overtones are not necessarily simple multiples of the fundament frequency. Such notes usually occur with percussion instruments, *e.g.* cymbals, drums, etc., and owing to their physical nature, they are of shorter duration than ordinary musical notes, hence they are termed "transients." The faithful reproduction of transients will depend upon maintaining the correct duration time of each of the tones, for if unduly prolonged they lose quality and the reproduction becomes "blurred." A further condition for true rendering is the correct phase relationship between the constituent overtones.

Compression and expansion

In the electrical reproduction of sound the contrasts obtainable in speech and music are limited by the relative value of the intensity of the strongest sound which can be reproduced without distortion, to that of the inevitable unwanted noise, interference or hum incurred in the different links of a mechanical or electrical system. In other words, for reproduction to be satisfactory it is essential that the weakest acoustical vibrations should be rendered at a sound intensity level appreciably higher than that of the unwanted sound, yet at the same time the most intense sounds should not produce overloading of the reproducing apparatus.

The human ear can perceive intensities over a range of approximately 130 db. between the audibility limit and the threshold of pain, but such a range is found to be unnecessary for satisfactory reproduction, and for the transmission of music an intensity range of 60 db. suffices. This range, however, is in excess of that permissible in most electrical transmission systems, *e.g.* it is of the order of 40 db. for broadcasting transmitters, hence means have to be adopted to artificially reduce the contrasts in the signal to be transmitted. This result may be achieved, for example, by controlling the amplification between the microphone and the transmitter, so that it is large for the feebler sound intensities but small for the strong intensities. This

type of regulation is known as compression, and for a correct reproduction of the original intensity spectrum the regulation at the receiving end should be the exact inverse of the compression, and is correspondingly called expansion.

As few receivers are furnished with "expansion" apparatus the form of the compression regulation, *i.e.* its characteristic, has usually, therefore, to be chosen so that the reproduction closely simulates the original sound, even in the absence of expansion The form of this compression characteristic will depend upon the actual character of the music; the regulation is often, however, carried out manually, the degree of amplification being adjusted by hand to suit the intensity of the sound. In this connection the *rate* of regulation has to be carefully chosen in relation to the characteristics of the ear.

Sound recording and reproduction

The earlier types of sound record were cylindrical, and the indentations made in the outer surface of the cylinder corresponding to the sound recorded were radial in direction. The instrument used for playing this type of record is properly known as a phonograph. The reproduction was notably poor in bass, but considerable improvement was effected with the introduction of the flat disc record, although the piano records still showed much "tinniness." The instrument associated with the disc record is known as the gramophone. In these early methods of recording the energy required for the tool cutting the wax record was derived solely from the sound waves produced by voices or musical instruments. A conical horn, with the performers grouped closely around it, was used in order to concentrate the available energy, and furthermore, orchestral instruments, such as the "Strop" type of violin, which incorporates a diaphragm within a flared horn, were adopted with the same end in view. The main deficiencies of this method of recording may be summarised as a restricted frequency range, and the existence of resonances which cannot be lessened by damping, as the acoustic output is already quite small.

The advent of electrical recording, round about 1926, overcame many of the previous difficulties. The method depends upon the conversion of the sound energy, picked up by a microphone, into electrical currents which are then amplified to the required extent for the operation of an electromagnetic cutter, and furthermore, with this increase in available energy, it becomes no longer necessary to use special types of musical instruments, or to group the artists immediately around the sound receiver. The apparatus initially employed in the reproduction of sound from a record was essentially a mechanical device known as a sound-box. It was effectively a diaphragm actuated, through a bar, by the vibrations of the needle (or stylus) in the groove of the record. In the radio-gramophone the movements of the needle are converted, through a magnetic or crystal pick-up (cf. p. 241), into variations of electric current or potential respectively, which are then suitably amplified before conversion into sound by loud-speakers.

There are two types of sound-tracks, depending on whether the motion of the "cutter" (and hence the needle or stylus) is from side

to side, *i.e.* lateral, or up and down, *i.e.* vertical. The following analysis is applicable to both systems. It is shown (p. 42) that the energy associated with a S.H.M. is proportional to $a^2\omega^2$, where a is the amplitude and $\frac{\omega}{2\pi}$ the frequency of the vibration. Hence it follows if the amplitude A of the "cut" in the record is independent of the frequency of cutting but is $\propto a$, that $A \propto \frac{1}{\omega}$ or $A\omega$ = constant, for a given input energy, *i.e.* a constant value of $a^2\omega^2$. Now $A\omega$ is the velocity amplitude of the recording stylus, hence apparatus designed on this principle are known as *constant-velocity recorders*. The limitations imposed at low frequencies will be due to the large groove amplitudes and consequent wider spaces between grooves leading, in lateral recording, to a shorter playing time of the record. Higher frequencies will be limited by the dimensions of the groove becoming comparable with the random variations constituting surface noise. It should be noted that for perfect reproduction it is not essential for the form of the sound tracks to be consistent with the above property, provided the *reproducing* instrument could exactly compensate for any departure from this condition in the recording. In *constant-amplitude* recording $\frac{A}{a}$ is made directly proportional to ω, and so for a constant energy input, the amplitude A is made independent of frequency. In commercial practice constant-amplitude recording is sometimes employed below 300 c.p.s. and constant-velocity recording above this frequency.

Both the lateral and the vertical (known as the "hill-and-dale") cut records have their relative advantages and disadvantages. The former had become the more popular method, although there has been a recent development in "hill-and-dale" recording. Although the large amplitudes required to record the bass notes in constant-velocity recording are more easily attained when the groove cutting is horizontal, *i.e.* lateral, yet in order to prevent undue bending of the needle stylus, a limit is set to the size of cut made during the recording, *i.e.* there is an amplitude limitation which means that the complete intensity range of an orchestra may not be adequately covered. The design of a suitable *reproducing* unit for "hill-and-dale" recording that would follow faithfully the undulating surface of this type of record presented some difficulty, but was overcome by analysing the equivalent electrical circuit (Fig. 11.13*b*) of the mechanical system (Fig. 11.13*a*), the equivalent circuit elements in the two systems being readily noted from the diagrams (see also Chapter 15). The E.M.F. in the electrical circuit corresponds to the mechanical force exerted on the stylus by the revolving disc. It may be briefly stated here that the fundamental condition which has to be satisfied is that the product of the mechanical impedance of the unit and the *vertical* velocity of the stylus at its point of contact with the disc must always be less than the force holding it on the record. This version of "hill-and-dale" recording is said to give far superior renderings than even a wide-range sound-film, and its scratch is asserted to be much less than for the other type of disc record.

Electrical pick-up

This instrument is the agency by which the mechanical vibrations of the stylus in the groove of a gramophone record are converted into corresponding variations of electrical potential, these latter being then applied to the input of a valve amplifier, the output of which is used to energise a loud-speaker. In the electromagnetic type of "pick-up" the movement of the stylus is arranged to move an iron armature relative to a coil, thereby causing a varying E.M.F. to be induced in the latter; alternatively the coil of wire itself may move in a fixed magnetic field. In the electrostatic pick-up a Rochelle salt crystal element is employed which is completely sealed in a Bakelite housing, but the head as a whole is quite light in weight compared with the magnetic type; and so the gramophone records are subjected to less wear.

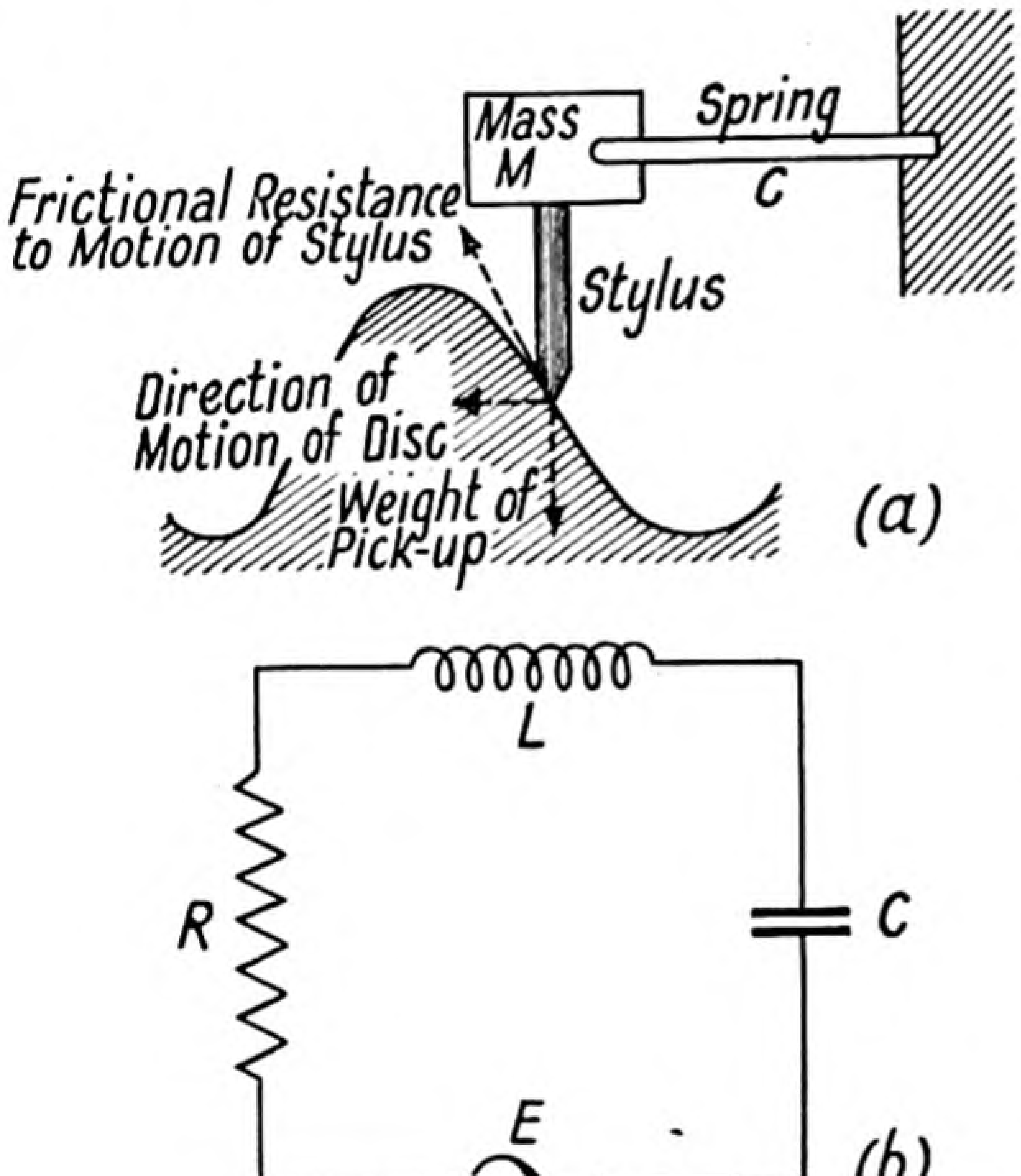

Fig. 11.13.

Sound-film

With the introduction of the modern "talkie" film the sound-film method, in which the sound is recorded photographically, has been almost universally adopted in preference to the gramophone-disc method of recording. An obvious advantage is that the sound-track may be photographed alongside the pictures themselves, as indicated in Fig. 11.14. Two main methods of procedure are used, the corresponding films being known as the variable-density and the variable-area (or width) track film respectively, but in either case, the underlying problem is to focus an illuminated slit upon a moving photographic film so that the light falling on the sensitised surface is modified in some way by the sounds to be recorded. In the variable-density method this effect is attained by amplifying the electrical output of a microphone by as much as 10^8 to enable a special gas (helium or neon) discharge tube to be actuated (Fig. 11.14). This lamp is operated at a gas pressure where the intensity of its illumination is particularly sensitive to variations of electric current. The narrow slit is very close to the film, so that an image only about $\frac{1}{10}$ in. long and $\frac{1}{1000}$ in. wide is focused on the moving film, and on the development of the latter the sound-track will be seen to consist of a series of narrow horizontal lines of constant length, but of different densities. The spacing apart of these lines will determine the pitch of the sound,

while the loudness will be defined by their density. In an alternative variable-density method a lamp of *constant* brilliancy is used to illuminate the film through a slit whose "opening" varies in step with the intensity of the sound being recorded. A typical arrangement is shown diagrammatically in Fig. 11.15*a*, where a loop of metal strip

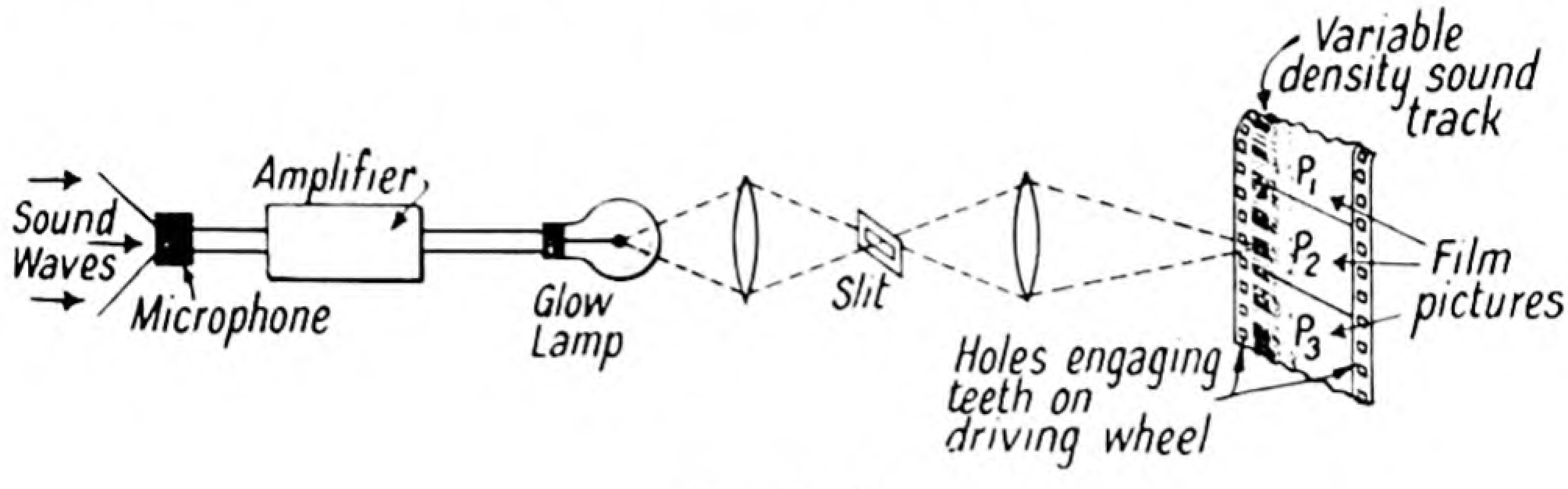

Fig. 11.14.

is maintained in tension so as to leave a very minute gap (of the order of $\frac{2}{1000}$ in.) between the inner edges. On the passage of the A.C. output from the microphone through the strip, the two portions of the loop move in opposite directions, so that the photographic film is exposed to a varying degree dependent upon the sound intensity. Fig. 11.15*b* illustrates the principle of the variable-area method; a metal wire or strip conducting the microphone current is set at an angle to the slit and is perpendicular to a magnetic field. It is evident that the force on the conductor and hence the area of slit covered depends on the current flowing, *i.e.* on the intensity of the source of sound.

The use of photographic film as a sound-recording medium brings into prominence the necessity for a very high standard of quality, for whereas the picture of a sound-film is enlarged by, say, 300 times when projected on the cinema screen, the minute photo-electric currents resulting from the light transmitted by the sound-track are magnified by a factor of 10^7 or 10^8 by the time they actuate the loud-speaker. Consequently the smallest particles of dust on the transparent part of the film will give rise to an appreciable background noise. In the case of variable-density records this defect is minimised by giving a preliminary short exposure to the whole of the sound-track portion of the film. To overcome the same difficulty with variable-area recording it is usual to employ double sound-tracks, obtained by giving a mask an up and down movement in synchronism with the sound vibrations. The mask has a **V** or **W** "cut-out" as indicated in Fig. 11.16, and by this means the

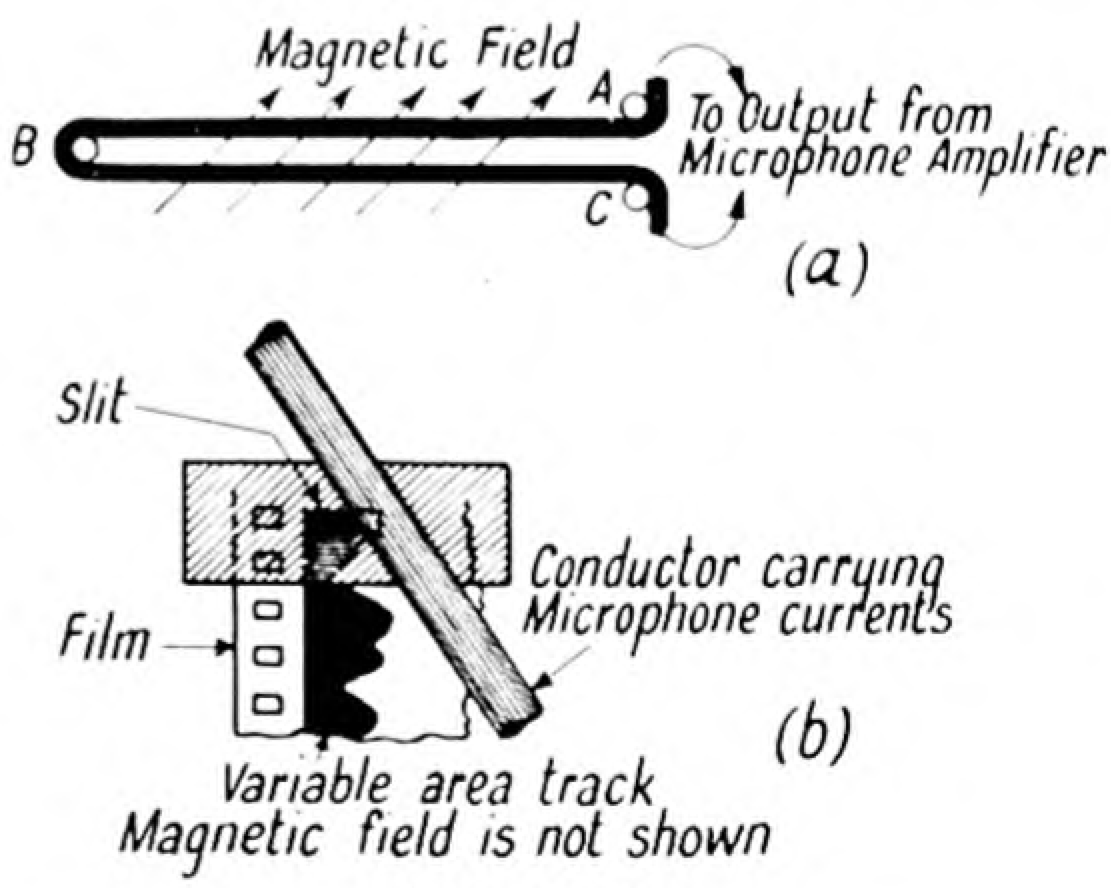

Fig. 11.15.

slit is illuminated to an extent dependent upon the amplitude of vibration of the mask, and hence of the sound waves. The width of the transparent gap in the sound record is automatically controlled, so that at any instant it is only just broad enough to contain the *relative* variations of the contours of the vibration curves. In this way the effect of the presence of dust particles is minimised, as otherwise they would have the greatest effect on the weaker signals, for in the normal sound record these possess the wider gaps.

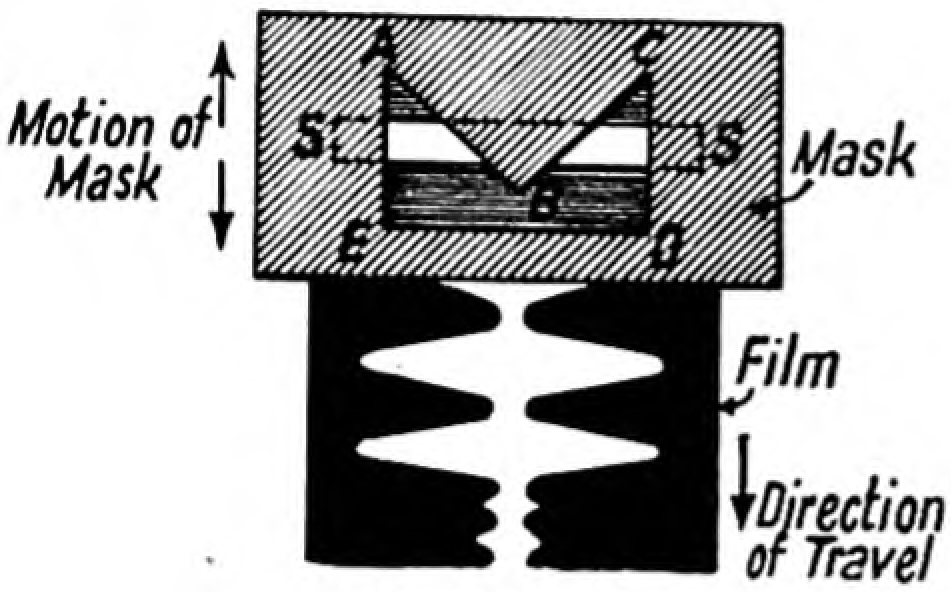

Fig. 11.16.

The sound projector section of a sound-film unit is shown diagrammatically in Fig. 11.17. The motion of the film through the picture projector (not shown) is intermittent, each picture remaining stationary for $\frac{1}{24}$ sec. so as to produce the illusion of movement, but through the sound projector the film must move continuously and at exactly the same speed at which the sound was originally recorded or otherwise the pitch of the projected sound will differ from that of the original. Hence, for this reason, the picture and its counterpart sound-track are separated.

An interesting application of sound-film technique is the speech-on-light signalling apparatus used by the Germans during the late war. The light beam is modulated after emission from the source, and the sending and receiving systems are combined in one apparatus, comprising the light source and modulating device with lens for focusing on the distant station, together with a photo-cell and amplifier for reception. By the use of an infra-red filter the risk of detection is lessened without any appreciable reduction of the range of communication, which is of the order of five miles.

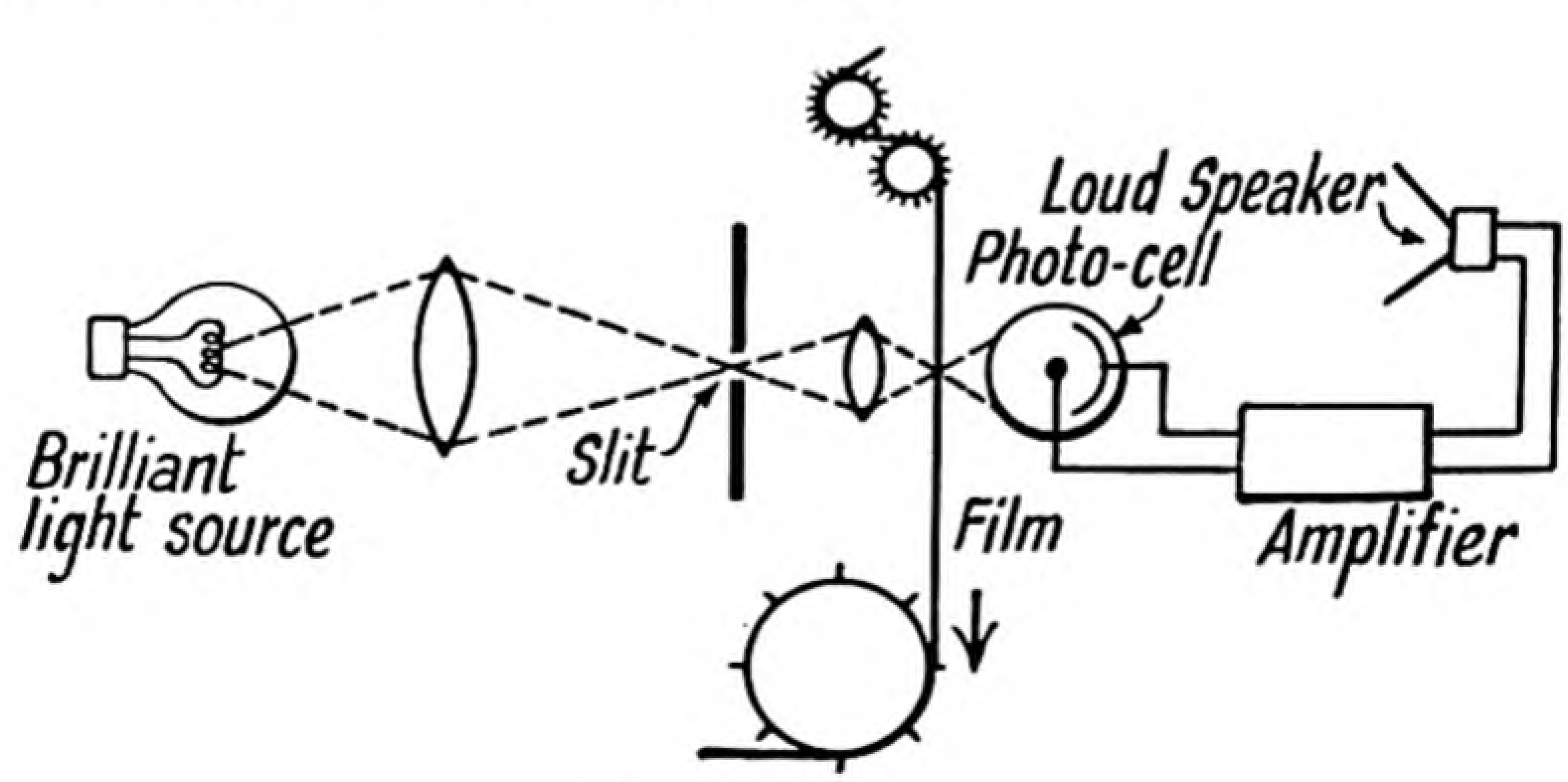

Fig. 11.17.

Magnetic recording

This type of recording is based on the principle of the early wireless magnetic detector, and in contrast to other methods, the recording medium, a steel wire or tape, may be used repeatedly for fresh recordings merely by passing the tape through a constant localised magnetic

field to erase the record.* In recording, the output from the microphone receiving the sound is fed into two small electromagnets of special design, and the steel tape is passed between the poles of these magnets at a speed between 3 and 9 ft. per sec., dependent on the nature of the sound being recorded. By this process the tape becomes magnetised to varying extents and degrees along its length in accord with the frequencies and intensities of the sounds recorded. In the reproduction of the sound the steel tape is passed, at the same speed and in the same direction as during the recording, between the poles of two similar, but non-energised, electromagnets to those used in the recording. The small electric currents induced in the electromagnets are suitably amplified and fed into the loud-speaker system. Plastic- or paper-base strip coated or impregnated with magnetic material is now finding considerable application by virtue of the ease with which it may be cut and spliced, thus permitting the rapid editing of unwanted parts of recordings. Magnetic recording is most useful in cases where material is required for re-broadcasting, and mention should be made here of another type of recording which has found considerable application in the radio broadcasting studio. In this Philips-Miller system, as it is called, the sound-track is cut into a layer of gelatine deposited on a celluloid base (Fig. 11.18), and as the film is moved along horizontally the wedge-shaped cutter vibrates up and down vertically in synchronism with the sound vibrations to be recorded. During this process a groove of varying width is cut in the gelatine layer, but as the latter is initially covered on its upper surface with a very thin black coating, it is evident that the act of cutting produces a transparent sound-track on a black background, which may be scanned in a similar way to an ordinary sound-film track.

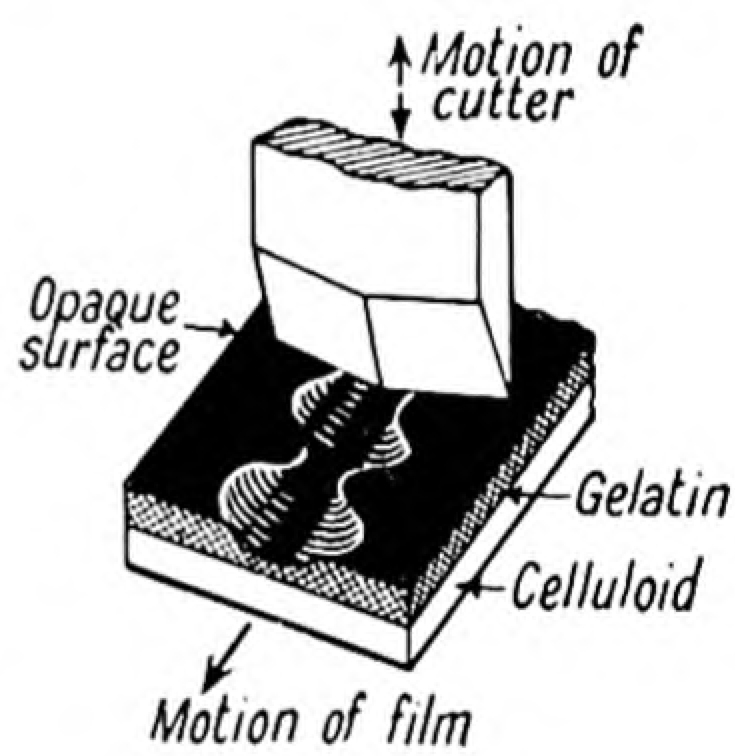

Fig. 11.18.

Tone-control systems

The frequency response of a reproducer, *e.g.* a loud-speaker, may be modified by employing suitable electrical networks in combination with it. The fundamental principles applied in the design of these circuits are that the reactances of an electrical condenser and an inductance respectively decrease and increase with frequency, while a simple resistance exhibits no such variation. The reactance of a loud-speaker (S in Fig. 11.19), may be assumed to be essentially inductive, *i.e.* $=\omega L$, where L is the self-inductance and $\omega/2\pi$ is the frequency. Hence, if a capacitance C is connected in parallel with the speaker (Fig. 11.19a), it follows that the high frequency signals will tend to by-pass the latter, for the capacitative reactance $1/\omega C$ decreases with frequency. On the other hand, if C is included in

* In effect the material is brought to magnetic saturation, but in American practice it approaches the recording-head in a magnetically neutral condition, which is brought about by feeding the wipe-out head from an A.C. supply of ultrasonic frequency, *i.e.* 30 Kc/s.

series with S (Fig. 11.19b), the bass part of the musical scale is reduced. A combined control system shows (Fig. 11.19c) that the position of the slider on the resistance R enables the relative degree of frequency control of C and the added inductance L to be widely adjustable.

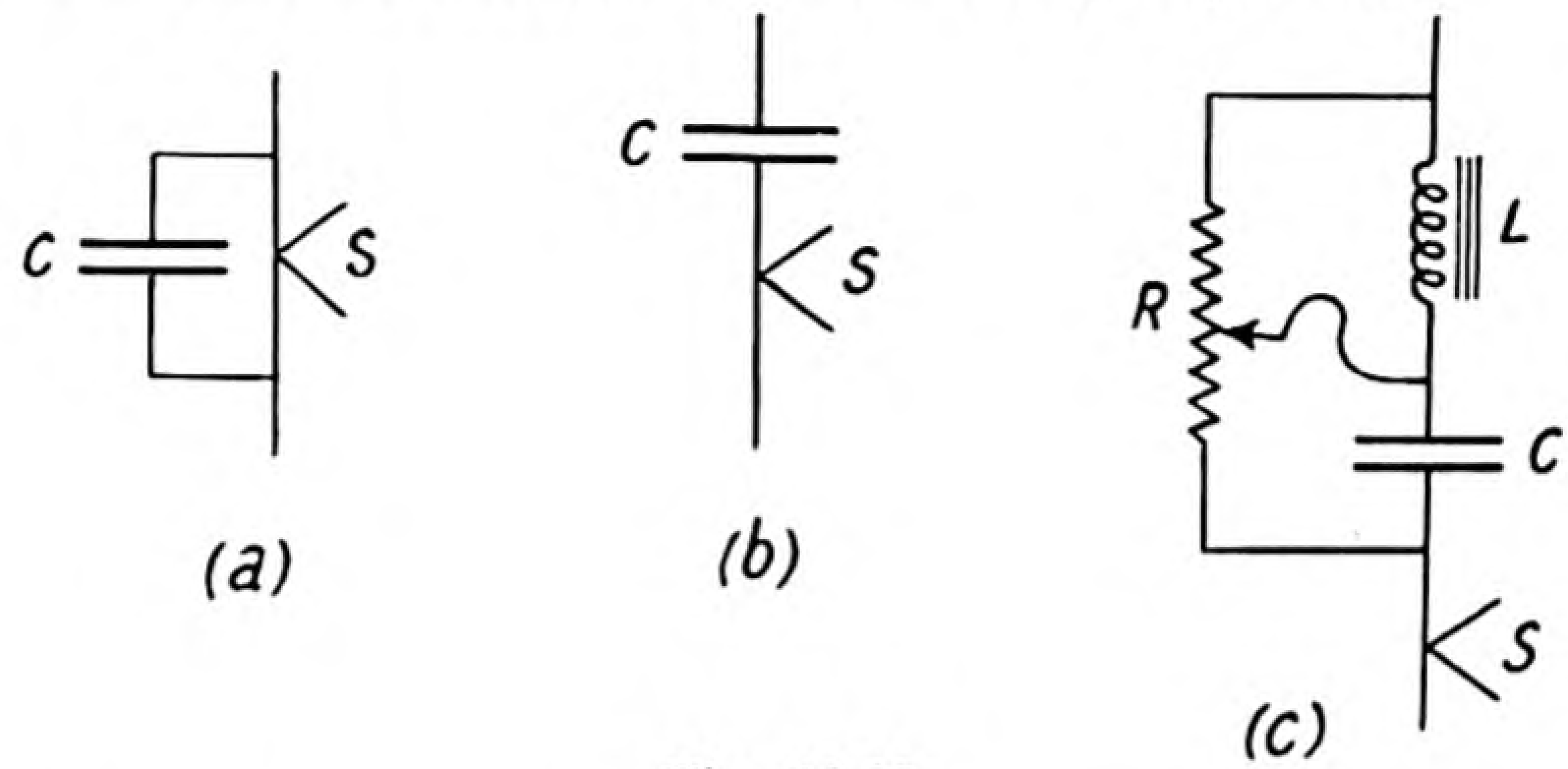

Fig. 11.19.

Combination speakers

The underlying principle of these instruments is the use of two different forms of speaker (Fig. 11.20), so that their combined output shows an even response over a wider frequency range than with either operating singly. Such a combination is that of the moving-coil speaker M, which is most effective in the reproduction of frequencies below 2 or 3 Kc.p.s., with a piezo-electric speaker P, which by the incorporation of a very small series inductance, maintains a uniform response over the higher audio-frequencies up to about 12 Kc.p.s.

The use of a polarising voltage (or magnetic field) in moving armature reproducing (or recording) systems

If a reproducing (or recording) system is dependent for its action upon the attraction of a movable element by a fixed element, then it may be shown that unless the electrostatic or magnetic element is polarised appreciable distortion may result. Consider Fig. 11.21, in which a light plate A and a fixed plate B form the plane electrodes of a condenser, which possesses a resilient dielectric such as rubber.

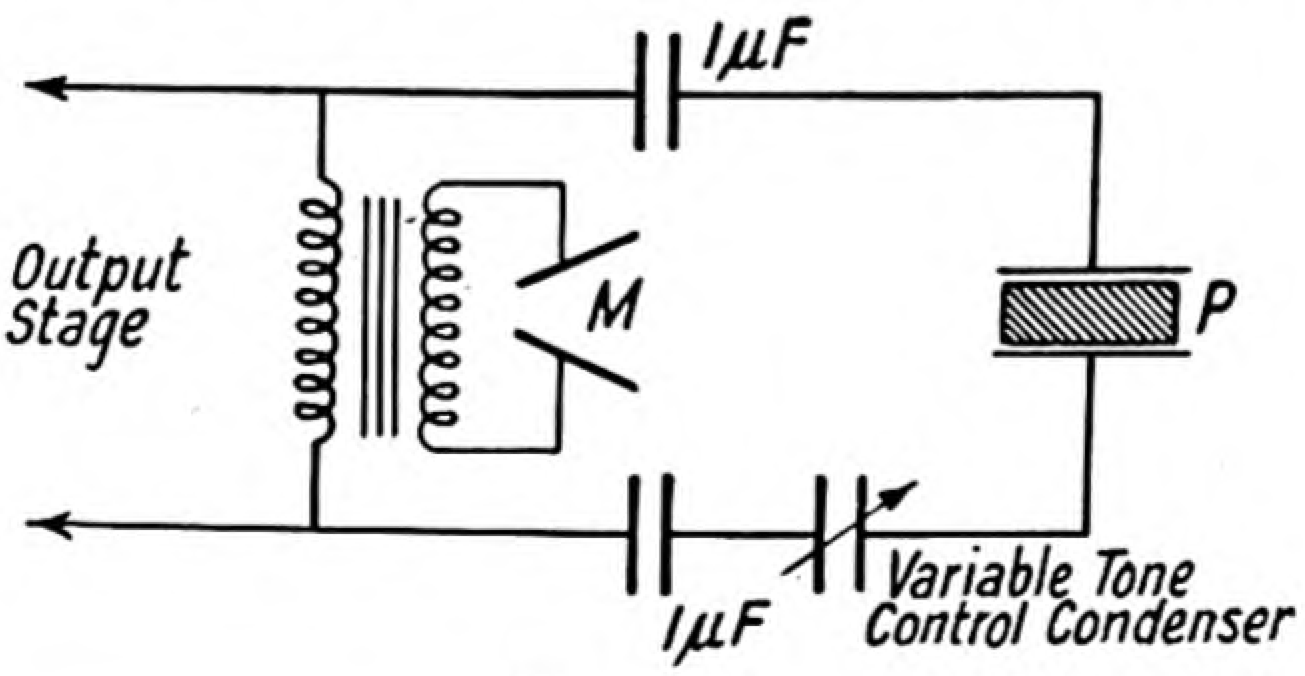

Fig. 11.20.

If a D.C. potential difference is applied to the condenser, A will move towards B, actual contact being prevented by the presence of the solid dielectric. The force between the plates will always be one of attraction for whichever plate is positively charged, hence for an alternating

potential difference electro-mechanical rectification will occur, for the movement of the armature plate A will be in the same direction in each half of a cycle. The frequency of this motion will, however, be *double* that of the external stimulus.

If now a D.C. potential difference is applied permanently to the condenser, *i.e.* it is polarised, then an application of an alternating potential difference will produce a variation in the magnitude of the attractive force about the steady value due to the polarising P.D., and the frequency of this variation will be the same as that of the alternating supply (see Fig. 11.13). The above reasoning will also apply to the case of a soft-iron armature and electromagnet system (Figs. 11.22 and 11.23), as in a telephone receiver, the polarising agent being either a permanent magnet, or a separate winding on the electromagnet to carry a direct current. Fig. 11.22 shows that for a non-polarised soft-iron diaphragm a rarefaction is produced whenever it *approaches* its extreme deflected position, which is twice per cycle of alternating current. In the case of the polarised diaphragm, however, there are two symmetrical extreme deflected positions on either side of the rest position, the approach to one obviously giving rise to a compression, and to the other a rarefaction, and this occurs once per cycle of alternating current.

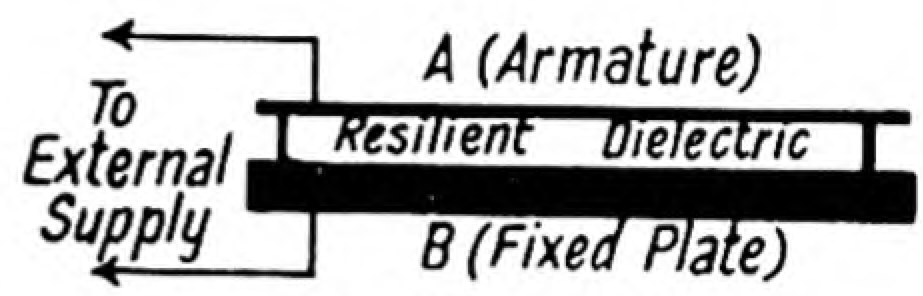

Fig. 11.21.

The above problem may be investigated theoretically as follows, the treatment is only approximate, but the same argument is applicable to both electrostatic and electromagnetic reproducers (or recorders).

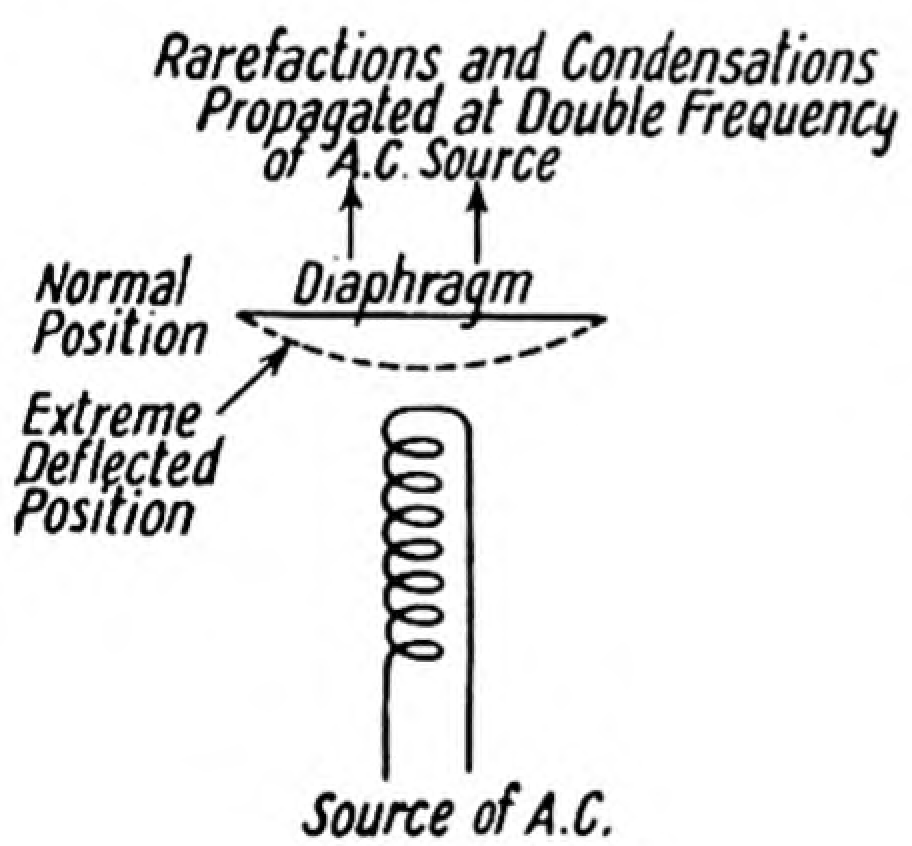

Fig. 11.22.

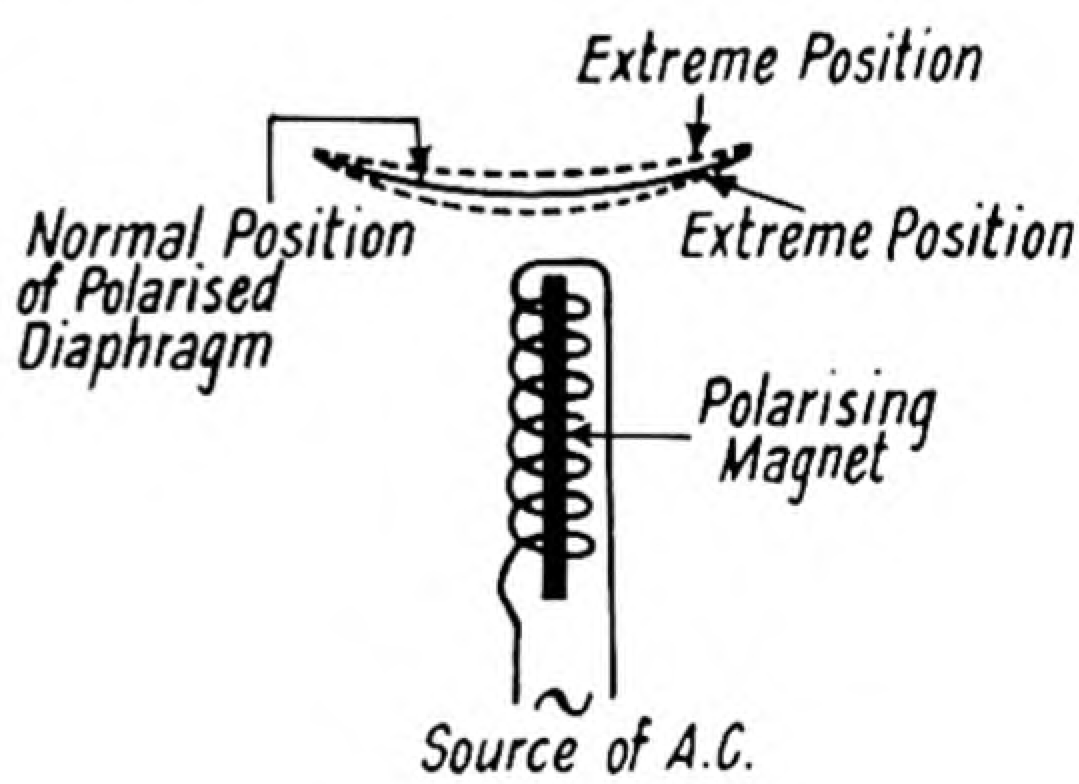

Fig. 11.23.

Let ϕ denote the magnetic flux (or electric charge) through the armature at any instant, then the pull on the armature will be $\propto\phi^2$, say $k\phi^2$, where k is a constant for the system. Suppose that this flux is due to the external stimulus in the form of an alternating current, *e.g.* a speech current $i=i_m \sin \omega t$; in the electrostatic case the charge on the armature will be the result of an applied P.D. $e=e_m \sin \omega t$. Then ϕ being $\propto i$ will be given by $\phi=Ki=Ki_m \sin \omega t$,

where K is a constant of the system, so that the attractive force on the armature is given by

$$F=kK^2i_m{}^2 \sin^2 \omega t=\frac{kK^2i_m{}^2}{2}-\frac{kK^2i_m{}^2}{2} \cos 2\omega t \quad . \quad . \quad (84)$$

If now a constant uni-directional flux be imposed on the system it follows that the attractive force now becomes

$$F'=k(\phi_c+\phi)^2=k(\phi_c+Ki)^2=k(\phi_c{}^2+2\phi_cKi+K^2i^2) \quad . \quad (85)$$

or

$$F'=k\left(\phi_c{}^2+2\phi_cKi_m \sin \omega t+\frac{K^2i_m{}^2}{2}-\frac{K^2i_m{}^2}{2} \cos 2\omega t\right) \quad . \quad (86)$$

Hence it follows from (86) that the resultant force acting on the receiver diaphragm comprises four components, two of which, viz. $k\phi_c{}^2$ and $\frac{kKi_m{}^2}{2}$, are constant forces of attraction. The force $2k\phi_cKi_m \sin \omega t$ is proportional to the product of the strength of the polarising magnets and of the instantaneous value of the current, whereas the last term in the expression (86) is seen to be the cause of distortion, having a frequency double that of the speech current. In order to achieve the minimum amount of distortion the importance of a large flux ϕ_c from the permanent magnet is made evident from (85), for the value of the ratio $\frac{\text{Amplitude of desired frequencies}}{\text{Amplitude of undesirable frequencies}}$ will be very approximately given by $\frac{2\phi_cKi}{K^2i^2}=\frac{2\phi_c}{Ki}=\frac{2\phi_c}{\phi}$.

Visible speech

A sound spectrograph recently perfected by the Bell Telephone Laboratory shows great promise of providing a fool-proof method of registering every type of inflexion known to man. In brief, the complex sound is analysed into twelve frequency bands by means of filters, the outputs from which energise corresponding glow lamps arranged in a vertical line. The fluctuations of intensities from the lamps produce corresponding variations in the respective tracks "traced out" by the light beams on a revolving fluorescent screen. The different patterns can receive interpretation into speech, and the vast field of application of the apparatus is at once apparent.

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CHAPTER 12

MUSIC

Musical scales

The frequencies of the notes in an octave bear a definite numerical relationship to each other, the actual values in the major diatonic scale, which is the commonest, being

Doh	Ra	Me	Fa	So	La	Te	Doh′
24	27	30	32	36	40	45	48

This sequence may be conveniently generated by using a form of Savart's toothed wheel (Fig. 12.1), in which eight wheels are mounted on a common axle, each wheel being fashioned after the manner of a circular saw, the first having 24 equidistant teeth on its periphery, the second 27, and so on. The axle is rotated at a uniform speed

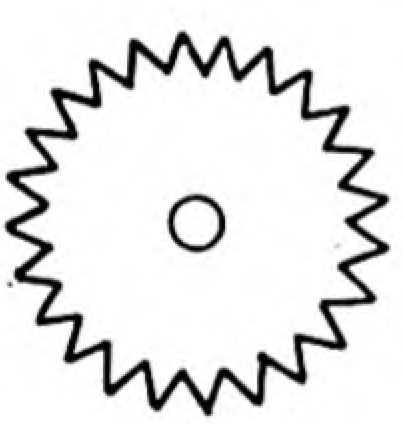


Fig. 12.1.

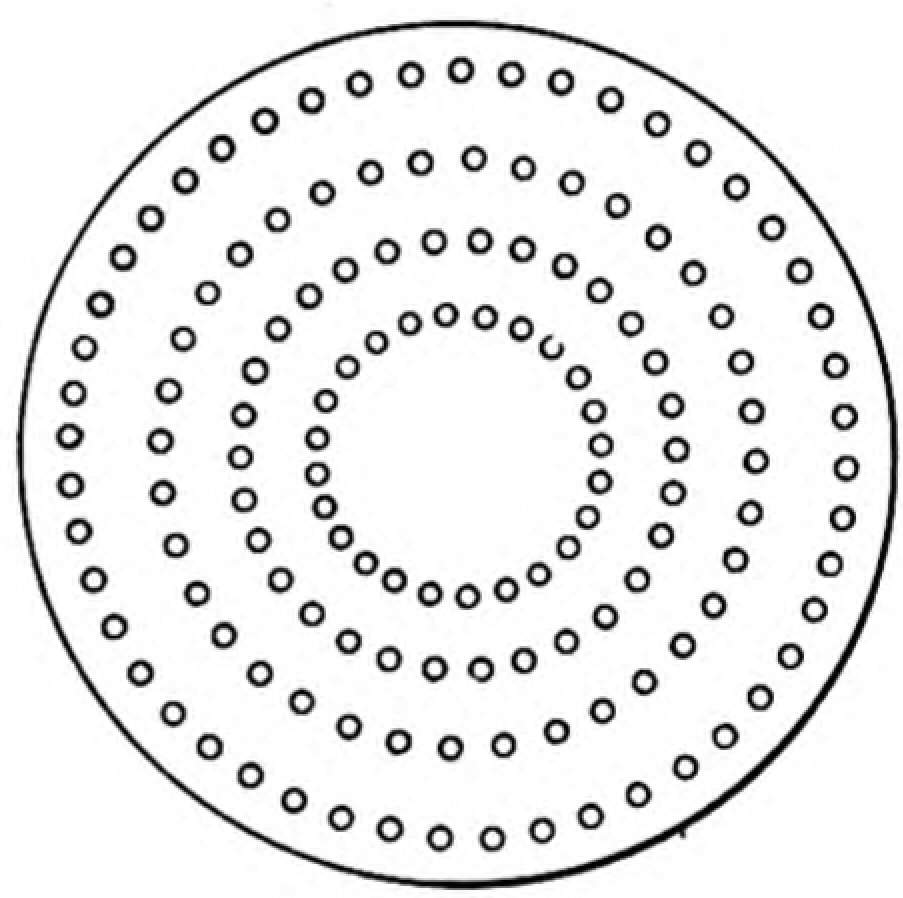


Fig. 12.2.

of, say, 10 revolutions per second, and a card is held lightly against the teeth of each wheel in turn. The series of impulses obtained from each wheel is easily recognised as being in the sequence given above. The pitch of the scale is defined by that of Doh, termed the key note, which in this case is $24 \times 10 = 240$ c.p.s.

Alternatively, a disc siren (Fig. 12.2) may be employed. It consists of a single disc with equidistant holes on the circumferences of circles concentric with the axle, the number of holes per circle ranging from 24 to 48. A jet of air directed against a particular ring of holes gives a sequence of puffs of the required frequency when the disc is rotating at the necessary uniform speed. This Seebeck type of siren does not give rise to a very pure note, and Milne has overcome this difficulty by using a rectangular jet and shaping the holes so that the area of the jet exposed varies sinusoidally with time. The sound generated by this improved form is remarkably free from overtones, and by

driving the disc with a variable-speed motor, having a suitable speed indicator, the pitch of the note may be directly ascertained.

On doubling the speed of rotation of either piece of apparatus the frequency of the impulses is doubled for each circle, and so the sequence of frequencies becomes:

D′	R′	M′	F′	S′	L′	T′	D″
48	54	60	64	72	80	90	96

in which D′ is the same as Doh′ of the first table. The sequence is D′ to D″ as indicated, each note being the octave of the corresponding note of the previous table. Thus if the frequency of each note is doubled, the higher frequencies are *increased* by a greater amount than the lower, *e.g.* D to D′ is an increase of 24, F to F′ by 32, but the *ratio* of corresponding notes is unchanged, and, whatever the pitch, a trained ear should recognise the scale. The ratio between two notes is termed the *interval*; the interval between Me and Doh is $\frac{5}{4}$, that between So and Me is $\frac{6}{5}$, and that between So and Doh is $\frac{3}{2}$, which is equivalent to $\frac{5}{4}\times\frac{6}{5}$, *i.e.* the product of the first pair of intervals. The ratios and intervals are set out in the table, which also shows the Helmholtz notation and the philosophical scale, in which C is made, for convenience, 256, *i.e.* 2^8.

Tonic sol-fa	Doh	Ra	Me	Fa	So	La	Te	Doh′
Vibration numbers . .	24	27	30	32	36	40	45	48
Interval between adjacent notes		$\frac{9}{8}$	$\frac{10}{9}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{9}{8}$	$\frac{16}{15}$
Interval from key note . .		$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Philosophical scale . .	256	288	320	$341\frac{1}{3}$	384	$426\frac{2}{3}$	480	512
Helmholtz scale; Key C (Doh is key note) . .	C	D	E	F	G	A	B	C′

Notice that the vibration numbers are multiples of 2, 3 or 5.

The intervals between adjacent notes are either $\frac{9}{8}$, $\frac{10}{9}$ or $\frac{16}{15}$. The first two are termed major and minor tones respectively, and the smallest, $\frac{16}{15}$, is a semitone, so that the scale is composed of tone, tone, semitone, tone, tone, tone, semitone. The tones are divided by using sharps and flats, so that there are twelve *semi*-tones in an octave, the five whole tones being divided in a piano keyboard by the "black notes." This augmented scale is termed the chromatic ("coloured") scale. Other scales are the minor diatonic scales, and these are obtained by reducing a major interval in a major diatonic scale by a semitone; *e.g.* if E is replaced by E flat, a minor diatonic scale is the result, and this is composed of tone, semitone, tone, tone, tone, tone, semitone.

Scales arose by intuition, reasoning and chance, and are often characteristic of a particular race. Pythagoras introduced the scientific aspect by pointing out that the ratios of the frequencies of notes which blended were in a *simple* ratio, and this gave rise to the major diatonic scale. This blending, or harmonising, is most satisfying if octaves (ratio 2), are sounded together; the same occurs with the fifth (ratio). It is interesting to notice that if male and female voices are in unison, the males are usually an octave lower; also, that pairs of street singers

have been found to sing in fifths, apparently a natural thing to do. This is clear when the vibration numbers are considered in relation to the overtone, common to the key note and the other notes in the octave.

Vibration number	24	27	30	32	36	40	45	48
Relative frequency of common overtone	—	216	120	96	72	120	360	48
Overtone in octaves	—	9	5	4	3	5	15	2

It will be seen that the order of *simplicity* is: octave, 5th, 4th, 6th and 3rd, 2nd, 7th. Actually the 2nd and 7th are discordant. The intervals between notes are expressed in music in this way, the range C to E being termed a major third to distinguish it from C to E flat, which is a minor third. The harmonic scale is mentioned later in this chapter (p. 212).

Yet another scale is the pentatonic, a five-note scale, which in key C consists of C, D, E, G, A, C′; F and B are omitted, so this scale is composed of tone, tone, tone and a half, tone, tone and a half. This is a scale natural to somewhat primitive peoples. Its intervals coincide with the black notes of a piano, which may be used to play the tune of the hymn "There is a happy land," which is actually a melody of an Indian hill tribe. Bagpipe music resembles pentatonic music in so far as some of it can be played on the black notes, *e.g.* "Auld Lang Syne" and "The Campbells are coming," but the two scales are distinct.

Musical instruments

These may be classified under the headings: (*a*) percussion, (*b*) wind, and (*c*) strings.

(*a*) Percussion instruments are either tuned (bell, triangle, kettle-drum) or untuned (big drum, side-drum, bones).

(*b*) Wind instruments are blown mainly by expired air, and the means of initiating and maintaining the vibrations are by the lips (bugle, posthorn, cornet, trombone), and reeds (clarinet, saxophone, bagpipes, reed organ pipe). The flute organ pipe and the ordinary tube blown across the end (*e.g.* a hollow key) are classified as simple tubes.

(*c*) Strings are plucked (harp, harpsichord), bowed (violin), hammered (pianoforte) or blown (aeolian harp).

Of these instruments, only the violin family (including the mandolin, banjo, etc.) and the trombone can give an unlimited number of notes within their respective ranges, for the length of the vibrating string in the former, and of the air column in the latter, are controlled by the player, whereas in the other stringed instruments each of the strings is of a fixed length, and the remaining wind instruments have either a fixed resonant length (bugle), or a series of fixed resonant lengths which are controlled by depressing pistons (cornet), or covering holes in the side of the tube (piccolo, saxophone).

This restriction to particular notes presents a serious handicap in the case of the instruments in an orchestra. This handicap will be discussed with reference to the pianoforte, but the discussion applies to all instruments which have a fixed range of notes in the diatonic scale. If a composer writes in the key of C, C is Doh, but if he changes to key E, E is Doh, and the other notes in the octave must have frequencies which conform to the ratio 24, 27, 30, etc. This means that, in the scientific scale, as the frequency of E is 320 c.p.s., the values of those from A to E′ must be as given in the first line of figures below:

A	B	C	D	E	F	G	A′	B′	C′	D′	E′
216·3	240	266·7	300	320	360	400	426·7	480	533·3	600	640
216·3	240	256	288	320	341·3	384	426·7	480	512	576	640

The lower line of figuresgives the actual frequencies, and although these are modified by the use of sharps, or black notes, they are not true. The trouble is due to the existence of major and minor intervals, and if a piano were constructed to cope with this, some 50 keys per octave would be necessary. This is impracticable, and the difficulty is removed, practically, by compromise. This is achieved by arranging the intervals between adjacent semitones to be equal, and the scale is then said to be tempered, and is termed *the scale of equal temperament.* Tempering is usually attributed to Bach, although there is evidence to show that it was suggested by Aristoxinus, 300 B.C.

The relative frequencies in this scale may be deduced as follows: Let the notes in an octave, including semitones, have frequencies represented by $K, L, M, \ldots, V, W$ respectively. As the intervals are equal, $\frac{L}{K}=\frac{M}{L}=\frac{N}{M}\cdots=\frac{V}{U}=\frac{W}{V}=p$, say, the semitone interval. Multiplying these fractions together eliminates all symbols but W and K, thus $\frac{W}{K}=p^{12}$, as there are twelve semitone intervals. As W is the octave of K, $\frac{W}{K}=2$, i.e. $p^{12}=2$, or $p=1{\cdot}0595$, and the full tone interval is $p^2=1{\cdot}12$.

The frequencies in the two scales in key C, and the intervals in each are shown in the table:

Note	C	D	E	F	G	A	B	C	D	E
Diatonic	256	288	320	341·3	384	426·7	480	512	576	640
Interval		1·125	1·11	1·067	1·125	1·11	1·125	1·067	1·125	1·11
Equal Temperament	256	287·4	322·5	341·7	383·6	430·6	483	512	575	645
Interval		1·12	1·12	1·06	1·12	1·12	1·12	1·06	1·12	1·12

It will be seen that the frequencies are practically the same in each case for each of the octaves has an interval of 2, so that, whatever key is used, the intervals are the same. Hence, by tuning pianos and other fixed-note instruments to mean-tone temperament and so sacrificing accurate tuning, the difficulties which would occur when transposing from one key to another are overcome. However, orchestras may "wander" in the course of a concert to the extent of one per cent. in frequency, and as the instruments themselves may not agree after having been played for some time, any discrepancy caused by tempering is not usually noticeable. In passing, it may be noted that the "tootling" in an orchestra that precedes a concert is necessary as it warms the instruments to the temperature at which they will be required to play, for as the velocity of sound in air increases with temperature, the frequency rises likewise in tubes of fixed length.

Frequencies quoted above are based on the philosophical pitch of C, 256 c.p.s., and A is thus 426·7 c.p.s. Since the days of the early master composers the gradual rise in the standard of pitch was becoming a serious problem, for example at one period in the last century a pitch of 467 c.p.s. for A was being used by the London Philharmonic Orchestra. However, musical pitch was standardised at 440 c.p.s. for A at an international conference in London, May 1939, and this frequency is now broadcast daily by the Bureau of Standards, Washington, U.S.A. It is appropriate to mention here that pitch is concerned with the position of a note in a musical scale, and so the judgment of the pitch of a note is essentially physiological. That a small difference in the numerical value of the pitch and the frequency of a note may exist is therefore not surprising.

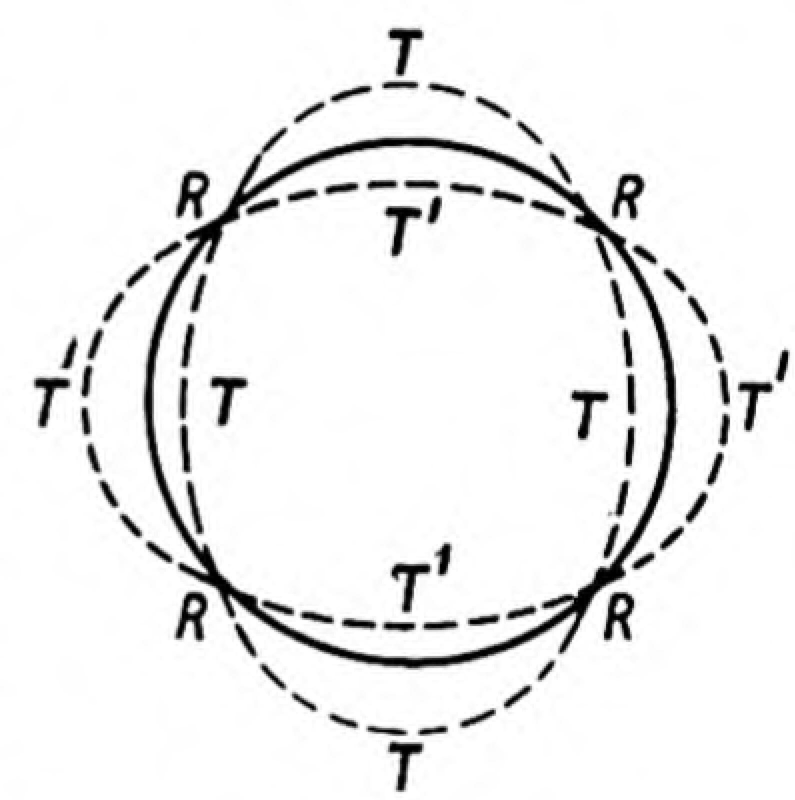


Fig. 12.3.

Brief details of instruments

(*a*) *Percussion*. The bell is an important member of this family, and its mode of vibration in an axial cross-section can be likened to that of a tuning-fork. In a similar section at right angles, the vibrations are 180° out of phase with the former, as the cross-section of the circular plan of the vibrating bell is made to assume an elliptical shape as shown in Fig. 12.3.

The dotted lines T, T' indicate the limits of motion, and the full curve the plan of the bell before striking. The points R where the curves cross, indicate nodes. Bells are made of a special alloy consisting of 4 parts of tin to 13 of copper, by weight, and they are tuned by machining the waist.

Chladni's work, and its modern treatment by Waller (p. 90), show the different modes of vibration that are possible in a vibrating plate fixed at its centre, and how the point of excitation decides the position of radial and circular nodal lines. In a similar way the modes of

vibration in a bell are influenced by the point of impact of the clapper and by variation in the thickness of the material.

In general, the different partials of a bell form a non-harmonic series, and it is therefore surprising that an agreeable tone should be forthcoming. The makers define the pitch of a bell by the so-called "strike" note, which is very prominent on striking but dies away rapidly, and it corresponds to the component of highest pitch. An installation of bells chosen to give chromatic intervals and which may be struck from a keyboard is called a carillon.

(*b*) *Wind instruments.* The flue organ pipe is an example of a simple tube instrument. It is, in effect, a resonating column of air with a source of vibration much as that described in Chapter 9, in which the vibrations are maintained by a tuning-fork, but at this point the resemblance ceases. The reason for this is that the organ pipe is an example of *coupled vibrations*, the components being the natural vibration of the column of air in the tube, and the natural vibrations of the jet of air in the mouth. When these are in agreement, the pipe is in tune, but when somewhat out of agreement a compromise occurs which results in a frequency between those of the components. For example, if the natural frequency of the pipe is 300, and that of the jet 290, they "couple" to give a frequency of about 295 c.p.s. This subject of coupled vibrations is discussed more fully in the Appendix.

The posthorn resembles an organ pipe, but the vibrations are caused by the air blown through the pursed lips of the player; by altering the blowing pressure and the tension in the lips, different frequencies can be produced, and any of the harmonics which fit the pipe can be elicited. The bell-shape (flare) of the wide end tends to eliminate the end-correction which is present in a uniform tube, and so the overtones harmonise with the fundamental, giving rise to the *harmonic scale*. This scale is one in which the *wave-lengths* of the notes are in the ratio of $1 : \frac{1}{2} : \frac{1}{3} : \frac{1}{4} : \frac{1}{5}$, etc., and it is the characteristic feature of the music of bugle bands and trumpet calls. Expressed as the ratio of frequencies the scale will be 24 : 48 : 72 : 96 : 120, etc., *e.g.* C, C′, G′, C″, E″. The effective length (*i.e.* of the air column) can be altered in a trumpet by means of pistons which, when depressed, bring extra branches of the tube into use, and thus make other sets of harmonic scales available, and so the complete scale is possible.

In the reed type of instrument, the player's lips are replaced by a reed, which vibrates when blown (saxophone) or, in the organ pipe, the familiar mouth is replaced by a reed which can be tuned (reed-pipe). The latter is described in the section on organ pipes.

(*c*) *Stringed instruments.* The motion of a vibrating string is usually of large amplitude, but as the vibrating surface is small, the damping, and hence also the rate of loss of energy, are both small. In a stringed instrument, however, the strings pass over a wooden bridge standing on the body of the instrument; this bridge communicates the vibrations of the string to the body. The vibrations of the latter set a large volume of air into motion and so increases the audibility of the string note (cf. p. 85). The air within the instrument is also set into vibration

through the movement of the body, and exercises an influence upon the quality of the note. The subject, however, is imperfectly understood; the "qualities" of each of two apparently identical instruments are not necessarily the same, and, at the moment, no satisfactory explanation has been given.

In the piano the back (sound-board), vibrates and acts in a manner similar to a table on which a tuning-fork is standing.

The total force exerted by the tensions of the strings of a piano amounts to about 10 tons.

The quality of instrumental sound is determined by the relative strengths of the overtones present, the higher overtones being responsible for brilliance and the lower ones for richness. This is noticeable in the violin family, the contrast between the violin and the double bass being particularly marked. The effect of "muting" a violin is to render the upper harmonics inaudible and the lower ones less intense, and thus to give rise to a subdued tone. The mute—a three-pronged clip attached to the bridge between the strings—loads the bridge and thereby reduces the amplitudes of vibration.

The violin bow, so named after the original shape, maintains the string in vibration by what has been termed slip-stick action; actually it is a type of relaxation oscillation (see Appendix 6). The velocity of the string is much larger when moving against the bow than in the forward direction with it, and since the smaller the *relative* velocity the larger the coefficient of solid friction, then excess of work done by the bow over that done by the string against it forms the source of energy for the maintenance of the oscillations. If the string coincides with the natural frequency of the air in the belly of the violin, then the supply of energy may not be sufficient to make up the increased demand for maintenance, and the octave becomes predominant in the vibration of the string. The free vibration of the belly will now die out since it is no longer in resonance, and consequently when the bow takes command again the cycle will repeat itself, and its continued repetition gives rise to a beating tone, known as the *wolf note*. Raman and Backhaus have worked on this subject, and more recently Vollmer, and the latter has established (in the case of the violoncello) that the wolf note is the beat note between the coupled vibration of the string and the "wood" of the instrument. Vollmer draws an interesting analogy between the violoncello and a valve transmitter, the "feed-back" in the electrical circuit being compared with the bow-pressure on the 'cello. The discussion of the coupling and damping in the two systems leads to suggestions on how the wolf note may be eliminated, by correct damping of the body of the instrument.

Vortex formation

Whenever a stream of fluid (liquid or gas) is flowing at an appreciable velocity with respect to the surrounding medium then a region of low pressure is created in its immediate neighbourhood, and conversely a high pressure region is located where the velocity is small. This effect may be illustrated by dropping a celluloid (*i.e.* table-tennis) ball on to a fountain of water. The ball will be seen to become "supported" by the jet, any tendency for it to move out of the stream being resisted,

due to the greater pressure outside of the main jet compared with that in the immediate neighbourhood of the fluid. Again, when a golf-ball is hit with an inclined club-face, a rotation is imparted to the ball in addition to a translational velocity; consequently the opposite sides of a diameter of the ball, perpendicular to the direction of flight, will possess different relative velocities with respect to the air. It

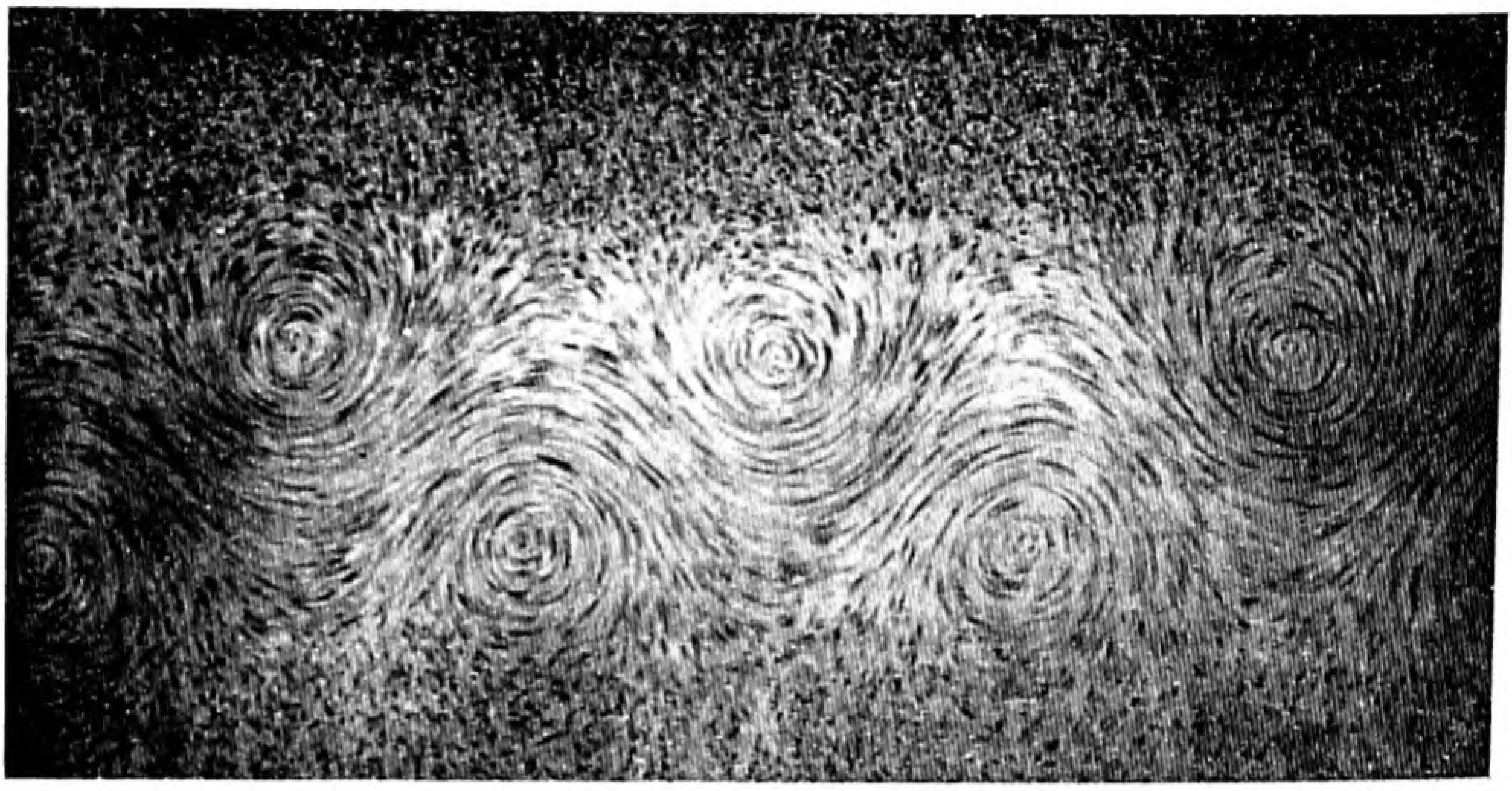


Fig. 12.4. *Richards. Phil. Trans. A.*

follows that a resultant *sideways* pressure is exerted on the ball causing it to "duck" or "rise," or be "pulled" or "sliced," according to the axis of and direction of spin.

If instead of a rotating solid body a cylindrical portion of the fluid medium is rotating with uniform angular velocity in the midst of the surrounding fluid, which has only a translational motion, then this cylinder of rotating fluid is termed a *vortex filament*. Such vortices, eddies, or whirls may be observed whenever a fast-flowing stream of water encounters a rock or similar obstruction, or alternatively when an object such as a ship is moving through still water innumerable eddies will be noted astern. Fig. 12.4 shows the regular procession of eddies in the wake of an obstacle drawn through still water, the motion of which was rendered visible by light diffused from minute drops of milk suspended in alcohol and carefully introduced in the

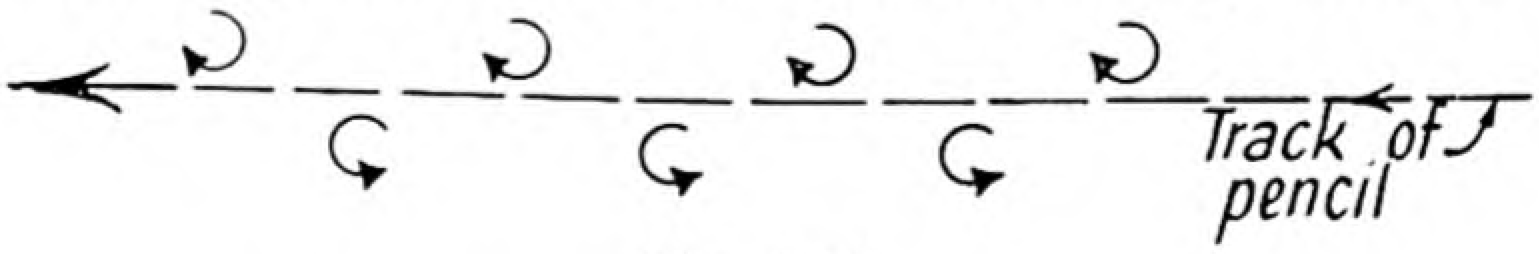


Fig. 12.5.

stream. Kármán showed theoretically that there is one stable arrangement of vortices behind a steadily moving cylinder such that the eddies are formed on the two sides of the obstacle in turn, as shown in the photograph (Fig. 12.4).* Such a system of vortices is usually known as a Kármán "street," and as shown diagrammatically in Fig. 12.5 the vortices on the opposite sides of the street are staggered and rotate

* According to Levy and Forsdyke a symmetrical (unstable) system of vortices is formed for low values (30 to 70) of Reynold's number.

in opposite senses. This latter property is made evident by the *lateral* vibrations felt when a pencil, held in the hand, is drawn through the liquid. If h is the distance between successive vortices on one side of the Kármán "street," and l is the distance between the lines of the centres of the vortices on opposite sides of the street, then Kármán showed that the equation governing this stable arrangement was $\cosh \frac{h\pi}{l} = \sqrt{2}$, which reduces to the form $h = 0{\cdot}283l$. It may be noted that if the rear part of the cylindrical body is "tailed-off" there is a considerable reduction in the eddy formation, and hence a much-reduced resistance to the motion of the body; this effect indicates the purpose of the stream-lining of fast moving air, land, or sea transport. The whole system of vortices in the Kármán "street" follows the motion of the obstacle, but only at a fraction of its speed.

When a stream of air is flowing with sufficient velocity past a stretched wire, vortices are formed in the air alternately on opposite sides of the wire. It follows that alternations of pressure will arise and cause the wire to vibrate transversely to the air stream.

Aeolian tones

When air is streaming past a wire a double series of vortices revolving in opposite directions is set up on alternate sides of the wire. The periodic production and detachment of these eddies, which are carried along by the stream, produce an alternating transverse force on the wire. If the air flow and wire dimensions are correctly adjusted these pressure fluctuations will give rise to an audible note. Stronhal rotated a vertical wire stretched on a frame about an axis parallel to the wire so that the latter continuously described the curved surface of a cylinder. He found that the frequency N of the note produced was related only to the relative velocity V of the wire through the air and the diameter d of the wire, and was such that $\frac{V}{nd} = 0{\cdot}19$ approx. The absence of terms involving the tension T, the mass m per cm. of the wire, and its clamped length l is to be noted, but if these quantities are such that $n = \frac{1}{2l}\sqrt{\frac{T}{m}}$ then a considerable enhancement of the sound will occur and similarly the response will be greater for the harmonics of the wire. These effects are similar if the obstacle is stationary and the air is streaming past, and wind blowing (whistling) past telephone wires is a typical example of these aeolian notes, as they are termed. The flapping of a flag in the breeze, the swish of a cane through the air, and the moaning of the winds in the woods are other everyday examples of the phenomenon. The **aeolian harp** is of ancient origin and founded upon the effect just discussed, but it is not a serious musical instrument. It comprises a series of short wires under the same tension stretched across an aperture exposed to the moving of an air stream. The probability of a wire resonance for any given air speed is enhanced by making the wires of different diameters.

Sensitive flames are jets issuing from a small nozzle under a controlled and initial pressure, which by their periodic changes of length

under the influence of sound waves were greatly used at one time as detectors, *e.g.* of a standing wave system. G. B. Brown has recently worked with these flames using a constant tone oscillator and loud-speaker as the sound source, and obtained some beautiful photographs of vortices passing up smoke jets issuing from linear slits. The Rayleigh relation that the velocity of vortices is one-half that of the jet was found not to hold for the shorter slits. Professor Andrade* has shown that the jet is unresponsive if the sound is directed along its length, and concludes that its sensitivity is due to a relative *transverse* motion of the jet and the surrounding medium.

Edge tones

Vortices are formed in a similar manner when a jet of air is forced through a slit. If a wedge is placed in the stream with its edge parallel to and facing the slit, however, a complication arises, for the spacing of the vortices is governed by the distance between the slit and the wedge. This is shown in Fig. 12.6*a*—as one vortex passes the lip, a second appears at the slit, and on the same side of the stream. By decreasing the distance, the number of vortices generated per second

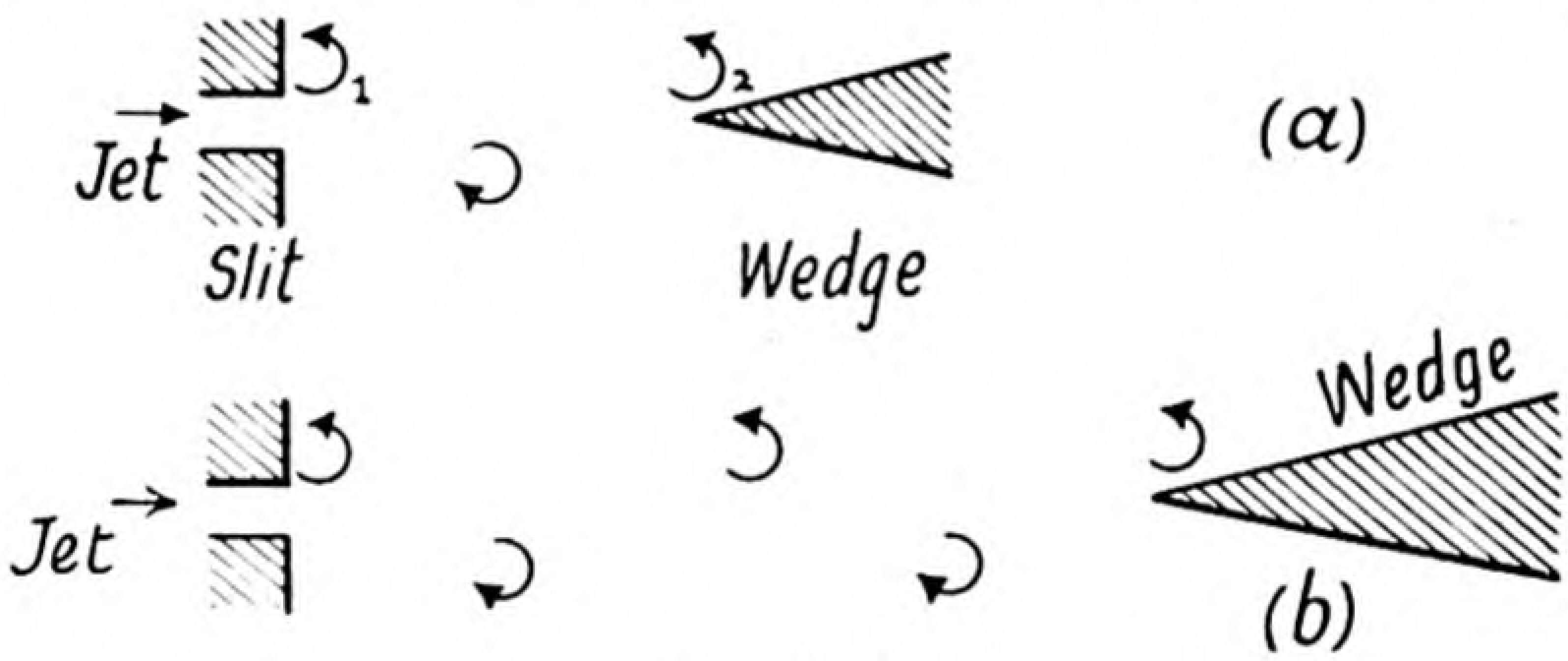


Fig. 12.6.

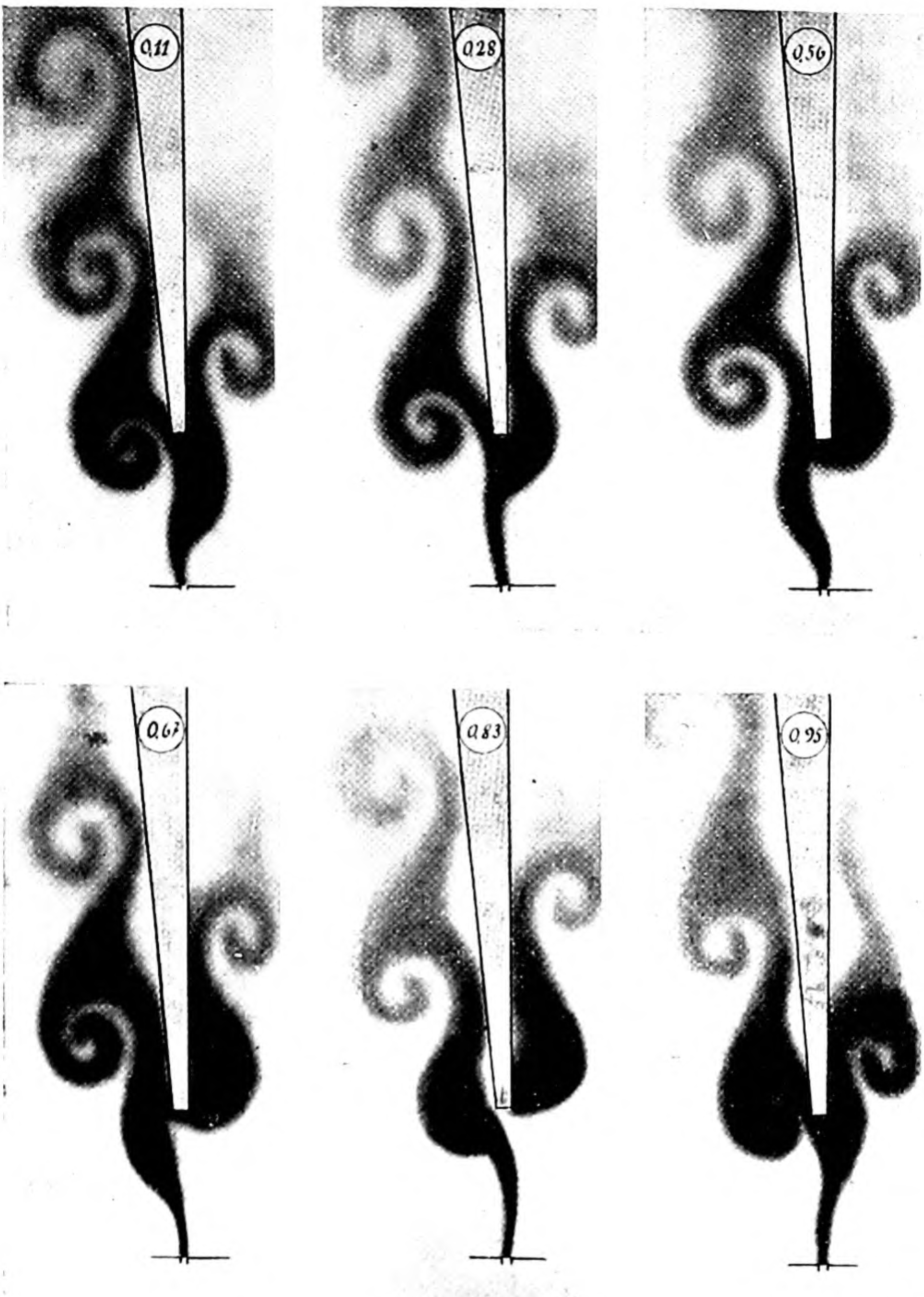
increases, and this causes a rise in the frequency of the "whistle" or edge-tone, as it is called. Alternatively, an increase in the velocity of the jet produces a rise in frequency. There is a minimum distance for the production of sound, but if the distance be increased, the vortex system is "stretched" until it becomes unstable, breaks down, and rearranges as in Fig. 12.6*b*, giving rise to a higher note. By increasing both distance and velocity, the frequency can be maintained constant, for the relationship between the velocity v, frequency n, and distance from slit to edge h, is closely represented by the expression: v/nh=constant when stable; the magnitude of the constant is governed largely by the dimensions of the slit, and the angle of the wedge appears to have little effect up to about 30°.

Flute (or flue) organ pipes

When a stream of air from a wind chest (at a pressure of approx. 0·5 lb. per sq. in.) strikes the lip of an organ pipe a series of eddies, or vortices, are formed. In turn these give rise to pressure pulses of a fairly well-defined frequency, the pitch of the faint "edge-tone"

* See "The Sensitive Flame" by E. N. da C. Andrade, Guthrie Lecture in *Proc. Phys. Soc.*, **53**, 329, 1941.

(1) (2) (3)



(4) (5) (6)

Fig. 12.7.

Drawings by Carrière showing eddies formed at the lip of a flue organ-pipe at successive intervals of approximately one sixth of a period (fractional values are marked on the drawings). The pipes are to the right of each drawing and fine smoke was used to render visible the motion of the air.

produced rising with the speed of the escaping air-jet. Should this frequency coincide with the fundamental or overtone of the air in the pipe then resonance takes the place and the pipe will be said to "speak." The presence of a procession of eddies was first observed by Carrière by utilising stroboscopic illumination and the changed density of moist air near the edge. Fig. 12.7, due to him, shows the development and progress of these eddies during a complete vibration of the pipe. The alternate vortices produce pressure changes of opposite signs, and so a complete cycle takes place during the time vortex (1) takes to replace vortex (2) (Fig. 12.6*a*). The translational velocity of the vortices is one-half that of the air-jet, *i.e.* $\frac{v}{2}$, therefore the period of the note $\frac{1}{n}=\frac{h}{\left(\frac{v}{2}\right)}$ or $\frac{v}{nh}=2$.

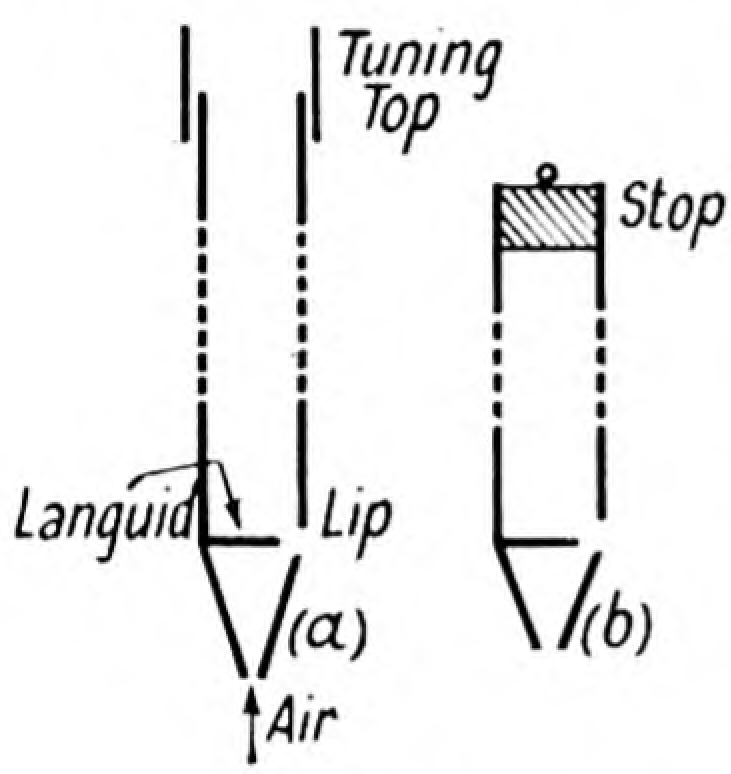


Fig. 12.8.

The tuning (Fig. 12.8) of such a pipe is performed by adjustment of the height of the mouth (termed "cut-up" by the organ builder), and also of the effective length of the pipe. The first process is known as "voicing." The length of the pipe, if stopped, is altered by moving the stop. If the pipe is open, its note can be "flattened" by partly shading it with an additional piece of metal left at the open end for this purpose. Frequently the open pipe is fitted with a tightly fitting sleeve at the end, by which the length may be changed. The end-correction at the mouth may amount to as much as three times the radius of the pipe.

The majority of pipes are made of tin, zinc, or some soft alloy, but wood is often used for the largest pipes. Miller, in the case of the flute, made an exhaustive investigation of the effect of the material of the instrument upon the harmonic content of its tones. The density and elasticity of the material are both involved, and Miller found that wooden flutes have fewer harmonics than metal ones.

The tonal qualities of an organ pipe can be modified by altering various constructional features such as the ratio of diameter to length of pipe, contour of the lip, and the shape of the pipe. Fig. 12.9 shows how the harmonic content is altered in respect of two particular stops, as they are termed.

The possible modes of vibration of a pipe have been discussed in Chapter 9, in which the wave-length of the first overtone of a *closed* pipe is shown to be one-third that of its fundamental, whereas the wave-length of the first overtone of an *open* pipe is one-half the wave-length of its fundamental, etc. The diagrams in Fig. 12.10 show that a similar relationship exists between open and stopped organ pipes. The difference in quality between the tones from a stopped pipe and an open pipe of the same frequency is due to this difference, *i.e.* the even harmonics are missing in the stopped pipe.

Reed pipes

A tongue or reed of flexible metal (Fig. 12.11) is fixed at one end to cover, but not close, a rectangular aperture through which air is blown. This wind displaces the reed which closes the aperture and thus interrupts the flow of air, but the reed springs back again, allowing more air to pass. In this way a series of puffs is generated at the required frequency. The air column is usually conical. The effective

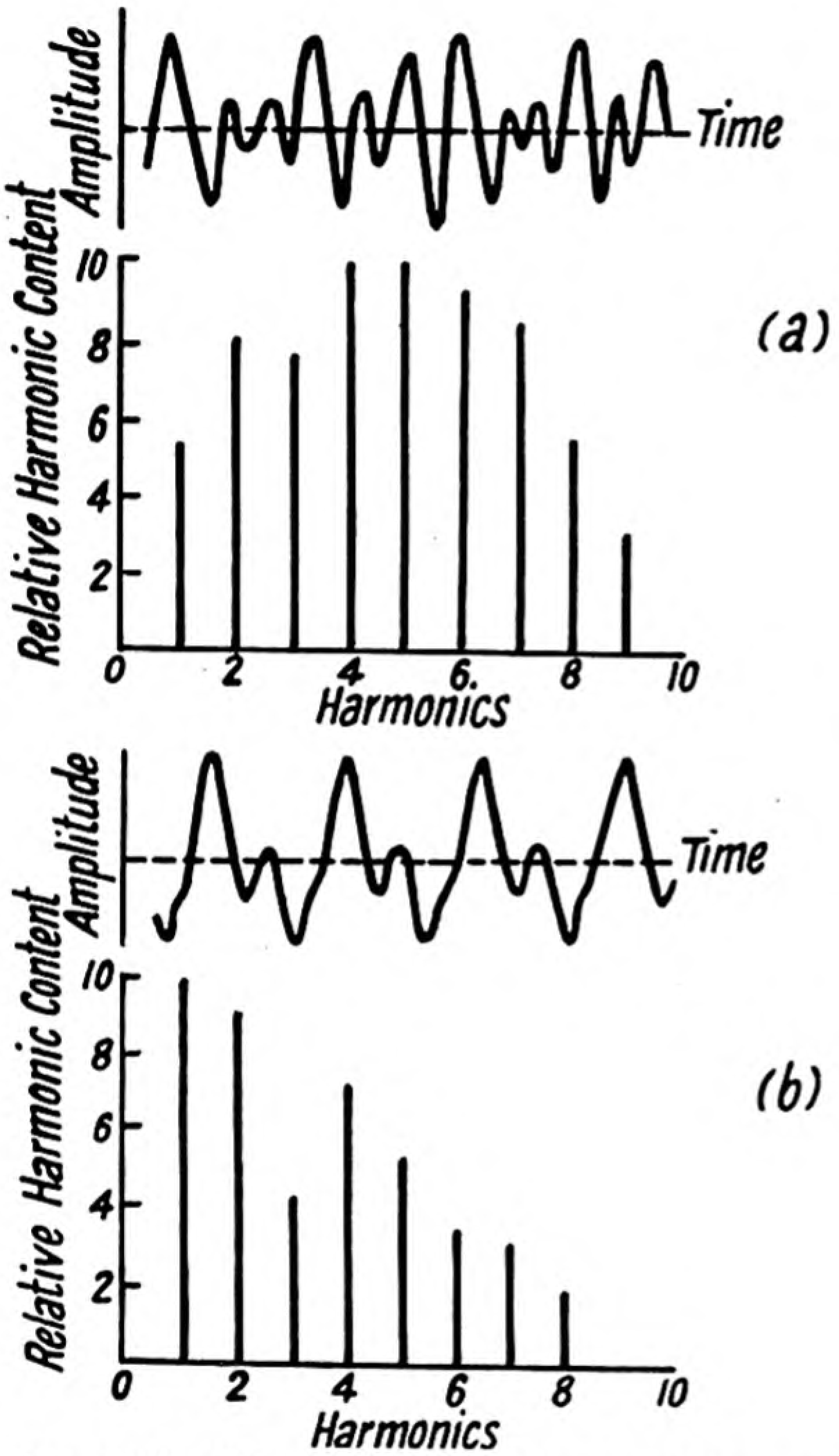


Fig. 12.9. Wave-forms and Spectra of Organ Tones (Culver).

(*a*) Viola d'orchestre. (*b*) Open Diapason.

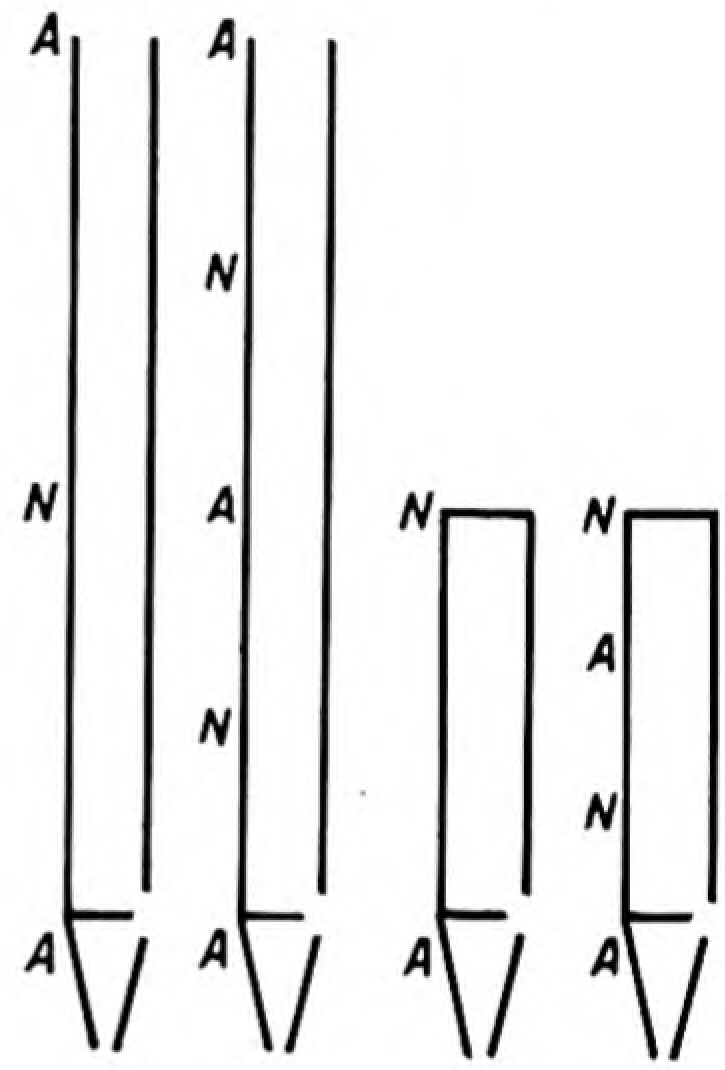


Fig. 12.10.

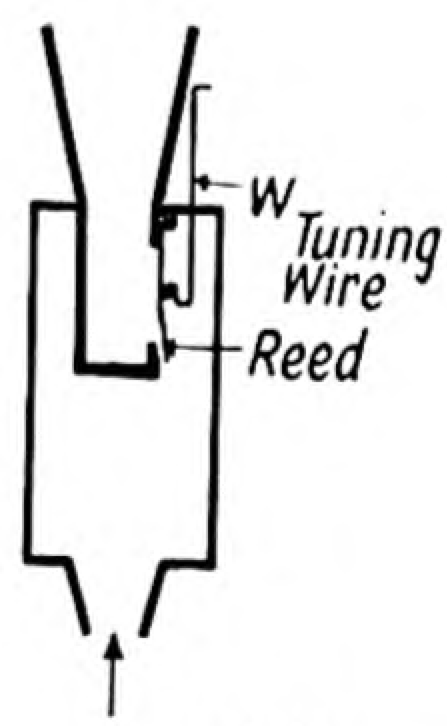


Fig. 12.11.

length of the reed is altered by means of the wire W, which enables the pipe to be tuned.

Effect of a change of temperature on frequency

(*a*) *Organ pipe*. The velocity of sound in air increases with temperature, so that as the wave-length of a pipe is governed by its length, and $v = n\lambda$, the frequency is proportional to the velocity (the increase

in the pipe itself is neglected), hence the ratios at $t°$ C. and 0° C. are

$$\frac{v_t}{v_0}=\frac{n_t}{n_0}=\sqrt{\frac{t+273}{273}}=1+\frac{t}{546} \text{ approx.}$$

In a pianoforte, however, the tension of the wires diminishes with the tendency to increase in length with temperature, so that the pitch tends to fall in a piano, but to rise in an organ with increase of temperature.

(*b*) *Tuning-fork.* Now the frequency of transverse vibrations of a clamped-free bar has been shown to be given by $n=\sqrt{\frac{E}{\rho}}\cdot\frac{k}{\pi l^2}\cdot m^2$, where E is Young's modulus, and ρ the density of the material; k is the radius of gyration of the bar, whose length is l, while m is a constant factor depending on the particular mode of vibration excited. Thus denoting the frequencies of vibration at $\theta_0°$ C. and $\theta_1°$ C. by n_0 and n_1 respectively, it follows that $\frac{n_1}{n_0}=\sqrt{\frac{E_1}{E_0}}\cdot\sqrt{\frac{\rho_0}{\rho_1}}\cdot\frac{k_1}{k_0}\cdot\frac{l_0^2}{l_1^2}$, where the suffixes 0 and 1 refer to values of the factors at the appropriate temperatures.

But $k\propto t$, the thickness of bar in the direction of vibration,

therefore $$\frac{k_1}{k_0}\cdot\frac{l_0^2}{l_1^2}=\frac{1}{[1+\alpha(\theta_1-\theta_0)]}, \text{ approx.}$$

where α is the coefficient of linear expansion of the material.

Furthermore $$\frac{\rho_0}{\rho_1}=1+3\alpha(\theta_1-\theta_0),$$

and hence it follows that

$$\left(\frac{n_1}{n_0}\right)^2=[1+\alpha(\theta_1-\theta_0)]\frac{[E_0\{1+\beta(\theta_1-\theta_0)\}]}{E_0},$$

i.e. $$\frac{n_1}{n_0}=1+\left(\frac{\beta+\alpha}{2}\right)(\theta_1-\theta_0) \text{ approx.},$$

where β is the temperature coefficient of Young's modulus. For steel $\beta\simeq-2\times10^{-4}$ and $\alpha\simeq+1\times10^{-5}$, hence it is evident that the change of frequency with temperature of a tuning-fork is primarily dependent upon the variation of Young's modulus of steel.

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CHAPTER 13

FORCED VIBRATIONS AND ACOUSTICAL MEASUREMENTS

Forced vibrations

A free vibration is maintained solely by the energy stored up in the system at the commencement of the vibration, and owing to the inevitable presence of damping, however small, the amplitude will gradually decay with time. Consequently in order to maintain a constant amplitude, energy must be continually supplied to the system so as to overcome the effect of frictional resistance, and if the external agency of energy supply is in the nature of a periodic force, the system is said to be in a state of *forced* vibration. This driving force in the case of sound reproducing apparatus will be of electromagnetic origin, whereas in recording or acoustic measuring apparatus it will be the fluctuating pressure acting on the diaphragm of the receiving instrument.

If M is the mass of the vibrating system, S the elastic restoring force per unit displacement, R the frictional resistance per unit velocity, and $F \sin ft$ is the applied periodic force, then the equation of motion is

$$M\ddot{x}+R\dot{x}+Sx=F\sin ft \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where x denotes the displacement at any time t.

This equation implies that if the motion is restricted to small amplitudes and velocities, the elastic and resistance forces may be taken as proportional to the displacement and velocity respectively. The complete solution of equation (1) comprises two terms, known respectively as the complementary function and the particular integral.

The complementary function is the solution of the equation for the damped free vibration, $M\ddot{x}+R\dot{x}+Sx=0$, and is given by (App. 2)

$$x=Ae^{-\frac{R}{2M}t}\sin\left(\sqrt{\frac{S}{M}-\frac{R^2}{4M^2}}.t-a\right) \quad . \quad . \quad . \quad . \quad (2)$$

for the case where $\frac{R}{2M}<\sqrt{\frac{S}{M}}$. The natural frequency of the system, when R is small, is given by $n=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{S}{M}}$, and the rate of decay of the free vibrations is expressed by the logarithmic decrement which is defined by $\delta=\log\frac{X_n}{X_{n+1}}$. X_n and X_{n+1} are successive *amplitudes* on the *same* side of the equilibrium position, *i.e.* the zero, of the system (Fig. 13.1). These amplitudes will correspond to maximum values of the expression (2), and are accurately deduced by equating $\frac{dx}{dt}=0$, which leads to the condition that

$$\tan\left[\left(\frac{S}{M}-\frac{R^2}{4M^2}\right)^{\frac{1}{2}}.t-a\right]=\frac{\left(\frac{S}{M}-\frac{R^2}{4M^2}\right)^{\frac{1}{2}}}{\frac{R}{2M}}.$$

In most practical cases, however, the times t_M at which these peaks occur are only slightly earlier than the times at which the curve $x=Ae^{-\frac{R}{2M}t}$ is tangential to the curve representing equation (2). These tangencies obviously occur at the values of t obtained by equating $\sin\left[\left(\frac{S}{M}-\frac{R^2}{4M^2}\right)^{\frac{1}{2}}.t-\alpha\right]=1$, and the corresponding values of x are usually taken as approximating to the amplitudes of the free vibration. Fig. 13.2*a* shows the usual practical case of small damping, where the period of the vibration approximates to that of the undamped free vibration (T_0), and the tangency points $\frac{T_0}{4}$, $\frac{5T_0}{4}$, etc., approximate, with sufficient accuracy, to the successive values of T_M. In the case of large damping, however, as shown in Fig. 13.2*b*, there is an appreciable disparity in the values of T_M and the times corresponding to the

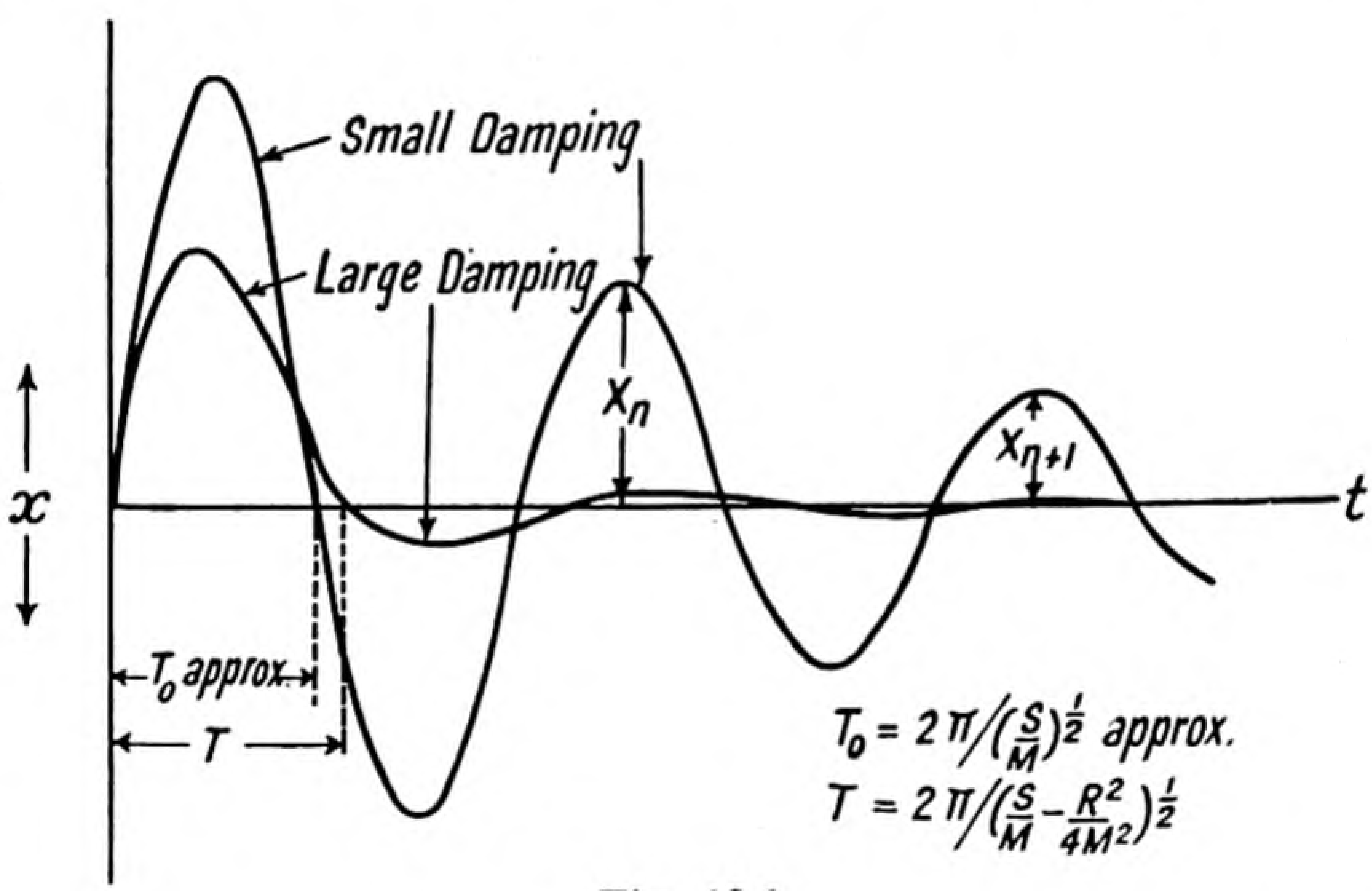


Fig. 13.1.

tangency points $\frac{T}{4}$, $\frac{5T}{4}$ etc., where T now assumes a value $2\pi\Big/\left(\frac{S}{M}-\frac{R^2}{4M^2}\right)^{\frac{1}{2}}$ which no longer approximates to the undamped free vibration period $T_0=2\pi\Big/\left(\frac{S}{M}\right)^{\frac{1}{2}}$. The effect of damping in increasing the period, and decreasing the amplitude, of the motion is emphasised in Fig. 13.1. The degree of damping of a mechanical system may be expressed by means of the logarithmic decrement (δ), defined by

$$* \quad \delta=\log_e\frac{X_n}{X_{n+1}}=\log_e\left[\frac{e^{-\frac{R}{2M}.t_n}}{e^{-\frac{R}{2M}(t_n+T)}}\right]=\log_e e^{\frac{TR}{2M}}=\frac{TR}{2M} \quad . \quad (3)$$

* If δ is defined in terms of successive amplitudes on *opposite* sides of the equilibrium position, $\delta=TR/4M$.

where X_n and X_{n+1} are successive maximum displacements on the *same* side of the zero position of the system. The above expression for very small values of δ may be written as $\frac{X_n}{X_{n+1}}=e^{\delta}=1+\delta$, or $\delta=\frac{X_n-X_{n+1}}{X_{n+1}}$. An alternative method of deducing the logarithmic decrement is given on p. 225.

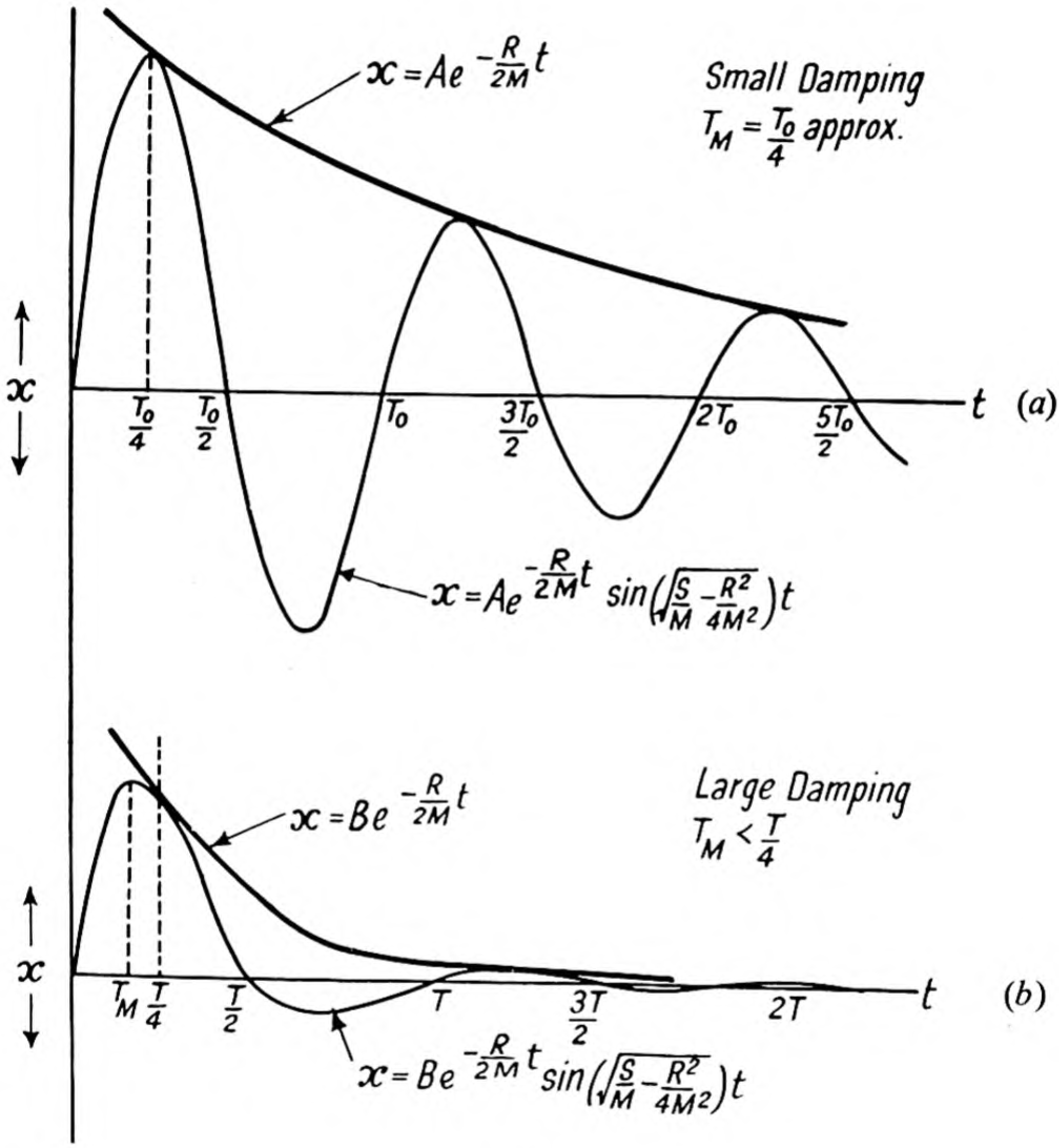


Fig. 13.2.

The second term, or particular integral of the complete solution of equation (1) may be shown to be (see Appendix)

$$x=\frac{\frac{F}{M}}{(\omega^2-f^2)^2+4\gamma^2f^2}[(\omega^2-f^2)\sin ft-2\gamma f\cos ft]$$

$$=\mathrm{B}\sin(ft-\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $$\omega^2=\frac{S}{M},\quad 2\gamma=\frac{R}{M},\quad \tan\beta=\frac{2\gamma f}{(\omega^2-f^2)},$$

and
$$B=\frac{\frac{F}{M}}{[(\omega^2-f^2)^2+4\gamma^2 f^2]^{\frac{1}{2}}}.$$

The complete solution of equation (1) is therefore

$$x=Ae^{-\frac{R}{2M}t}\sin\left(\sqrt{\frac{S}{M}-\frac{R^2}{4m^2}}.t-a\right)$$

$$+B\sin\left[ft-\tan^{-1}\left(\frac{\frac{R}{M}.f}{\frac{S}{M}-f^2}\right)\right] \quad . \quad . \quad . \quad (5)$$

Hence it follows that when the force $F\sin ft$ is first applied the ensuing motion is complicated, and is a combination of two harmonic motions of, in general, different frequencies. The first term, as its usual designation of "transient" denotes, dies out quite rapidly even with small damping, and the motion then assumes the steady state as given by the second term of the solution (5). This term represents a S.H.M. of constant amplitude, and of frequency equal to that of the driving force, and it may be re-written as

$$x=\frac{F}{M\left[\left(\frac{S}{M}-f^2\right)^2+\frac{R^2}{M^2}f^2\right]^{\frac{1}{2}}}\sin(ft-\beta)$$

$$=\frac{F}{f\left[\left(\frac{S}{f}-Mf\right)^2+R^2\right]^{\frac{1}{2}}}\sin(ft-\beta),$$

or $$x=\frac{F}{fZ}.\sin(ft-\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where $Z=\left[\left(\frac{S}{f}-Mf\right)^2+R^2\right]^{\frac{1}{2}}$ is known as the *mechanical impedance* of the vibrating system.

It is sometimes more convenient to plot an *amplitude-ratio* (G)—*frequency-ratio* (ν) curve (Fig. 13.3), where G is defined by $\frac{X}{X_s}$. X and X_s are the respective displacement-amplitudes at a given frequency $\frac{f}{2\pi}$ and at zero frequency (*i.e.* the static displacement), due to applied forces of the same amplitude. The frequency ratio is defined by $\nu=\frac{f}{\omega}$, and will be equal to unity at resonance, when the applied frequency is identical with the natural frequency of the system.

Now from (6) it follows that

$$G=\frac{\dfrac{F}{S}}{\dfrac{F}{S}\left[\left(1-\dfrac{M}{S}f^2\right)^2+\dfrac{f^2R^2}{S^2}\right]^{\frac{1}{2}}}=\frac{1}{\left[(1-\nu^2)^2+\dfrac{R^2}{SM}\nu^2\right]^{\frac{1}{2}}} \quad . \quad (7)$$

since

$$\omega^2=\frac{S}{M}.$$

Hence at resonance G has a maximum value G_ω, where $G_\omega=\dfrac{1}{\left(\dfrac{R^2}{SM}\right)^{\frac{1}{2}}}$.

If ν_1 and ν_2 are the frequency-ratios corresponding to values of $G=\dfrac{G_\omega}{\sqrt{2}}$ (see Fig. 13.3), then it can be shown that for small values of damping the log. dec. (δ) of the system is given by $\delta=\frac{\pi}{2}(\nu_2{}^2-\nu_1{}^2)$.

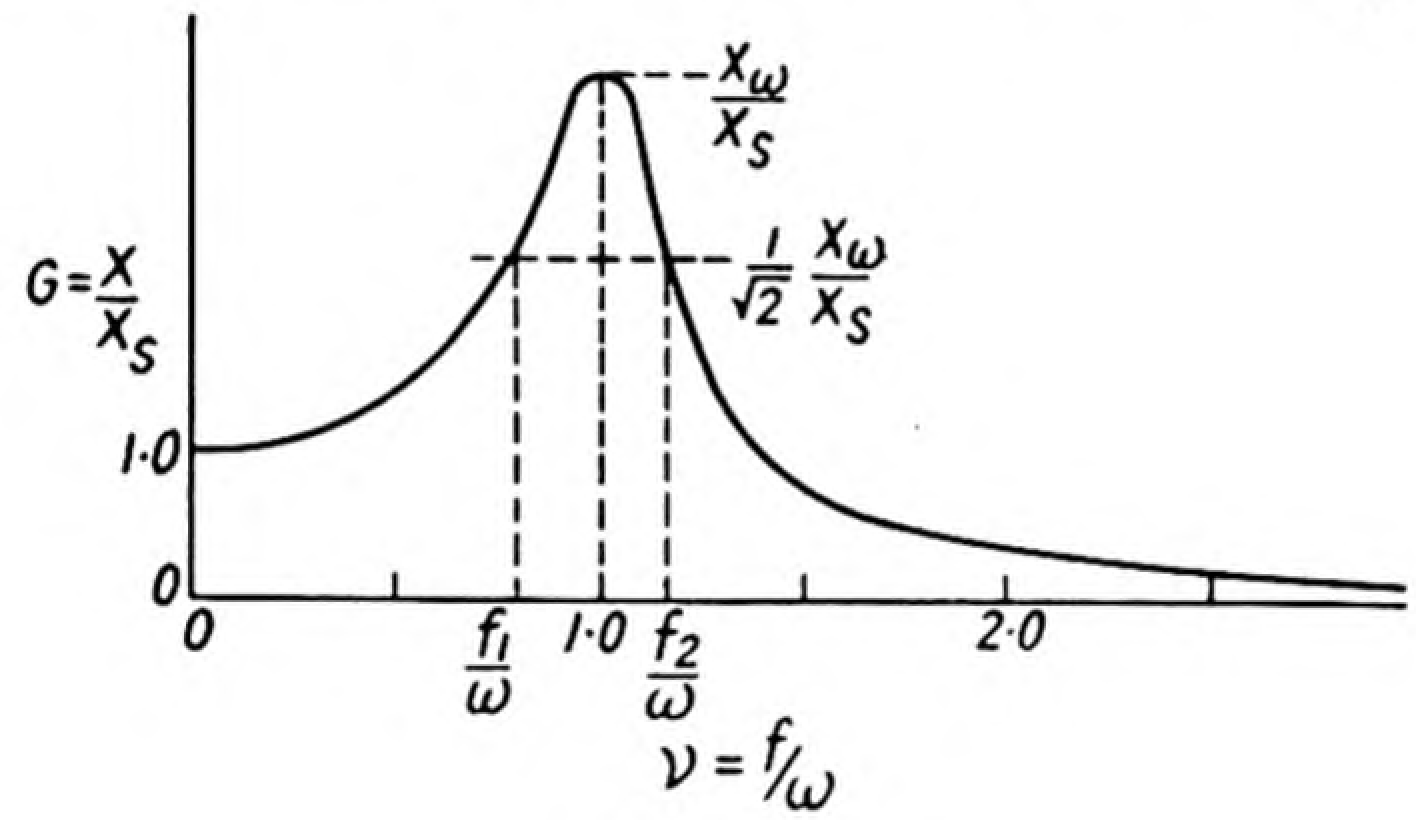


Fig. 13.3.

Phase and energy relations of a forced vibration in the steady state

From equation (6) it is seen that the displacement x *lags* behind the exciting force $F\sin ft$ by the angle

$$\beta=\tan^{-1}\left(\frac{f\dfrac{R}{M}}{\dfrac{S}{M}-f^2}\right)=\tan^{-1}\left(\frac{f\dfrac{R}{M}}{\omega^2-f^2}\right)$$

$$=\sin^{-1}\left(\frac{f\dfrac{R}{M}}{\left[(\omega^2-f^2)^2+f^2\left(\dfrac{R}{M}\right)^2\right]}\right)$$

$$=\cos^{-1}\left(\frac{(\omega^2-f^2)}{\left[(\omega^2-f^2)^2+f^2\left(\dfrac{R}{M}\right)^2\right]^{\frac{1}{2}}}\right) . \quad (8)$$

From the above expressions and for a given value of R, it follows that

(i) if $f<\omega$, β lies between 0 and $\frac{\pi}{2}$,

(ii) if $f=\omega$, $\beta=\frac{\pi}{2}$, and

(iii) if $f>\omega$, β lies between $\frac{\pi}{2}$ and π.

Hence, as the exciting frequency (f) is varied the driven (or excited) system will respond so that, at any instant, the displacement lags behind the applied force by the angle β, which varies in the manner shown in Fig. 13.4. When R is *very* small the phase angle β is practically zero for all values of $f<\omega$, but at $f=\omega$ there is a sudden phase change of π, and $\beta=\pi$ for all values of $f>\omega$. At these higher frequencies, therefore, it means that the system executing forced vibrations moves in the direction opposite to the applied force. It is

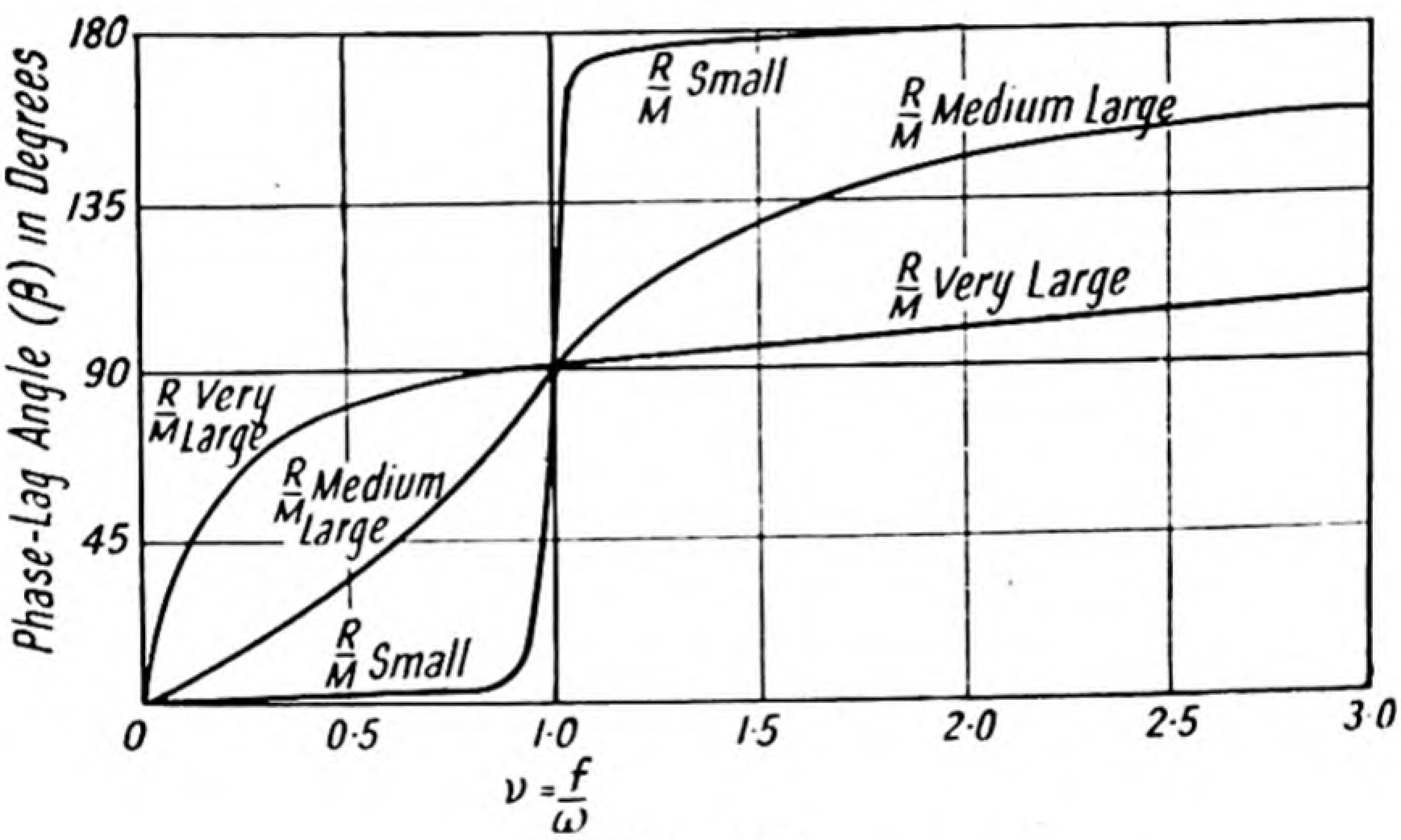


Fig. 13.4.

to be noted from Fig. 13.4 that $\beta=\frac{\pi}{2}$ at resonance for *all* values of damping, and hence from (6) the expressions for the displacement and velocity become respectively:—

$$x=\frac{F}{\omega Z}\sin\left(\omega t-\frac{\pi}{2}\right)=-\frac{F}{\omega Z}\cos\omega t \quad . \quad . \quad . \quad . \quad (9)$$

and

$$\dot{x}=\frac{F}{Z}\cos\left(\omega t-\frac{\pi}{2}\right)=\frac{F}{Z}\sin\omega t \quad . \quad . \quad . \quad . \quad (10)$$

An immediate deduction from these results is that at resonance the velocity of the driven system is in phase, but the displacement is in quadrature (*i.e.* $\frac{\pi}{2}$ out of phase), with the applied force. The significance of these phase changes is rendered more evident when considering the energy of the system, but the interpretation of the motion in terms of rotating vectors will be first undertaken as providing a good insight into the forces operating during the motion.

Since a S.H.M. of a vibrating particle may be represented by the projection of the extremity of a rotating vector [of length proportional to the *amplitude* (X) of the motion] upon a fixed axis through the centre of rotation, it follows that its velocity (fX) and acceleration (f^2X) may be represented likewise as shown in Fig. 13.5*a*. The corresponding force-vector diagram for a damped harmonic motion is shown in Fig. 13.5*b*, the direction of the forces being such as to resist the motion.

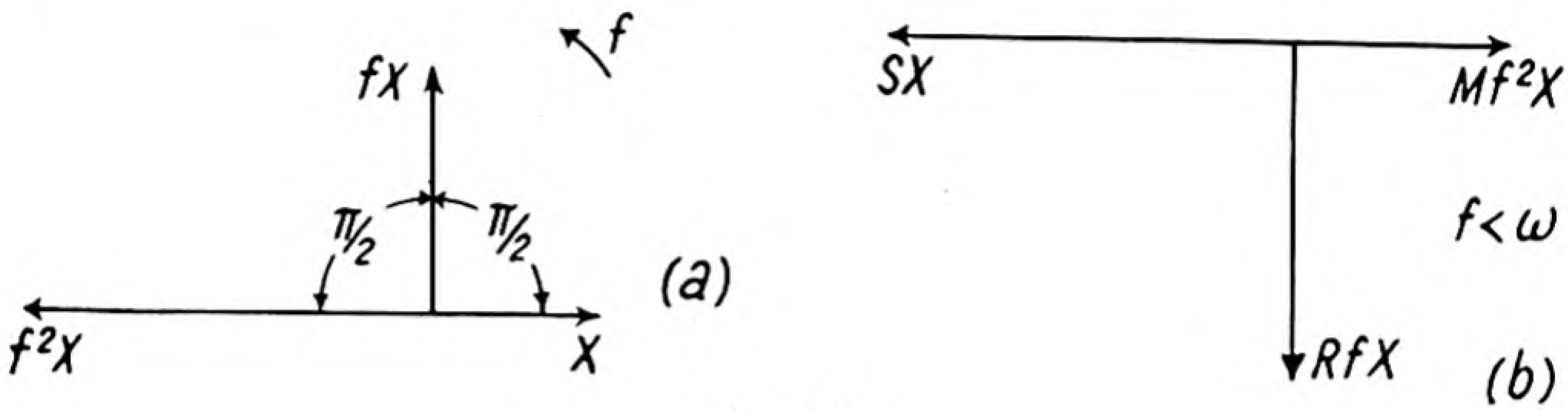


Fig. 13.5.

The diagram refers to the case when $f<\omega$ so that, since $SX = M\omega^2X$ from (p. 224), the restoring force SX will be greater than the inertia force Mf^2X. Such a system will not be maintained, unless an external force is applied which has resolved components to balance both the damping force and the excess of the restoring over the inertia force, *i.e.* the *driving* force F must be represented by a rotating vector whose magnitude F and phase angle β, with respect to the displacement X, will be given by the force polygon of Fig. 13.6*b*. At resonance $f=\omega$, and therefore $SX = M\omega^2X$, so that the applied periodic force F has only to balance the damping force $R\omega X$, *i.e.* the phase angle β at

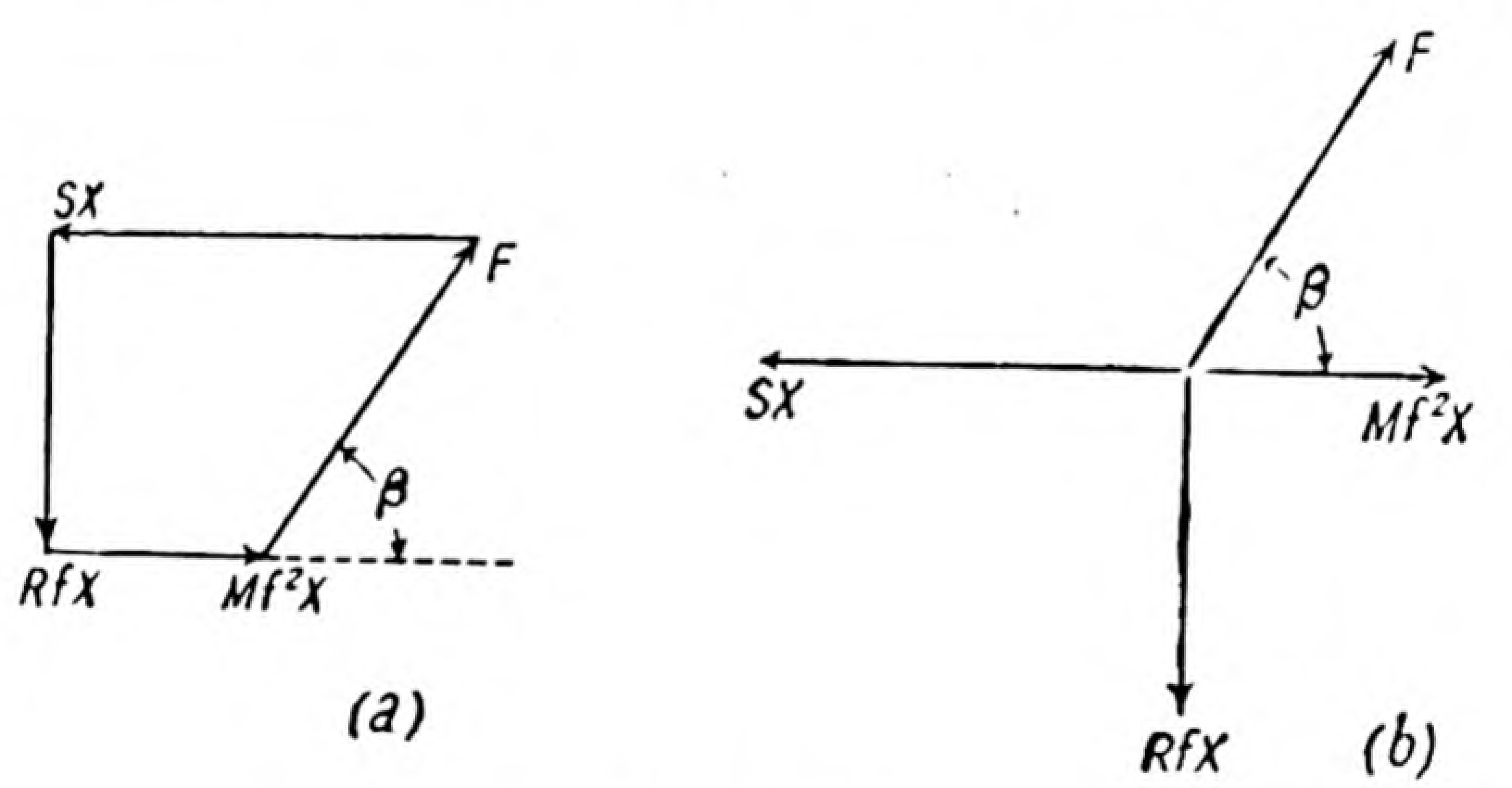


Fig. 13.6.

resonance will be $\frac{\pi}{2}$ as already shown analytically. When the frequency $\left(\frac{f}{2\pi}\right)$ of the driver is greater than the natural frequency $\left(\frac{\omega}{2\pi}\right)$ of the system it is easily deduced that β lies between $\frac{\pi}{2}$ and π (see Fig. 13.7).

Rate of energy supply to driven system

In the steady state at any given time t the work done by the driver on the system, when it moves through a distance dx is given by

$$dW = (F \sin ft)dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If this displacement takes place in a time dt sec., then from (6) it follows that

$$dW = (F \sin ft) . \frac{dx}{dt} . dt$$

$$= (F \sin ft) . \frac{F}{Z} \cos (ft - \beta) . dt \quad . \quad . \quad . \quad . \quad (12)$$

The *rate* of energy supply is therefore

$$\frac{dW}{dt} = (F \sin ft) . \frac{F}{Z} \cos (ft - \beta) \quad . \quad . \quad . \quad . \quad (13)$$

This expression (13) is analogous to the electrical formula for the power expended in an alternating current circuit, of supply frequency

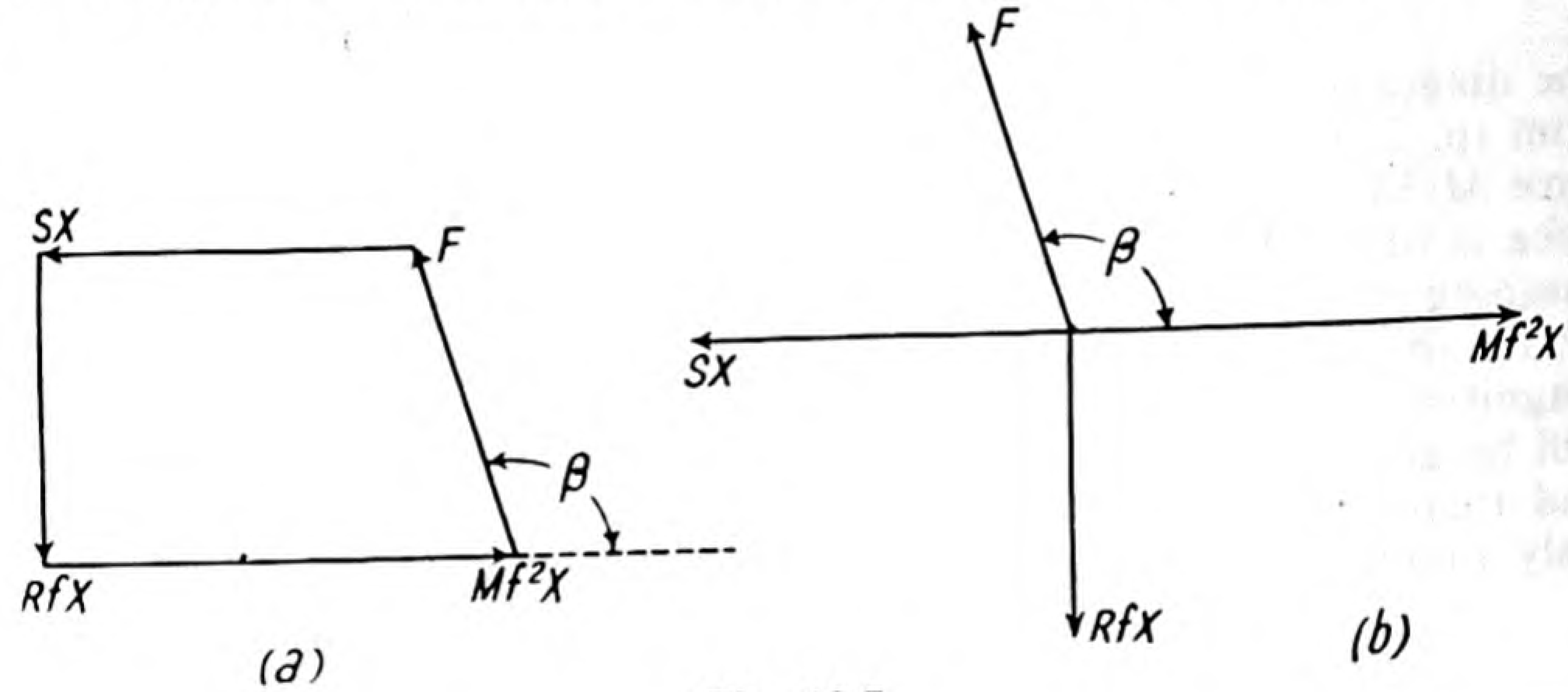


Fig. 13.7.

$\frac{\omega}{2\pi}$, viz. $P = \frac{dW}{dt} = E \sin \omega t \times I \cos (\omega t - \phi) = E \sin \omega t \times \frac{E}{Z} \cos (\omega t - \phi)$, where E, I and Z have their usual electrical significance, and $\cos \phi$ is known as the power factor of the circuit.

The *average rate* of energy supply, $\overline{\frac{dW}{dt}}$, to the driven system is therefore given from (13) by the average value, over a complete cycle, of

$$\frac{F^2}{Z} \sin ft \cos (ft - \beta),$$

i.e. the average value of $\frac{F^2}{Z}[(\sin ft \cos ft) \cos \beta + \sin^2 ft \sin \beta]$.

Hence

$$\overline{\frac{dW}{dt}} = \frac{F^2}{Z} . \frac{1}{2} \sin \beta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$= \frac{F^2}{Z} . \frac{R}{2Z}$$

$$= \frac{1}{2} R \left(\frac{F}{Z}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

since $$\sin\beta=\frac{R}{Z} \text{ from (8).}$$

Alternatively, since the amplitude (X) of the motion is given by the maximum value of the displacement (x), it follows from (6) that $X=\frac{F}{fZ}$ and so (15) may be re-written as

$$\frac{\overline{dW}}{dt}=\frac{1}{2}Rf^2X^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Expression (16), it should be noted, is analogous to that giving the power P expended in an electrical resistance R by an alternating current of peak value $I=\frac{E}{Z}$, where E is the amplitude of the alternating potential applied to the circuit of total impedance Z. This electrical expression is

$$P=\frac{1}{2}R\left(\frac{E}{Z}\right)^2=R.\left(\frac{I}{\sqrt{2}}\right)^2,$$

where $\frac{I}{\sqrt{2}}$ is the root mean square value of the current. In the mechanical problem the energy supplied by the driver is used to overcome the frictional resistance opposing the motion of the driven system, and the *rate* of working against the frictional force $R.\frac{dx}{dt}$ will be $P=\left\{\left(R\frac{dx}{dt}\right).dx\right\}\div dt$, i.e. $R\left(\frac{dx}{dt}\right)^2$, since dx is the displacement during a time dt sec. Now the kinetic energy of the system at any instant t will be given by

$$T=\frac{1}{2}M\left(\frac{dx}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and the average value over a whole period will be

$$\overline{T}=\frac{T_{\max}}{2}=\frac{1}{4}M\left(\frac{dx}{dt}\right)^2_{\max}=\frac{M}{4}\cdot\left(\frac{F}{Z}\right)^2 \text{ from (10).}$$

Hence the average power $\overline{P}=R\left(\frac{dx}{dt}\right)^2_{\text{average}}=\frac{2\overline{T}R}{M}$ from (17),

i.e. $\overline{P}=\frac{2R}{M}\cdot\frac{M}{4}\left(\frac{F}{Z}\right)^2=\frac{1}{2}R\left(\frac{F}{Z}\right)^2$, which is identical with the average rate of energy supply from the driver, as given by expression (15). It follows, therefore, that the amplitude and phase of the driven system become so adjusted that the energy lost per cycle by friction (and which reappears as thermal energy within the system) is exactly equal to that supplied per cycle by the driver.

The *total* energy supplied per cycle will be given by the product of the average rate of energy supply times the period of vibration,

i.e.
$$=\frac{F^2}{Z}\cdot\frac{1}{2}(\sin\beta)\times\frac{2\pi}{f}$$

$$=\frac{\pi F^2}{fZ}\sin\beta \text{ or } \pi FX\sin\beta \quad . \quad . \quad . \quad . \quad (18)$$

since the amplitude $X=\frac{F}{fZ}$. In the *complete absence* of damping the energy supply would be zero ($\beta=0$ or π), except at resonance $\left(\beta=\frac{\pi}{2}\right)$ when energy would be absorbed in building up an infinitely large amplitude of motion.

Forced vibrations and mechanical acoustical systems

Now any acoustical instrument or apparatus may be regarded as a combination of the acoustical elements M, R and S, in an analogous manner to the combination of the corresponding elements L, R and C of an electrical circuit. As an example, consider the diaphragm of a telephone receiver in which the external force is supplied by the alternating excess pressure, $p=p_0 \sin ft$, of the impinging sound waves. Then, provided the dimensions of the diaphragm are small compared with the length of the sound waves, the resulting displacement at any point (from 6) will be given by

$$x=\frac{p_0 A}{fZ}\sin(ft-\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and the corresponding velocity by

$$\dot{x}=\frac{p_0 A}{Z}\cos(ft-\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where A=area of the diaphragm. The mechanical impedance Z of the vibrating system is given by $Z=\left[\left(\frac{S}{f}-Mf\right)^2+R^2\right]^{\frac{1}{2}}$, the term $\left(\frac{S}{f}-Mf\right)$ being known as the mechanical reactance.

In the case of sustained periodic sounds it is only the frequency characteristic which is important, and so it is the magnitude and not the phase of the displacement which is concerned. The *amplitude* (X) of the displacement (from 6) is given by the following alternative expressions:—

$$X=\frac{p_0 A}{[(S-Mf^2)^2+f^2R^2]^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad (21a)$$

$$=\frac{p_0 A}{M\left[(\omega^2-f^2)^2+\frac{R^2}{M^2}f^2\right]^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (21b)$$

$$=\frac{p_0 A}{Mf\left[\omega^2\left(\frac{\omega}{f}-\frac{f}{\omega}\right)^2+\frac{R^2}{M^2}\right]^{\frac{1}{2}}} \quad . \quad . \quad . \quad (21c)$$

Amplitude-frequency characteristics

The variation of amplitude with frequency is ascertained by differentiating (21b) with respect to f and equating to zero. The result obtained is $4f^3-2f\left\{2\omega^2-\left(\frac{R}{M}\right)^2\right\}=0$, i.e. $f=0$ or $f=\left[\omega^2-\frac{1}{2}\left(\frac{R}{M}\right)^2\right]^{\frac{1}{2}}$.

The value $f_\omega = \left[\omega^2 - \frac{1}{2}\left(\frac{R}{M}\right)^2\right]^{\frac{1}{2}}$ corresponds to a maximum amplitude of vibration, while at $f=0$ the amplitude is a minimum and equal to $\frac{p_0A}{M\omega^2} = \frac{p_0A}{S}$. This quantity $\frac{p_0A}{S}$ will represent the static displacement of the system, X_s, due to the application of the force p_0A. The displacement will also approach a minimum value as f increases indefinitely above f_ω. By substituting the value of f_ω in equation (21*a*) the corresponding amplitude (Fig. 13.8) is seen to be $\frac{p_0A}{R\left[\frac{S}{M} - \left(\frac{R}{2M}\right)^2\right]^{\frac{1}{2}}}$. The factor $\left(\frac{R}{2M}\right)$ which occurs in the denominator of this expression is the *decay* factor which determines the rate of disappearance of the transient term in a forced vibration (p. 224); in the curve under discussion it is a measure of the sharpness of the peak of the characteristic. This

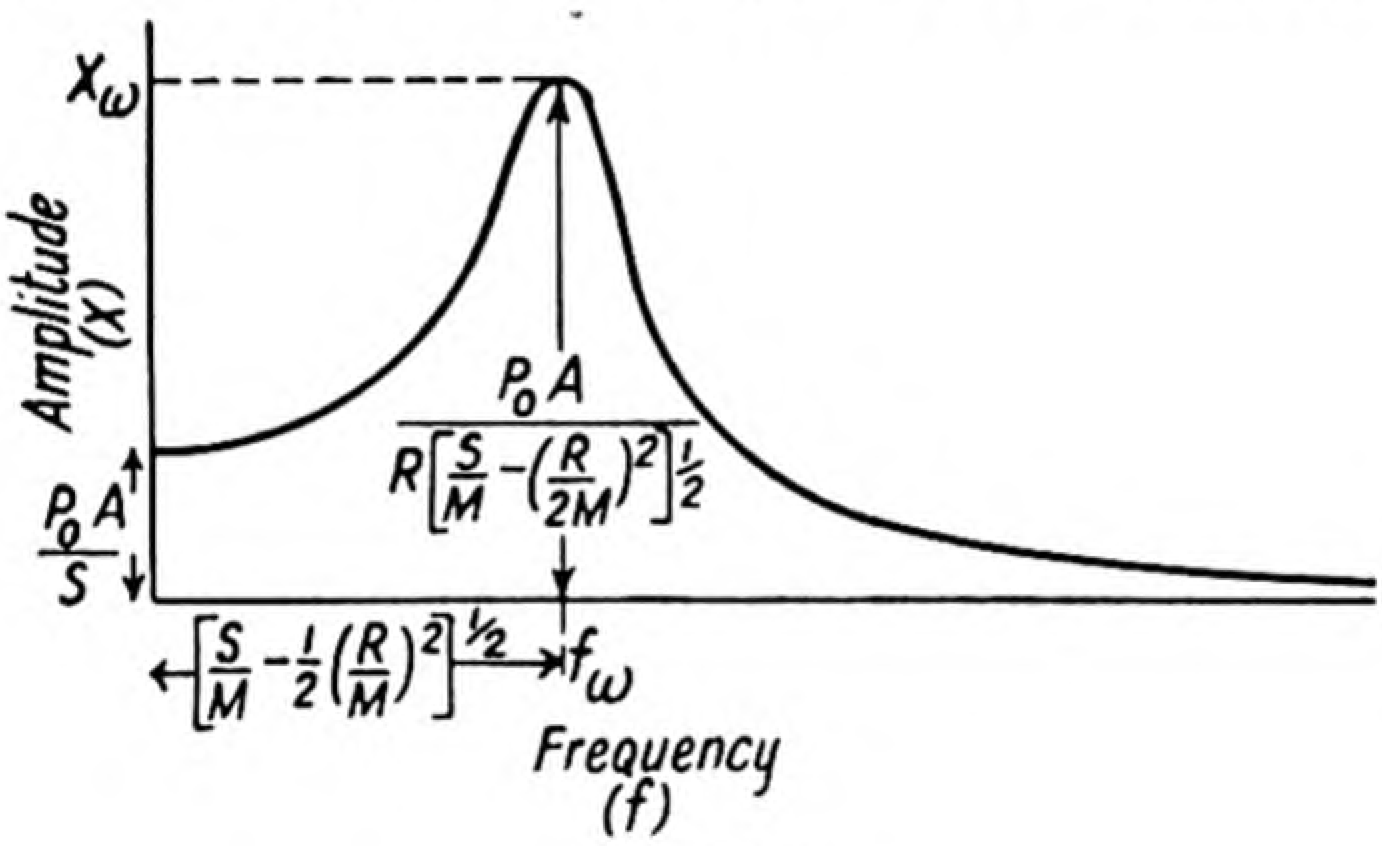


Fig. 13.8.

fact and the way in which the decay factor compares with the $\frac{1}{Q}$ of a tuned radio circuit are considered on p. 234.

The period of vibration for a free, undamped, simple harmonic motion is given by $T_0 = \frac{2\pi}{\omega}$, whereas for a resisted motion, i.e. R is appreciable, $T = \frac{2\pi}{\left[\omega^2 - \frac{R^2}{4M^2}\right]^{\frac{1}{2}}}$. It is to be noted, therefore, that the period $T_m = \frac{2\pi}{\left[\omega^2 - \frac{1}{2}\left(\frac{R}{M}\right)^2\right]^{\frac{1}{2}}}$ corresponding to frequency at maximum amplitude is greater than either T_0 or T.

Usually the effective values of the elements M, S and R of a practical system only remain constant over a limited frequency range, hence the distorting effects of resonance on the response of such a system might be overcome if these elements varied in such a way that the impedance Z itself remained invariable. As an example, if S varied

as f, while M varied as $\frac{1}{f}$, then the reactance $\left(\frac{S}{f}-Mf\right)$ of the system would be a constant at all frequencies, and an effect of this kind is presumed to happen with light conical sound radiators. In contrast to the "damping" of resonance by increasing the value of R, which has the effect of flattening the peak of curve (Fig. 13.9), the above procedure has been termed as "distributing" the resonance.

Response of a mechanical system

The response of such a system as considered above, at any given frequency and for the same applied pressure amplitude, is measured by the energy of the forced vibration in the steady state. It follows from (8) and (18) that the total energy supply (W) per cycle may be written as

$$\frac{\pi(p_0A)^2}{fZ}\sin\beta=\frac{\pi(p_0A)^2}{fZ}\cdot\frac{R}{Z}=\frac{\pi(p_0A)^2R}{fZ^2} \quad . \quad . \quad . \quad . \quad (22)$$

since here $F=p_0A$. Hence the *energy* of the forced vibration will be a maximum when Z^2 is a minimum, *i.e.* when $\left[\left(\frac{S}{f}-Mf\right)^2+R^2\right]$ is a minimum. The required condition for a given value of R is therefore that $\frac{S}{f}-Mf=0$, i.e. $\frac{S}{f}=Mf$ or $f^2=\frac{S}{M}=\omega^2$, whence $f=\omega$.

This resonance condition is known as *velocity resonance*, the reason being readily seen by noting from (10) that the expression for the velocity amplitude of the displacement, viz. $\dot{X}=\frac{p_0A}{Z}=\frac{p_0A}{\left[\left(\frac{S}{f}-Mf\right)^2+R^2\right]^{\frac{1}{2}}}$, is a maximum when $f=\omega$. The condition for velocity resonance, which corresponds to that for *current* resonance in an electrical circuit, is therefore that the frequency of the applied force should be identical with the *undamped* frequency of the driven system, which differs from the condition for *amplitude* or *displacement* resonance derived previously. The maximum value of W will be given by $W_{\max}=\frac{\pi(p_0A)^2R}{\omega R^2}=\frac{\pi(p_0A)^2}{\omega R}$.

In order to determine how the energy of the forced vibration varies with the frequency and damping, Barton derives an expression for the energy of the system at the particular instant when the displacement is zero, *i.e.* when the energy is wholly kinetic and is given by $\frac{1}{2}M\dot{x}_{\max}^2$. This energy per unit driving force is termed the response N_f of the system, when the driving frequency is $f/2\pi$.

From (10) it follows that

$$N_f=\frac{M}{2Z^2}$$

$$=\frac{1}{2M\left[\left\{\omega\left(\frac{\omega}{f}-\frac{f}{\omega}\right)\right\}^2+\left(\frac{R}{M}\right)^2\right]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The *sharpness of resonance* of the system is measured by the ratio

$$\frac{N_f}{N_{\max}}=\frac{\left(\frac{R}{M}\right)^2}{\left[\left\{\omega\left(\frac{\omega}{f}-\frac{f}{\omega}\right)\right\}^2+\left(\frac{R}{M}\right)^2\right]} \quad . \quad . \quad . \quad . \quad (24)$$

An inspection of the above formulae leads to the immediate deduction that the decrease of the vibrational energy of the driven system, due to the non-coincidence of the applied $\left(\frac{f}{2\pi}\right)$ with the natural $\left(\frac{\omega}{2\pi}\right)$ frequency of the system, is the same for a given ratio q of frequencies, whether f is too great or too small. This result follows from the symmetry of the term $\left(\frac{\omega}{f}-\frac{f}{\omega}\right)^2$ in the denominator, and the ratio $\frac{\omega}{f}=q$ measures the *musical interval* between the frequency of the

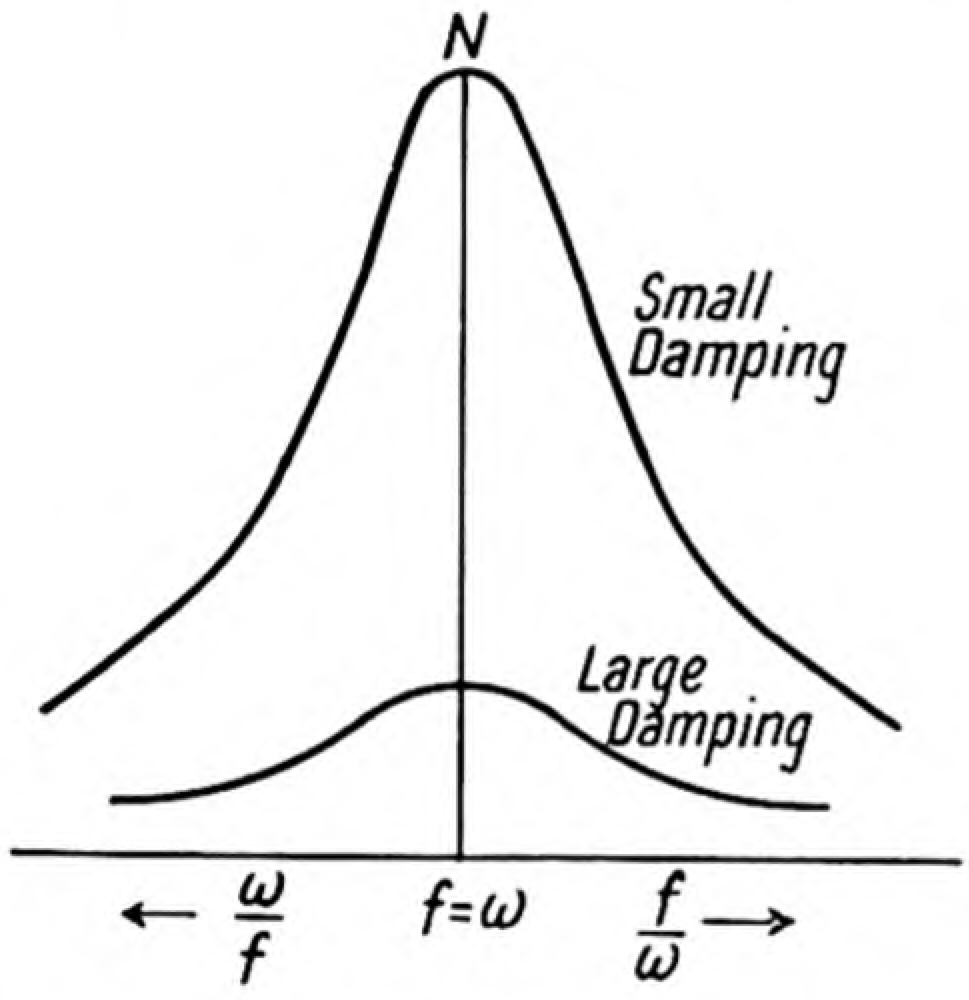


Fig. 13.9.

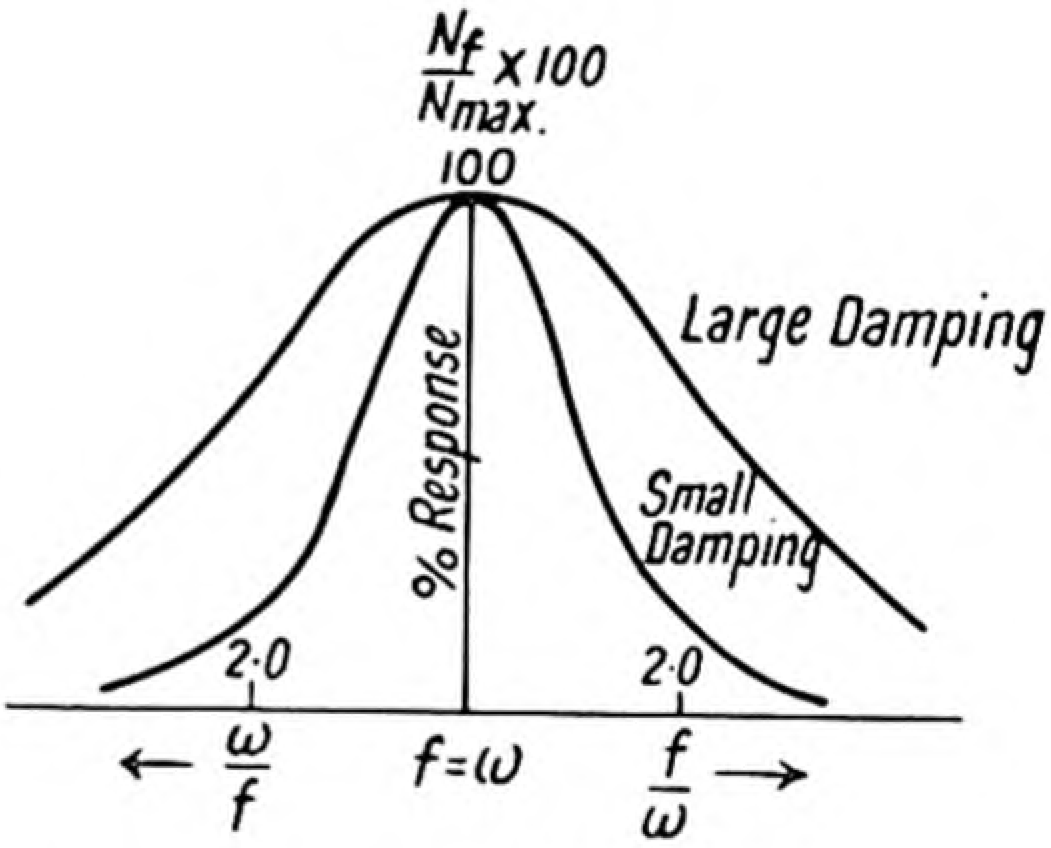


Fig. 13.10.

applied force and that of the undamped frequency of the system. Figs. 13.9 and 13.10 show respectively the response and sharpness of resonance curves and how they depend upon the damping of the driven system.

The whole process of sound recording involves forced vibrations, and every part of the recording apparatus such as membranes, lever mechanisms, etc., is a possible resonator. Hence, in order to avoid the undue enhancement of any particular component of a complex wave-form which is being recorded, it is necessary to make the lowest natural frequency of the recording system at least equal to that of the highest frequency component of the wave. Greater fidelity of recording will be obtained if, in addition, the natural frequencies of the recording apparatus are strongly damped so that the response curve is very flat (see Fig. 13.11). The attainment of this condition will involve a loss of sensitivity which, however, may be overcome in electrical recording by suitable valve amplifying circuits. The exact rendering

of the relative phases of the various components of the complex wave occurs only at very low frequency ratios (Fig. 13.4), so that the natural frequencies (ω) must be very large, which means that purely mechanical methods of recording even moderately high frequencies, are rather out of the question.

The "Q" of a mechanical system

The complementary function (2) of the solution of equation (1) may be rewritten as

$$x=A_e{}^{-\delta t/T} \sin\left(\sqrt{\frac{S}{M}-\frac{R^2}{4M^2}}.t-\alpha\right) \quad . \quad . \quad . \quad (25)$$

where δ is defined by (3).

The form of the amplitude term in (25) indicates that the amplitude

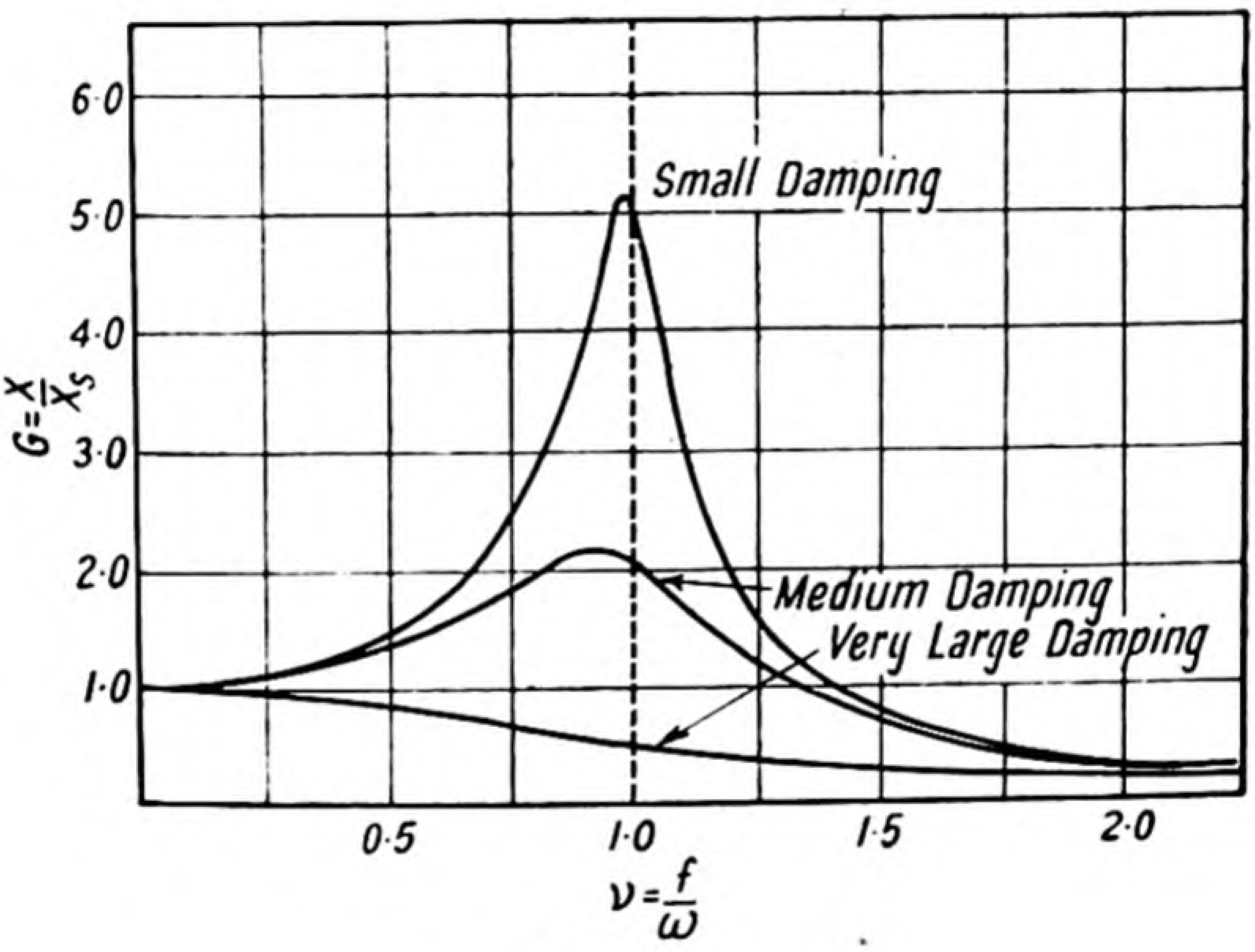


Fig. 13.11.

decreases by a factor $\frac{1}{e}$ in a time $\frac{T}{\delta}$ secs., which is therefore a measure of how rapidly the vibrations are damped by friction and is called the *modulus of decay* of the motion. An alternative way of describing this quality of a mechanical system is by means of the "Q" of the system, a quantity originally employed in electrical circuits.

In the case of a series resonant electrical circuit the "Q" is given by $\frac{\omega_0 L}{R}$, where $\frac{\omega_0}{2\pi}$ is the resonant frequency, L the self-inductance, and R the electrical resistance of the circuit. By analogy the "Q" of the mechanical system will be $\frac{\omega_0 M}{R}$, but $\frac{R}{2M}=\frac{\delta}{T}$, and hence

$$Q=\frac{\omega_0 T}{2\delta}=\frac{\pi}{\delta} \text{ approx.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Now, on page 225 it is stated that $\delta=\frac{\pi}{2}(\nu_2{}^2-\nu_1{}^2)$, where $\nu_1=\frac{f_1}{\omega}$, and $\nu_2=\frac{f_2}{\omega}$ (see Fig. 13.3). Hence it follows that

$$\frac{1}{Q}=\frac{\delta}{\pi}=(\nu_2-\nu_1)\left(\frac{\nu_2+\nu_1}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

or for small damping
$$Q\simeq\frac{\omega}{f_2-f_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

since
$$\left(\frac{f_2+f_1}{2\omega}\right)\simeq 1.$$

The "Q" of a mechanical system is therefore inversely proportional to the band-width, f_2-f_1, of the resonance curve (Fig. 13.3), and it is a measure of the selectivity of the system as given by the sharpness of the curve. The resonance curves connecting current and frequency for a series electrical circuit will be of similar form to the curves of Fig. 13.10, that of *small damping*, *i.e.* small R, obviously corresponding to a circuit or system of *high* "Q." The power factor of a coil in an A.C. circuit is given by $\cos\theta=\frac{R}{\sqrt{R^2+\omega^2L^2}}$, which reduces to $\frac{R}{\omega L}$, *i.e.* $\frac{1}{Q}$, when the resistance of the coil is small compared with its reactance. In other words, $1/Q$ is a measure of the power dissipation in a circuit or mechanical system, the higher "Q" corresponding to a lower energy dissipation as is to be expected from the smaller electrical or mechanical resistance involved.

Characteristic properties of a mechanical system

The nature of the forced motion (5) of a mechanical system is dependent upon the relative values of M, R and S, as the following analysis shows.

(i) *M large.*

If
$$fM>\frac{S}{f}, \text{ i.e. } f>\left(\frac{S}{M}\right)^{\frac{1}{2}} \text{ or } f>\omega$$

and
$$fM>R, \text{ i.e. } f>\frac{R}{M}.$$

Then
$$Z\rightarrow fM \text{ and therefore } x=\frac{F}{f^2M}\sin(ft-\beta) \quad . \quad . \quad . \quad (29a)$$

$$\dot{x}=\frac{F}{fM}\cos(ft-\beta) \quad . \quad . \quad . \quad . \quad (29b)$$

$$\ddot{x}=-\frac{F}{M}\sin(ft-\beta) \quad . \quad . \quad . \quad (30)$$

(ii) *R large.*

If $$R>\frac{S}{f}, \text{ i.e. } f>\frac{S}{R}$$

and $$R>fM, \text{ i.e. } f<\frac{R}{M}.$$

Then $$Z\to R \text{ and therefore } x=\frac{F}{fR}\sin(ft-\beta) \quad . \quad . \quad . \quad . \quad (31a)$$

$$\dot{x}=\frac{F}{R}\cos(ft-\beta) \quad . \quad . \quad . \quad . \quad (31b)$$

(iii) *S large.*

If $$\frac{S}{f}>fM, \text{ i.e. } f<\left(\frac{S}{M}\right)^{\frac{1}{2}} \text{ or } f<\omega$$

and $$\frac{S}{f}>R, \text{ i.e. } f<\frac{S}{R}.$$

Then $$Z\to\frac{S}{f} \text{ and therefore } x=\frac{F}{S}\sin(ft-\beta) \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Deductions. These results are important when considering the design of mechanical and acoustical apparatus. For example, it follows directly from the above analysis that there is a *lower* limit to the frequency range in which a driven vibrating system is *mass* (M) controlled, and this limit is well above its natural frequency. For a system to be *stiffness* (S) controlled, however, there is an *upper* frequency limit which is much below the natural frequency. In the case of *resistance* (R) control the frequency range is limited by both upper and lower values, which are, however, both in the region of the natural frequency. By suitable choice of the mechanical constants the different ranges of control may be varied, but every driven system will be resistance controlled near its natural frequency, and respectively stiffness and mass controlled at frequencies much below and above resonance.

ACOUSTICAL MEASUREMENTS

In dealing with the problem of acoustical measurements it is essential to be thoroughly familiar with the fundamental theory and the order of magnitude of the quantities involved. It is proposed now, therefore, to summarise the relevant theory, although certain sections may receive development in other parts of the book. Compressional wave motion only being considered, the elastic modulus concerned is the bulk modulus K, which is defined by $K=-\left(\dfrac{p}{\frac{\Delta V}{V_0}}\right)$, where p is the *excess* pressure required to produce a decrease ΔV in a volume V_0 of material. The ratio $\dfrac{\Delta V}{V_0}$ is known as the *dilatation D*, so that $V=V_0(1+D)$. Correspondingly the *condensation* s is defined by $s=\dfrac{\Delta\rho}{\rho_0}$, or $\rho=\rho_0(1+s)$,

where $\Delta\rho$ represents the change in the density ρ_0 of the medium.

If a constant mass of gas is being considered, then $\rho V=\rho_0 V_0$ or $(1+s)(1+D)=1$, i.e. $s=-D$ approx. or $-\frac{\Delta V}{V_0}$ approx.

Hence it follows that for small values of condensation (cf. p. 67)

$$p=-K\frac{\Delta V}{V_0}=Ks \qquad (33)$$

Consider a plane wave of sound travelling in the direction of the x-axis, in which the displacement η, at time t, of the particles in a plane perpendicular to x, is given by

$$\eta=a \sin \omega\left(t-\frac{x}{c}\right) \qquad (34)$$

a is the amplitude of motion of any particle, $\frac{\omega}{2\pi}$ is the frequency of the sound source, c is the velocity of propagation of the sound waves, and x defines the position of the plane in the path of the waves.

Let p be excess pressure in a plane at x, at time t, and suppose that after a small interval of time Δt the displacement of the particles at x has become $\eta+\Delta\eta$.

The work done per unit area during this displacement is $p\times\Delta\eta$, and the rate of working, *i.e.* the instantaneous value (W_i) of the power (per unit area) will therefore be $\frac{p\times\Delta\eta}{\Delta t}=p\frac{d\eta}{dt}$ in the limit,

i.e.
$$W_i=p\frac{d\eta}{dt} \qquad (35)$$

But from (33) $\quad p=Ks$ and $s=-\frac{d\eta}{dx}=\frac{1}{c}\frac{d\eta}{dt}$ from (34) $\qquad (36)$

$$\therefore\ W_i=Ks.\left(\frac{d\eta}{dt}\right)=\frac{K}{c}\left(\frac{d\eta}{dt}\right)^2=\rho c\left(\frac{d\eta}{dt}\right)^2 \qquad (37)$$

since $c=\sqrt{\frac{K}{\rho}}$ where ρ is the density of the medium. The variation of $\left(\frac{d\eta}{dt}\right)^2$ over a complete cycle will be expressed by the variation of $\omega^2a^2\cos^2\omega\left(t-\frac{x}{c}\right)$ (from 34). Now the *mean* value of $\cos^2\omega\left(t-\frac{x}{c}\right)$ over *any* number of *complete* cycles is $\frac{1}{2}$, hence the *mean* value of

$$\left(\frac{d\eta}{dt}\right)^2=\frac{1}{2}\omega^2a^2=\frac{1}{2}\left(\frac{d\eta}{dt}\right)^2_{\max}=\left[\frac{1}{\sqrt{2}}\cdot\left(\frac{d\eta}{dt}\right)_{\max}\right]^2 \qquad (38)$$

where $\left(\frac{d\eta}{dt}\right)_{\max}$ is the maximum value of the particle- (or displacement-) velocity.

It follows, therefore, that the *mean* value of the power conveyed by the sound waves per unit area is

$$\overline{W}=\rho c\left[\frac{1}{\sqrt{2}}\left(\frac{d\eta}{dt}\right)_{\max}\right]^2 \qquad (39)$$

This expression for the transmission of acoustical energy is analogous to that for the average electrical power consumed in a resistance R by an alternating current $I=I_0 \sin \omega t$, viz.

$$\text{Mean electrical power}=R\left(\frac{I_0}{\sqrt{2}}\right)^2 \quad . \quad . \quad . \quad . \quad (40)$$

Comparing the expressions (39) and (40), it appears that the particle-velocity is analogous to the current in an electrical circuit, and the product ρc is comparable to the electrical impedance of the circuit.

It should be noted, however, that if the electrical circuit contains an inductance or capacitance (or both), the current and voltage are, in general, not in phase with one another, and an instrument of the wattmeter type has to be employed in which the indications are a function of both the current and voltage at any instant. In acoustical systems of plane waves as seen from (36), the excess pressure, p, and particle-velocity, $\frac{d\eta}{dt}$, are always in phase, so that to determine the acoustical power it is only necessary to measure either of them separately, provided the properties of the medium are also known.

From previous expressions it is easily seen that

$$p=KS=\rho c\left(\frac{d\eta}{dt}\right), \quad \text{i.e.} \quad \frac{d\eta}{dt}=\frac{p}{\rho c} \quad . \quad . \quad . \quad . \quad (41)$$

This equation bears a strong resemblance to Ohm's law in electricity, $I=\frac{E}{R}$, if p is assumed to correspond to the applied E.M.F. (E).

It is both impracticable and undesirable to make direct measurements of excess pressures (or displacement amplitudes) in the medium of the sound propagation, and it is, therefore, necessary to introduce a suitable receiving surface incorporated in the measuring apparatus. The problem is thus intimately connected with that of the incidence of sound waves upon a plane surface, and its mathematical aspect will be considered later in the chapter.

The chief difficulties inherent with the measurement of acoustic energy are that (a) unless high intensities are employed with the undesirable accompanying distortion, the power generated is very small, being of the order of micro-watts (Kaye has calculated that a cup-final crowd of 100,000, all talking continuously and rather loudly throughout the match, would have expended an amount of acoustical energy equivalent to the thermal energy required to make one cup of tea) and only a fraction of this total output is available for measurement, since it is unusual for acoustical energy to be "confined" in the same way as electrical currents transmitted along conductors or electromagnetic radiation propagated in wave-guides; (b) the dimensions of the measuring apparatus are comparable with the average length of the audio-frequency waves, so that the reaction of the instrument upon the wave system (see p. 132) complicates the interpretation of the experimental observations, and the possibility of interference effects arising will result in appreciable variations of instrumental readings with a small change of position in the sound-field;

(c) the amount of energy abstracted or absorbed by the instrument from the incident acoustic energy.

It will be advantageous to give a brief consideration at this stage to the general problem of sound reception. The primary function of any sound receiver is to absorb some fraction of the incident sound energy and transform it into electrical, mechanical or thermal energy, but the method of abstracting this energy is determined by a number of conditions, viz. (i) the character of the medium which is transmitting the sound waves, *i.e.* it will involve the density of, and the velocity of sound within the medium, (ii) the properties of the sound waves themselves as regards frequency, amplitude and wave form, and (iii) the ultimate use for which the sound-energy is required.

In any physical measuring instrument it is eminently desirable that consistent with the accuracy of observation, the minimum amount of energy should be absorbed from the system under quantitative investigation. Now in the case of electrical measurements, good grade voltmeters and ammeters are characterised by the fact that they require for operation only small currents and small potential differences respectively, thus satisfying the criterion that the energy absorption within the instrument, proportional to the product of current and voltage, is a minimum. In an analogous way metrical sound receivers may be conveniently divided into displacement and pressure receivers, corresponding to ammeters and voltmeters respectively. The classes are not completely separable, however, for some acoustic energy $\left(p\frac{d\eta}{dt}\right)$ must be absorbed to operate the measuring mechanism, and therefore a displacement (or velocity) receiver must experience a small pressure variation, and a so-called pressure receiver must suffer a small displacement. A purely pressure receiver would completely reflect, while a purely displacement receiver would completely transmit all incident sound, and they thus correspond by analogy to electrical insulators and conductors (of infinite conductivity) respectively.

The quantity $Z=\rho c$ is known as the characteristic impedance of the medium, and if ρ is measured in grams per cubic centimetre and c in centimetres per second, the value of Z will be expressed in acoustical ohms, and values for a number of typical substances are given below.

	c.g.s. (acoustical ohms)
Hydrogen	$1{\cdot}1\times10^{1}$
Air	$4{\cdot}2\times10^{1}$
Vulcanised Rubber.. ..	5×10^{3}
Cork	$1{\cdot}2\times10^{4}$
Water	$1{\cdot}5\times10^{5}$
Ice	$2{\cdot}9\times10^{5}$
Sand	$3{\cdot}2\times10^{5}$
Brick	$5{\cdot}0\times10^{5}$
Magnesium	$8{\cdot}0\times10^{5}$
Mercury	$1{\cdot}9\times10^{6}$
Iron	$4{\cdot}0\times10^{6}$

It should be noted that, in general, the acoustic impedance will be a function of both temperature and frequency, so that the above values can only be regarded as approximate.

Another acoustic characteristic of a sound material is termed its *acoustic hardness*, and it is defined by $c\rho\omega$ which is equal to $\frac{p_{\max}}{a}$ for $\left(\frac{d\eta}{dt}\right)_{\max}=a\omega$ and $\frac{p_{\max}}{\left(\frac{d\eta}{dt}\right)_{\max}}=c\rho$.

The acoustical impedance, as stated above, depends upon the frequency of the sound waves, which is an important consideration, for example, in the use of porous materials for sound absorption in rooms. The relative values of the acoustic impedances of two media are the chief factors in determining the energy transfer from one to the other, the *reflection* at the interface being *smaller* the more nearly equal are the acoustic impedances. It is only the sudden discontinuities of properties in distances small compared with a wave-length that cause reflections, but it is a problem which has many analogies in other branches of physics. In the field of electricity the transformer is an example of an impedance matching device (see p. 327). Again, when a light wave is incident upon the interface between two media, as at an air-glass boundary surface, reflection occurs. If, however, the two surfaces are separated by a quarter-wave-length film of material whose refractive index $\mu=\sqrt{\mu_1\mu_2}$, where μ_1 and μ_2 are the respective indices for the two media, then reflection is practically eliminated for this wave-length.

Transducers

Acoustical or mechanical vibrations are usually derived from electrical sources of power by the use of electro-mechanical converting elements, or transducers as they are often termed. These systems may be dependent upon either of the following physical properties or qualities: (*a*) electromagnetism, (*b*) electrostatics, (*c*) magnetostriction, or (*d*) electrostriction. Under (*a*) the mechanical force is derived from the magnetic attraction or repulsion between an electric current and a permanent magnet, or between two current-carrying circuits. In systems dependent upon (*b*) the force results from the electrostatic repulsion between a steady polarising electric charge and the variable applied charge. Magnetostriction refers to the change in length of a specimen when placed in a magnetic field parallel to its length, a phenomenon exhibited by most magnetic materials. This application is dealt with on p. 335. Electrostriction (or electrotraction as it is sometimes termed) is the electrical phenomenon analogous to magnetostriction. The effect is most pronounced in certain types of glass and rubber, and the change of length being proportional to the *square* of the applied potential gradient it is necessary to apply a steady polarising voltage. The system is not widely used in practice, although a particular application is given on p. 141. In certain crystals, however, the change of length is proportional to, and reverses with, the sign of the applied potential difference. This *piezo-electric* effect, as it is termed, receives considerable application in the use of

quartz crystals (and also to a lesser degree those of Rochelle salt) as driving elements in mechanical systems.

The choice of system to be employed is governed largely by the nature of the medium in which the vibrations are to be propagated. It follows from considerations of "matching" (p. 240, etc.), that for the efficient setting-up of disturbances in air, driving systems of low mechanical impedance, *i.e.* those possessing a low value of the ratio $\frac{\text{force}}{\text{current (or charge)}}$ are best suited, and so systems under (*a*) and (*b*) find most favour. On the other hand, high impedance systems, which are characteristic of devices under (*c*) and (*d*), are the most efficient for the propagation of acoustic waves in liquids or solids. The mechanical impedance of a vibrating piezo-electric crystal may be considerably reduced by gluing together two crystals (usually 45° **X**-cuts of Rochelle salt), and so oriented that the applied electric field causes one to contract longitudinally, while the other expands. Fig. 13.12 *a* and *b* indicates the separate crystal units, and Fig. 13.12*c* shows the combined *bimorph* unit, as it is termed. On applying the electric field the combined unit undergoes a much larger *flexural* motion than the longitudinal motion of a single crystal, the bending movement being similar to that of the bimetallic strip of a thermostat (see dotted line in Fig. 13.12*c*). If one end of the bimorph unit is clamped a considerable movement is obtained at the other end, and conversely, if the unit is combined with a suitable voltage amplifier and relevant apparatus it can be used to detect such small changes of length as 10^{-6} in. An additional advantage of such units is that the hysteresis effects of the separate crystals tend to cancel out.

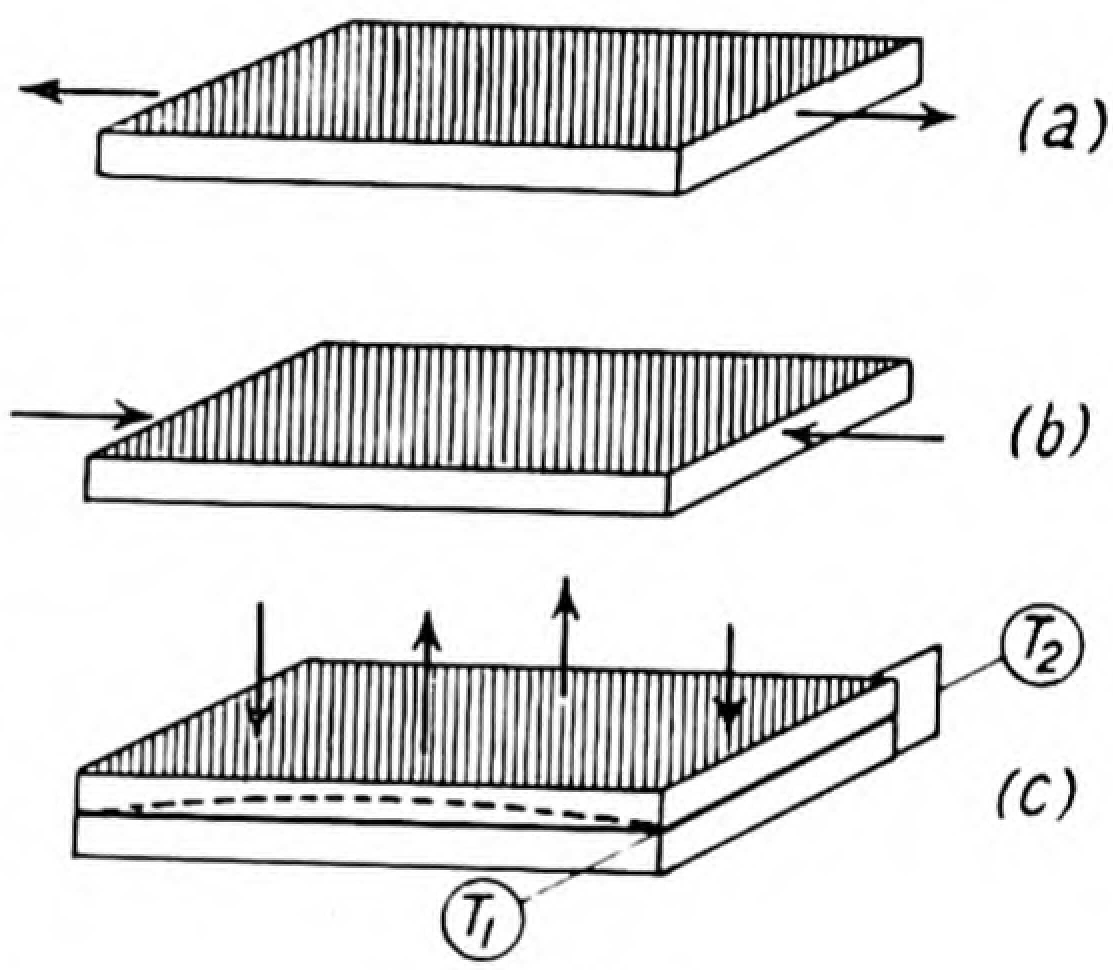


Fig. 13.12.

Piezo-electric pressure-voltage converters are employed in radio and acoustic reproduction in the form of headphones and loudspeakers, and in the form of microphones or vibration "pick-ups" as sound or vibration indicators respectively.

Vibration pick-up

The crystal type of pick-up is an instrument widely used for the detection and measurement of vibrations. It may be either of the displacement or of the inertia type, with the former the output voltage, at a constant amplitude of vibration, is a function of the frequency. In the inertia type, the crystal, when subjected to vibration, is deflected by its own inertia, and so the voltage generated is proportional to the

total force exerted on the crystal, and this force is in turn proportional to the acceleration. When a pick-up of this latter type is used with a vibration meter it is necessary to provide an electrical integrating circuit to provide a response giving the velocity of the vibrating body (since $v=\int a dt$), while a further integrating stage is needed to obtain a response proportional to displacement (since $x=\int v dt=\int\int a dt$). x, v and a respectively refer to the displacement, velocity and acceleration of the vibration. When possible it is preferable to use an instrument recording velocity, since only one mathematical operation, in either direction, is required to obtain displacement or acceleration. Fig. 13.13 shows response of electrical circuit excluding "pick-up."

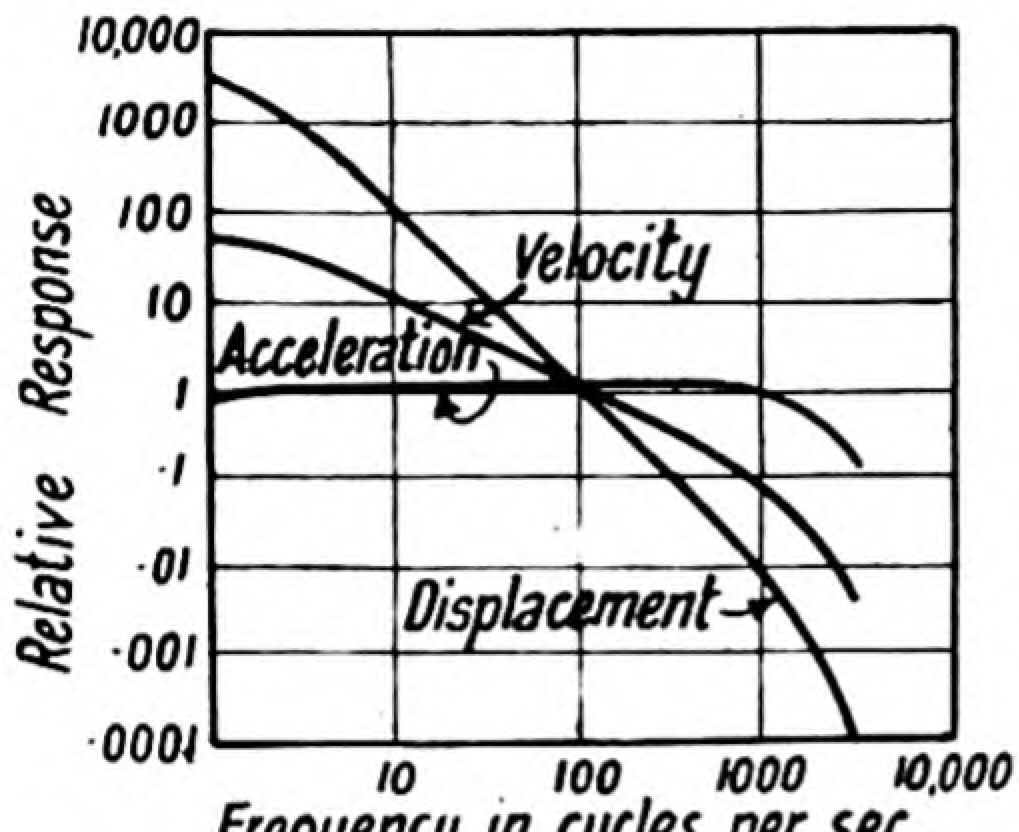


Fig. 13.13.

Vibration-measuring instruments

The amplitude of the vertical vibrations of an oscillating surface may be measured by means of a vibrometer of the type shown schematically in Fig. 13.15. A solid frame R, from which a heavy mass M is suspended at the end of a suitable spring S, is maintained in firm contact with the vibrating surface V so that the motion of the latter is strictly followed.

Suppose that at any instant the vertical displacements of the spring and mass are y_S and y_M respectively, then the total elongation of the spring will be $y=y_M-y_S$. Now assuming that the motion of the vibrating surface is S.H.M., of period $2\pi/\omega$, and represented by the

(a) (b) (c)

Fig. 13.14.—Illustrating effect of electrical integrating circuits on a square waveform: (a) shows a square wave as transmitted by the amplifier when set for acceleration measurements; (b) shows the wave after one stage of electrical integration for velocity measurements; (c) shows the result after two stages of integration as used for displacement measurements.

equation $y_S=F_S \sin \omega t$, then it follows that the motion of the mass M is given by $M\ddot{y}_M+ky=0$, k being the restoring force exerted by the spring for unit displacement.

The last equation may be rewritten as $M\ddot{y}_M+k(y_M-y_S)=0$ or $M\ddot{y}_M+ky_M=Y \sin \omega t$, where $Y=kF_S$. Consequently the steady motion of M is seen to be harmonic and the solution is readily deduced by

making R vanishingly small in (5) on p. 224, and is given by

$$\left.\begin{aligned} y_M &= F_M \sin \omega t \\ F_M &= \frac{1}{1-r^2} F_S \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

where $r=\frac{\omega}{\omega_n}$, and $\frac{\omega_n}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{k}{M}}$ is the natural frequency of vibration of the spring loaded with the mass M.

The quantity which is measured by the instrument, and is usually shown by a dial indicator, is $y=y_M-y_S$. Now both displacements y_M and y_S vary simple harmonically, and with the same frequency, hence the displacement (y) of the spring also moves with a S.H.M. which may be expressed by the equation $y=F \sin \omega t$, where from equation (42)

$$F=F_M-F_S=\left(\frac{1}{1-r^2}-1\right)F_S=\left(\frac{r^2}{1-r^2}\right)F_S \quad . \quad . \quad . \quad (43)$$

The ratio $\frac{r^2}{1-r^2}$ is known as the *absolute transmissibility* and is seen to approach unity as r becomes large. In such a case the measured amplitude F may be taken without serious error to be equal to the required amplitude (F_S). This desirable result may be achieved by using a spring of soft material so that ω_n is small, which means that $r=\frac{\omega}{\omega_n}$ will be large. Should, however, r not be sufficiently large it will require to be known, and so ω must be evaluated. This determination is effected by recording the relative displacement (y) on a moving paper strip, on which a time-base is also marked.

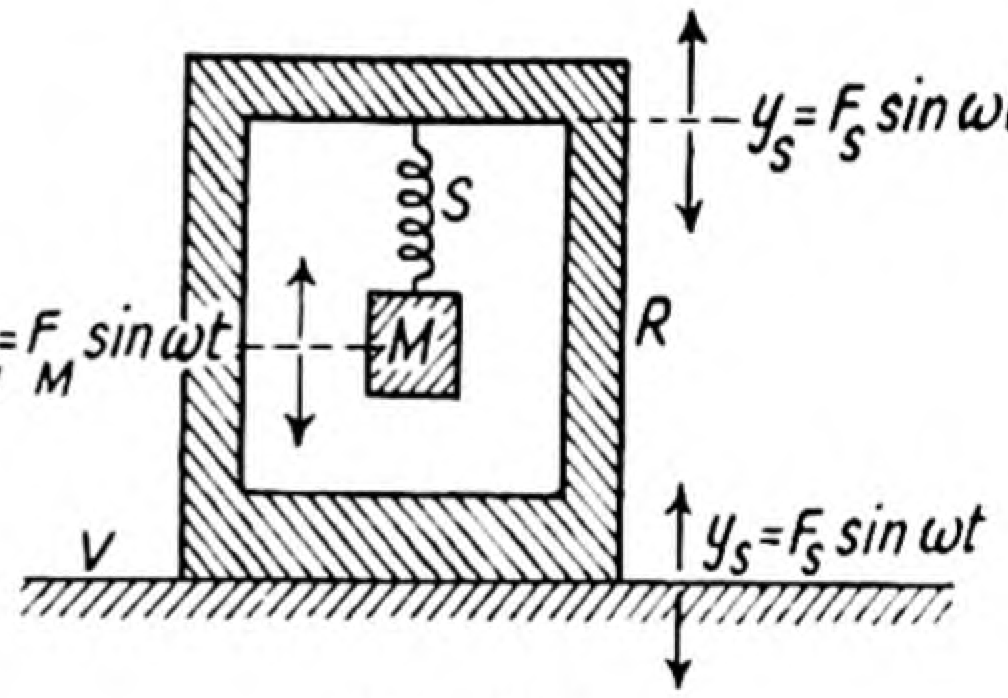


Fig. 13.15.

If the natural period of the mass-spring system is, on the other hand, less than that of the body under test, then the relative movement between the mass M and the body will be controlled by the amount of extension or compression of the spring, *i.e.* it will be a measure of the acceleration. This follows since by Hooke's law stress (or force) is proportional to strain, and by Newton's law of motion acceleration is proportional to the applied force. *Accelerometers* are therefore constructed in a similar way to vibrometers, except that their natural frequencies are high and not as low as possible.

The above result follows from (43), for when $\omega_n \gg \omega$ then $r \ll 1$ and therefore

$$F=r^2F_S \text{ approx.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

But $y=F \sin \omega t$, $y_S=F_S \sin \omega t$, and $r^2=\frac{\omega^2}{{\omega_n}^2}$, and on substitution in (44)

it is easily found that $y=\frac{1}{\omega_n^2}\cdot\omega^2 y_S=\frac{1}{\omega_n^2}\cdot$(accn. of body). The factor $\frac{1}{\omega_n^2}$ will be a constant of the instrument.

In the class of instruments known as *tachometers* there is a series of spring-mass systems of different natural frequencies, and the system which vibrates with the largest amplitude will possess the frequency nearest to resonance with the vibrating surface. These instruments usually take the form of a set of different reeds, *i.e.* small cantilever beams weighted at their ends, and a similar construction is also adopted in one form of frequency meter for use with alternating currents, the reeds in this case being excited electromagnetically.

Reflection and transmission of sound waves at an interface, and the application to acoustical receiver measuring apparatus

In the previous discussion of the forced vibrations of a mechanical-acoustical system, the latter was supposed to be energised by the impact of the incident sound waves and the absorption of energy assumed to be complete. Actually the problem is a more complicated one, owing to the *moving* diaphragm itself acting as a radiator of sound waves. Theoretical analysis of this problem will now be made, after the manner suggested by Drysdale.

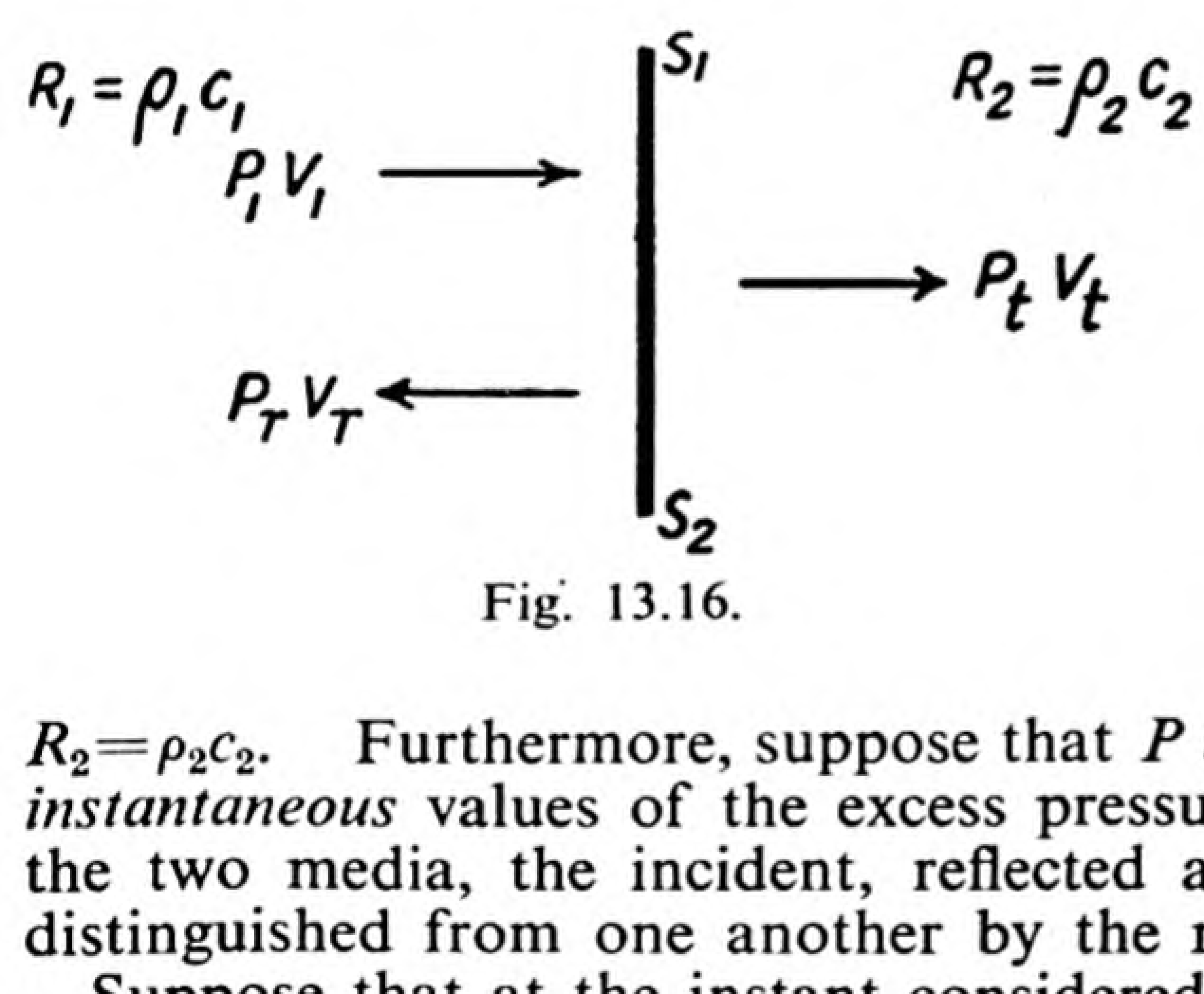


Fig. 13.16.

Let S_1S_2 (Fig. 13.16) be a thin plane surface of area A and mass M, which separates two media of acoustical resistances $R_1=\rho_1c_1$, and $R_2=\rho_2c_2$. Furthermore, suppose that P and V respectively denote the *instantaneous* values of the excess pressures and particle-velocities in the two media, the incident, reflected and transmitted waves being distinguished from one another by the respective suffixes i, r and t.

Suppose that at the instant considered, the diaphragm is displaced a distance x_t to the right of its equilibrium position S_1S_2 (Fig. 13.16) so that the restoring force due to the elasticity of the diaphragm may be assumed to be βx_t per unit area, where β is a constant, and that the fluid damping force per unit area is $\alpha V_t=\alpha\left(\frac{dx}{dt}\right)_t=\alpha\dot{x}_t$, where α is a constant. The instantaneous equation of motion, writing $m=\frac{M}{A}$ for the mass per unit area, is therefore

$$P_i+P_r=m\left(\frac{dV_t}{dt}\right)+\alpha V_t+\beta x_t+P_t$$

$$=mx_t+\alpha\dot{x}_t+\beta x_t+P_t$$

or

$$m\ddot{x}_t+\alpha\dot{x}_t+\beta x_t=(P_i+P_r-P_t) \quad . \quad . \quad . \quad . \quad (45)$$

Alternatively $$P_i+P_r=m\dot{V}_t+\alpha V_t+\int\beta\dot{x}_t dt+P_t \quad . \quad . \quad . \quad . \quad (46)$$

Now it is evident that $$V_i=V_r+V_t$$

or $$V_r=V_i-V_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

and also that and
$$\left.\begin{aligned}P_i=\rho_1c_1V_i=R_1V_i;\quad P_r=\rho_1c_1V_r=R_1V_r\\ P_t=\rho_2c_2V_t=R_2V_t\end{aligned}\right\} \quad . \quad . \quad . \quad (48)$$

Hence $R_1V_i+R_1V_r=R_1(V_i+V_r)=R_1(2V_i-V_t)$, and therefore
$$P_i+P_r=R_1(V_i+V_r)=R_1(2V_i-V_t) \quad . \quad . \quad . \quad . \quad (49)$$

Equation (45) now becomes on substitution
$$\begin{aligned}m\ddot{x}_t+\alpha\dot{x}_t+\beta x_t&=R_1(2V_i-V_t)-R_2V_t\\ &=2R_1V_i-(R_1+R_2)V_t\\ &=2R_1V_i-(R_1+R_2)\dot{x}_t,\end{aligned}$$

i.e. $$m\ddot{x}_t+(\alpha+R_1+R_2)\dot{x}_t+\beta x_t=2R_1V_i \quad . \quad . \quad . \quad (50)$$

It is evident from the above expression that the damping term has been increased by the factor (R_1+R_2), due to the diaphragm acting as a radiator, and this part of the total damping is usually known as the external damping factor r_E. The force term $\alpha\dot{x}_t$ will *correspond* to the energy absorbed within the system, and α is correspondingly denoted by r_I, the internal damping factor. In the case of *resonant* receivers it may be shown that the *rate* of energy absorption is a maximum when r_I is as small as possible, and when the energy absorbed $=$the energy radiated, i.e. $r_E=r_I$. These results are seen to be in accordance with those of the analogous electrical problem.

If it is assumed that the diaphragm is vibrating sinusoidally, so that $\dot{x}_t=V_t=V_{tm}\sin\omega t$, then (50) becomes

$$2R_1V_i=\left\{\left(m\omega-\frac{\beta}{\omega}\right)\cos\omega t+(\alpha+R_1+R_2)\sin\omega t\right\}V_{tm} \quad (51)$$

since $$x_t=\int\dot{x}_t dt.$$

This expression may be transformed into the equivalent form, if $V_i=V_{im}\sin nt$, of

$$V_{tm}=\frac{2R_1V_{im}\sin nt}{\sqrt{\left(m\omega-\dfrac{\beta}{\omega}\right)^2+(\alpha+R_1+R_2)^2}}\cdot\frac{1}{\sin(\omega t+\gamma)} \quad . \quad . \quad (52)$$

Where $$\tan\gamma=\frac{\left(m\omega-\dfrac{\beta}{\omega}\right)}{(\alpha+R_1+R_2)}.$$

Special cases.

If the diaphragm is of *very* light mass, and the control and damping are *very* small, then

$$V_t=\left(\frac{2R_1}{R_1+R_2}\right)V_i,\text{ from (50), } V_r=V_i-V_t=\frac{R_2-R_1}{R_1+R_2}\cdot V_i,$$

$$P_t=R_2V_t=\left(\frac{2R_1R_2}{R_1+R_2}\right)V_i=\left(\frac{2R_2}{R_1+R_2}\right)P_i,$$

and $$P_r=R_1V_r=\left(\frac{R_2-R_1}{R_1+R_2}\right)R_1V_i=\left(\frac{R_2-R_1}{R_1+R_2}\right)P_i \quad . \quad . \quad (53)$$

Now it was shown from equation (35), on p. 237, that the instantaneous value of the power per unit area of the vibrating diaphragm is $W_{\text{inst.}}=$excess-pressure$\times$particle-velocity.

Therefore the fractions W_r and W_t of the incident power which are reflected and transmitted are respectively given by

$$W_r=\frac{P_rV_r}{P_iV_i}=\left(\frac{R_2-R_1}{R_1+R_2}\right)\left(\frac{R_2-R_1}{R_1+R_2}\right)=\left(\frac{1-\frac{R_1}{R_2}}{1+\frac{R_1}{R_2}}\right)^2$$

and
$$W_t=\frac{P_tV_t}{P_iV_i}=\frac{2R_2}{(R_1+R_2)}\cdot\frac{2R_1}{(R_1+R_2)}=\frac{4\frac{R_1}{R_2}}{\left(1+\frac{R_1}{R_2}\right)^2} \quad . \quad . \quad (54)$$

The above results will apply to the case of sound waves impinging *normally* on the plane interface between two media, and they will assume special forms according to the value of the resistance ratio $\frac{R_1}{R_2}$ of the media.

(i) $R_2=R_1$. In this case of perfectly *matched impedances* $\frac{R_1}{R_2}$ is unity, and therefore $W_r=0$ and $W_t=1$. In other words, the whole of the incident energy is transmitted into the second medium.

(ii) $\frac{R_1}{R_2}\to 0$ or R_2 is infinitely large compared with R_1. Hence $W_t=0$, but $W_r=1$, which means that the whole of the incident energy is reflected, also, since the pressure $P_r=P_i$ the total pressure on the interface is double that of case (i). As $\frac{R_1}{R_2}\to 0$ the interface approaches the condition for a perfectly rigid body, and corresponds to a "pressure receiver" as exemplified by a plate of quartz, or Rochelle salt crystal. Some movement of the crystal, however, must of necessity take place as power must be absorbed for its operation, just as a voltmeter, however high its electrical resistance, must "pass" an electric current of some small magnitude.

(iii) $\frac{R_1}{R_2}\to\infty$. This case leads to the result that $P_t=0$ and $V_t=2V_i$, and thus refers to the extreme case of a *displacement* receiver with a very flabby diaphragm. The values W_r and W_t will be unity and zero respectively, *i.e.* the whole of the incident energy is reflected, so that the arrangement, if practicable, would be extremely inefficient.

A further interest attaches to cases (ii) and (iii) as being applicable to the conditions at the closed and open ends respectively, of a resonance tube or organ pipe. In the former case the incident wave is reflected without change of phase, viz. $P_r=P_i$ and $V_r=V_i$, but at the open or free end, $P_r=-P_i$ and $V_r=-V_i$, so that there is a phase reversal.

The general behaviour of an actual membrane (*i.e.* one possessing mass, etc.) when exposed to sound waves, is deducible from the

expression (50) given above. Drysdale points out an interesting case when $m\omega$ and $\frac{\beta}{\omega}$ are very small compared with $(R_1+R_2+\alpha)$, or, in fact, if resonance occurs, i.e. $m\omega=\frac{\beta}{\omega}$. In this case it is easy to show that the phase angle is zero, and $V_t=V_i$, or in other words, no reflection exists, and a useful amount of power $(R_1-R_2)\overline{V}_i^2$, where $\overline{V}_i$ is the R.M.S. velocity, is absorbed by the diaphragm and is available for measurement purposes.

The Rayleigh disc

This instrument is the standard device for the measurement of sound intensities, and is essentially a light, thin circular disc of radius a which is suspended so that its plane is vertical, and the normal to the plane makes a chosen angle θ with the direction of the sound stream.

Now if such a disc is suspended in a fluid stream, of undisturbed density ρ_0, and which is moving with a velocity V, then the stream lines around the circular lamina will become disturbed in the manner suggested by Fig. 13.17, where θ is assumed to be 45°. The points P_1 and P_2 will be regions of maximum pressure, *i.e.* zero velocity, and in consequence a couple is exerted on the disc tending to set it broadside-on to the direction of streaming. The magnitude of this couple was calculated by König for the case of elliptical bodies, but an infinitely thin circular disc may be considered as a particular example, and the value of the couple C is given by

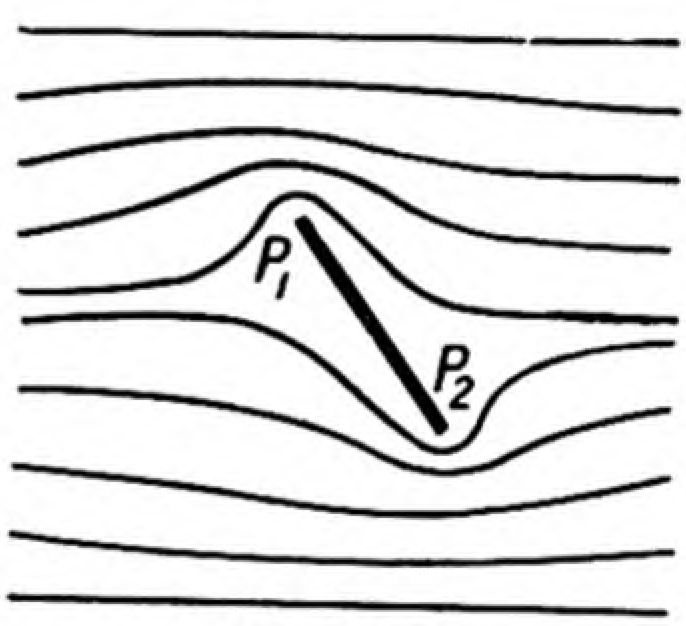


Fig. 13.17.

$$C=\tfrac{4}{3}\rho_0 a^3 V^2 \sin 2\theta \quad . \quad . \quad (55)$$

If the flow is an alternating one as produced by a sound source vibrating simple harmonically, viz. $V=V_m \sin pt$, where p is the frequency of the source and V is now the *particle*-velocity of amplitude V_m, then (55) becomes

$$\begin{aligned} C&=\tfrac{4}{3}\rho_0 a^3 V_m^2 \sin^2 pt \sin 2\theta \\ &=\frac{4}{3}\rho_0 a^3 V_m^2 \sin 2\theta\left[\frac{1-\cos 2pt}{2}\right] \quad . \quad . \quad . \quad . \quad (56)\end{aligned}$$

If $\theta=45°$ the expression for C simplifies to

$$C=\frac{4}{3}\rho_0 a^3\frac{V_m^2}{2}-\frac{4}{3}\rho_0 a^3\frac{V_m^2}{2}\cos 2pt \quad . \quad . \quad . \quad . \quad (57)$$

The latter term is seen to be one of double frequency, and the large resistance experienced by the light disc in moving at audio-frequencies makes this term negligible compared with the rectified (*i.e.* static) component $\frac{4}{3}\rho_0 a^3\left(\frac{V_m}{\sqrt{2}}\right)^2=\frac{4}{3}\rho_0 a^3\overline{V}^2$, where $\overline{V}$ is the R.M.S. velocity.

In the design and conditions of use of the Rayleigh disc, at any rate for *absolute* measurements, two important assumptions implicit in

the theory require to be complied with. Firstly the absence of other bodies in close proximity to the disc, and secondly the chosen diameter $2a$ of the disc should be small compared with the wave-length λ of the sound. The first condition was investigated by Zernow, who suspended discs of different sizes within a cylindrical tube and found that an error of approximately 3 per cent. was incurred in the velocity measurement when the ratio $\dfrac{\text{diameter of disc}}{\text{diameter of tube}}$ was about $\frac{1}{3}$. Again, if the disc has an appreciable thickness t, König's formula (55) assumes the corrected form

$$C=\left(1-0{\cdot}15\frac{t}{a}\right)\frac{4}{3}\rho_0 a^3 V^2 \sin 2\theta. \quad . \quad . \quad . \quad . \quad (58)$$

Further assumptions involved in the derivation of the expression (55) are that the fluid medium is incompressible, and that viscous forces and discontinuities of flow at the edges of the disc are absent or negligible. These latter conditions cannot be readily investigated theoretically, and therefore have only been tested experimentally, so that in this respect the Rayleigh disc is not strictly an absolute instrument. In the use of the disc for measurement of sound intensities in heavy fluids such as water, the formula (55) is no longer directly applicable since the disc, besides following the slow rotation as in air, will partake of the fluid motion itself. Hence the velocity V in König's formula must now be interpreted as the relative velocity $V=V_l-V_d$, where V_l and V_d are the R.M.S. velocities of the fluid and disc respectively. If the ratio $\dfrac{V_d}{V_l}$ of these velocities is denoted by G then it follows that $V=(1-G)V_l$. The effect of the heavier medium is to appreciably "load" the disc, and it is shown in treatises on hydrodynamics (*e.g.* Lamb) that this load for a disc set at 45° to the stream is given approx. by $\frac{4}{3}a^3\rho_l$, ρ_l being the density of the liquid. Hence, by equating momenta, $G=\dfrac{V_d}{V_l}=\dfrac{m_l+\frac{4}{3}a^3\rho_l}{m_d+\frac{4}{3}a^3\rho_l}$, where m_l is the mass of fluid displaced and m_d=mass of disc. König's formula (55) now becomes

$$C=\tfrac{4}{3}\rho_l a^3 V_l^2(1-G)^2$$

$$=\tfrac{4}{3}\rho_l a^3 V_l^2\left(\frac{m_d-m_l}{m_d+\frac{4}{3}a^3\rho_l}\right)^2. \quad . \quad . \quad . \quad . \quad (59)$$

The expression for the intensity I of the sound wave in the liquid is then given by

$$I=\rho_l c_l V_l^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

where c_l is the velocity of sound in the fluid and V_l is given by equation (59),

i.e.
$$I=\frac{\frac{3}{4}c_l C}{a^3\left(\dfrac{m_d-m_l}{m_d+\frac{4}{3}a^3\rho_l}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

A. B. Wood has calculated the magnitude of this correction for a mica disc where $a=0{\cdot}5$ cm., $t=0{\cdot}002$ cm., and $m_d=0{\cdot}005$ gm. When used

in air and in water the correction term $\left(\frac{m_d-m_l}{m_d+\frac{4}{3}a^3\rho_l}\right)^{-2}$ has the values 1·08 and 2500 respectively; these figures indicate the importance of using as heavy a disc as possible, especially with liquid media.

The Rayleigh disc usually consists of a thin mica disc, about ·002 cm. thick, and 1 cm. or less in diameter, suspended by a fine glass (or quartz) fibre or phosphor-bronze suspension, the upper end of the suspension being attached to a torsion head. A light mirror is fixed at the centre of the mica disc, or alternatively, it might be silvered directly. The torsional constant τ of the suspension fibre is determined before attaching the disc, by temporarily attaching an oscillator of known moment of inertia K, and observing the period T and the logarithmic decrement δ of small oscillations of the system. Then it is easily shown (p. 222) that

$$\tau=\frac{4K}{T^2}(\pi^2+\delta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Now if θ is the angle through which the torsion head is turned to restore the disc to its original position before the sound field was applied, it follows from (58) and (62) that the restoring couple (C) is given by

$$\frac{4K}{T_2}(\pi^2+\delta^2)\theta=\tau\theta=C=\left(1-0{\cdot}15\frac{t}{a}\right)\frac{4}{3}\rho_0 a^3 V^2 \sin 2\theta \quad . \quad (63)$$

whence V^2 may be found.

The apparatus is normally employed for quite moderate sound intensities, and under careful experimental conditions it will respond to sound intensities with excess pressures as low as 10^{-6} bar. An immediate consequence of such a sensitive system is that the effect of even small air currents will be quite considerable, and recourse is often made to the use of fine silk net to surround the disc and suspension thus minimising the effects of light draughts. A recent modified method of using the disc, which also largely overcomes the effects of unwanted air disturbances, has been devised by Siviam, in which he modulates the amplitude of the sound wave under test with a very low frequency (of the order of 25 cycles per *minute*), which, however, will be usually higher than any component frequency of circulating air currents. The Rayleigh disc system is designed to have a natural frequency equal to the modulation frequency, and the *amplitude* of its oscillations will be proportional to V^2, and, moreover, will be little affected by air currents.

Crystal microphones

In the diaphragm type the sound waves impinge upon a diaphragm which transmits the alternating pressure to the crystal via a mechanical system. The Rochelle crystal element (see p. 241) is usually enclosed within a bakelite case, the diaphragm being protected by a metal meshed-screen. These microphones do not require field currents or polarising voltages, and by reason of their high impedance (of the order of 10^5 ohms or more at 60 c.p.s.) may be connected directly between the grid and cathode of the input valve of a pre-amplifier. Their

quality and ruggedness are also superior to those of the best carbon microphones.

In the sound-cell type of microphone the sound waves act directly upon the crystals. These consist of two "bimorph" elements supported at two points within a rectangular bakelite frame, and sealed by flexible air-tight annuli, which allow the crystals to be distorted by the sound pressure variations. The crystal units are connected in parallel so that their respective outputs are additive when due to incident sound, but tend to cancel when caused by mechanical vibration or shock, so that elaborate methods of suspension are unnecessary. The two important factors governing the size of a single sound-cell are that its dimensions in any direction should be $<<$ wave-length of the shortest sound wave to be reproduced, and secondly, that the mechanical resonance of the crystal units should be above the highest frequency to be employed. The units are usually designed to resonate near 10^4 c.p.s., which means that the separate crystals of a bimorph unit are only about $\frac{6}{1000}$ in. thick. The consequent rise in the frequency characteristic curve at the higher frequencies may be compensated for in the design of the amplifier.

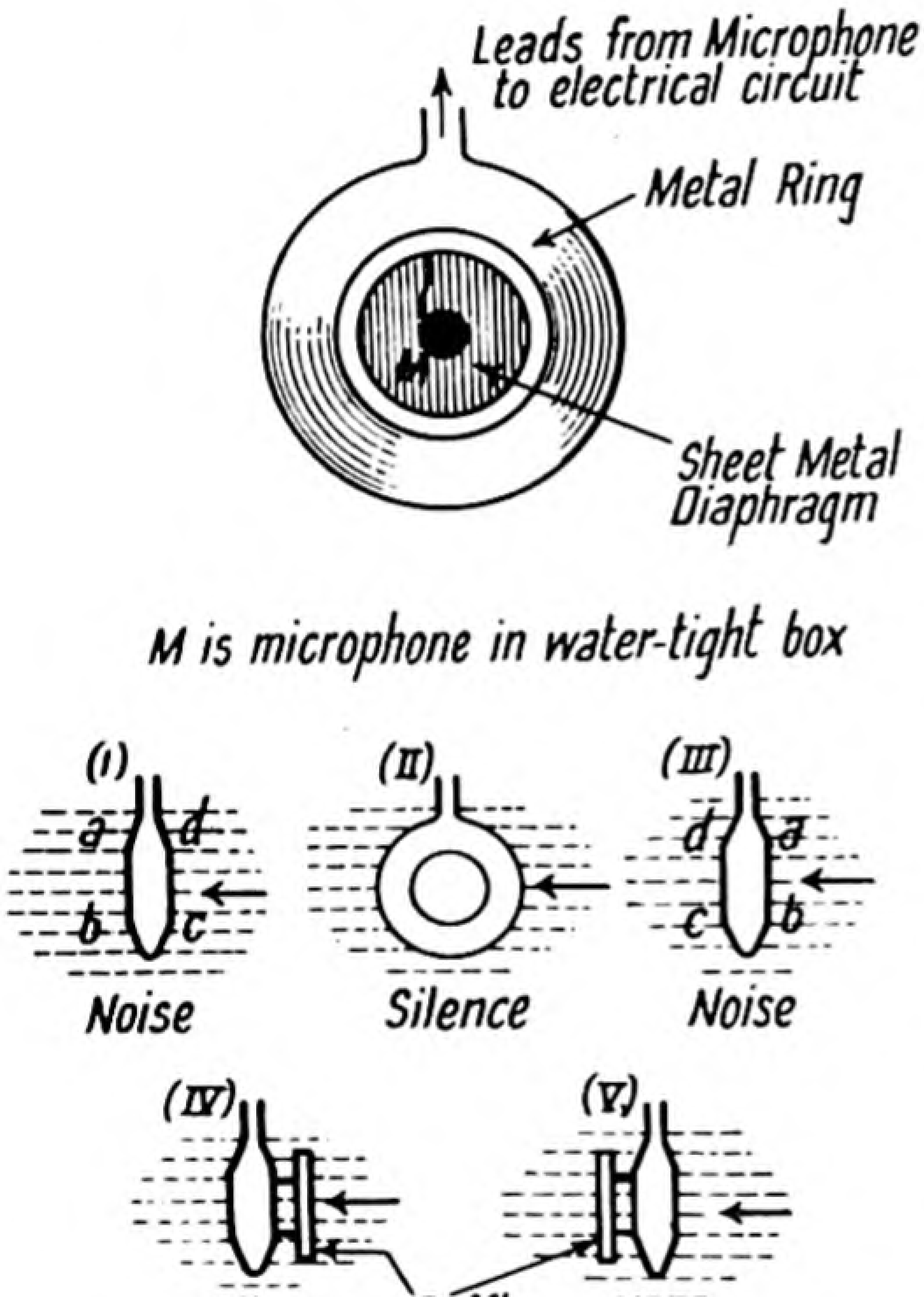


Fig. 13.18.

Hydrophone

This is an adaptation of the microphone for listening under water, and consists of a heavy metal ring which is closed by a sheet metal diaphragm. A small water-tight box containing a carbon button microphone is fixed at the centre of the diaphragm.

Now when a plane surface is placed edgewise-on to a distant source of sound, the pressures on either side of the plane will be equal and no response will be given by the microphone. Hence no sound will be heard in the phones when the edge of the hydrophone is set in the direction of the sound, but there is another possibility of zero response when the instrument is turned through a further 180°. In order to determine which of these two directions was the correct one, a baffle plate was fixed at a certain distance on the side of the hydrophone, as shown in Fig. 13.18, and the ambiguity of direction is immediately removed.

General principles governing acoustic generators

The energy resident in a sound-field may be in either of two forms:—

(*a*) "Wattless" energy, which is associated with the *movement of a body of air* (or other medium) in the immediate neighbourhood of

the vibrating body. This portion of the surrounding medium effectively "loads" the radiator (analogous to the hydration of ions in electrolysis), so that during acceleration periods it acquires kinetic energy which it tends to give back in the retardation parts of the cycles.

(*b*) "Useful" energy associated with the altered condition of the medium, viz. compression or rarefaction, which condition is propagated outwards to infinity.

The relative proportions of the "wattless" and "useful" energies in the case of a given radiator will depend on the frequency of vibration, the more rapid the *changes* of motion, *i.e.* the higher the frequency, the greater the fraction of the total energy radiated in sound waves.

Furthermore, the wave propagation involves the transference of energy from a solid medium, the vibrating surface, to the surrounding fluid medium, and the efficiency of such a process depends chiefly on the equality of the acoustic impedances of the two media (see p. 246). The methods adopted to *match* the large mechanical impedance of the source with the low acoustical impedance of the air are the use either (*a*) of a large light surface in the form of a paper or cloth diaphragm, or (*b*) of a horn of suitable design. In the case of the former the ideal procedure would be to spread the "drive" over the whole of the surface, *e.g.* as in electrostatic loud-speakers (p. 241). Actually it is easier in practice to "drive" at one point and endeavour to shape the surface so that it is substantially rigid at all frequencies, a condition which is reasonably satisfied if the paper or cloth diaphragm is not greater than about 10 in. in diameter. It should be noted that the problem becomes simplified if radiation at one frequency only is required, for an acoustical resonator may be employed to compensate for a poor radiation efficiency, *e.g.* as in the mounting of a tuning-fork on a resonator box. The horn is essentially an impedance matching device, *i.e.* an acoustic transformer, and it must not be regarded as an amplifying apparatus, for it does not embrace any auxiliary source of power. The theoretical treatment of the horn cannot be developed here, but from the design point of view it may be regarded as a filter system, and it may be shown that the smaller the rate of taper the lower the limit of frequencies that can be transmitted. The theory is, in fact, similar to the tapered electrical transmission line with its distributed circuit constants gradually changing with the length. Furthermore, the longer the horn the larger will be the corresponding area of the mouth and the smaller the reflection of sound waves from that end, thus minimising the possibility of resonance occurring from this cause. In such a case the piston source becomes equivalent to the conical element of a pulsating sphere (Appendix 16).

Measurement of vibration

In this section vibrations restricted to those taking place in solid bodies only will be considered, and their frequency range of interest extends below the lower limit of audibility, *i.e.* down to 2 or 3 c.p.s. The need for accurate measurements of vibration occur in such investigations as on the causes of noise, and on the location of possible centres of excessive vibration in machinery or in neighbouring

structures. If vibrations of large amplitude exist they are liable to set up stresses greater than the permitted safety values, and so become real sources of danger. It is due to a careful study of vibration problems that the present perfection in high-speed land-, sea-, or air-craft has been attained.

The magnitude of a vibration may be expressed in terms either of the displacement, the velocity, or the acceleration of the surface considered, the quantity chosen to be measured being the one appropriate to the problem in hand (p. 242). In the case of a surface vibrating simple harmonically with a frequency $\frac{\omega_0}{2\pi}$ and an amplitude A, then it follows that

(i) the displacement $x=A\sin\omega_0 t$,

(ii) the velocity $v=\frac{dx}{dt}=\omega_0 A\cos\omega_0 t$, and

(iii) the acceleration $a=\frac{dv}{dt}=\frac{d^2x}{dt^2}=-\omega_0{}^2 A\sin\omega_0 t=-\omega_0{}^2 x$.

If the periodic vibrations are complex, then they may be represented as a Fourier series of simple harmonic vibrations, and the expressions for x, v and a now become:—

$$x=A_1\sin(\omega_0 t+\beta_1)+A_2\sin(2\omega_0 t+\beta_2)+A_3\sin(3\omega_0 t+\beta_3)+\ldots \quad (64)$$

$$v=\omega_0 A_1\cos(\omega_0 t+\beta_1)+2\omega_0 A_2\cos(2\omega_0 t+\beta_2)$$
$$+3\omega_0 A_3\cos(3\omega_0 t+\beta_3)+\ldots \quad (65)$$

and $$a=-\{\omega_0{}^2 A_1\sin(\omega_0 t+\beta_1)+4\omega_0{}^2 A_2\sin(2\omega_0 t+\beta_2)$$
$$+9\omega_0{}^2 A_3\sin(3\omega_0 t+\beta_3)+\ldots\} \quad (66)$$

β_1, β_2, etc., denote the relative phase numbers of the harmonics of the fundamental frequency $\frac{\omega_0}{2\pi}$.

The effective readings of a vibration meter, *i.e.* the root mean square readings, corresponding to (64), (65) and (66) respectively, would be

$$\bar{x}=\frac{1}{\sqrt{2}}\sqrt{A_1{}^2+A_2{}^2+A_3{}^2+\ldots}, \quad \bar{v}=\frac{\omega_0}{\sqrt{2}}\sqrt{A_1{}^2+4A_2{}^2+9A_3{}^2+\ldots},$$

and $$\bar{a}=\frac{\omega_0{}^2}{\sqrt{2}}\sqrt{A_1{}^2+16A_2{}^2+81A_3{}^2+\ldots}.$$

It will be noted that the presence of higher frequency components in a complex vibration has the greatest effect in the case of acceleration measurements (since $a\propto\omega^2$), but even in observations of velocity ($v\propto\omega$) these components are more effective than in displacement measurements.

Displacement measurements are only rarely required, such as in cases where large amplitudes might lead to "rattling" of the moving parts, and then it may often be measured directly with a scale. In the measurement of small amplitudes of vibration as for those of a diaphragm, a device due to W. H. Bragg is worthy of mention here. In Fig. 13.19, H is a pointed hammer-head of small mass, fixed to the upper end of a metal spring S, which is fixed to a block B. This block

may be moved horizontally by means of the knob K, at the end of a screw thread T. The method of operation is to move the apparatus carefully until H is just in contact with the vibrating surface V, an adjustment which is indicated by the completion of an electric circuit if the surface of the diaphragm is rendered conducting. Suppose now that B is moved towards V through a distance A, then the acceleration of H towards V will be $A\omega_H^2$ (from above), where $\frac{\omega_H}{2\pi}=n_H$ is the natural frequency of the loaded spring. The maximum acceleration of the diaphragm is $a\omega_V^2$, where a is amplitude of its motion and $\frac{\omega_V}{2\pi}=n_V$ is the frequency of vibration. If $a\omega_V^2$ is greater than $A\omega_H^2$ then "chattering" will occur, but it will just *cease* when $A\omega_H^2$ becomes equal to $a\omega_V^2$. Hence for this critical setting $a=A\,.\frac{\omega_H^2}{\omega_V^2}=A\,.\frac{n_H^2}{n_V^2}$, and since n_V is large compared with n_H it follows that A will be very large compared with a. The value of n_H may be determined stroboscopically if too large for direct timing. The Bragg method is obviously applicable only to sinusoidal vibrations.

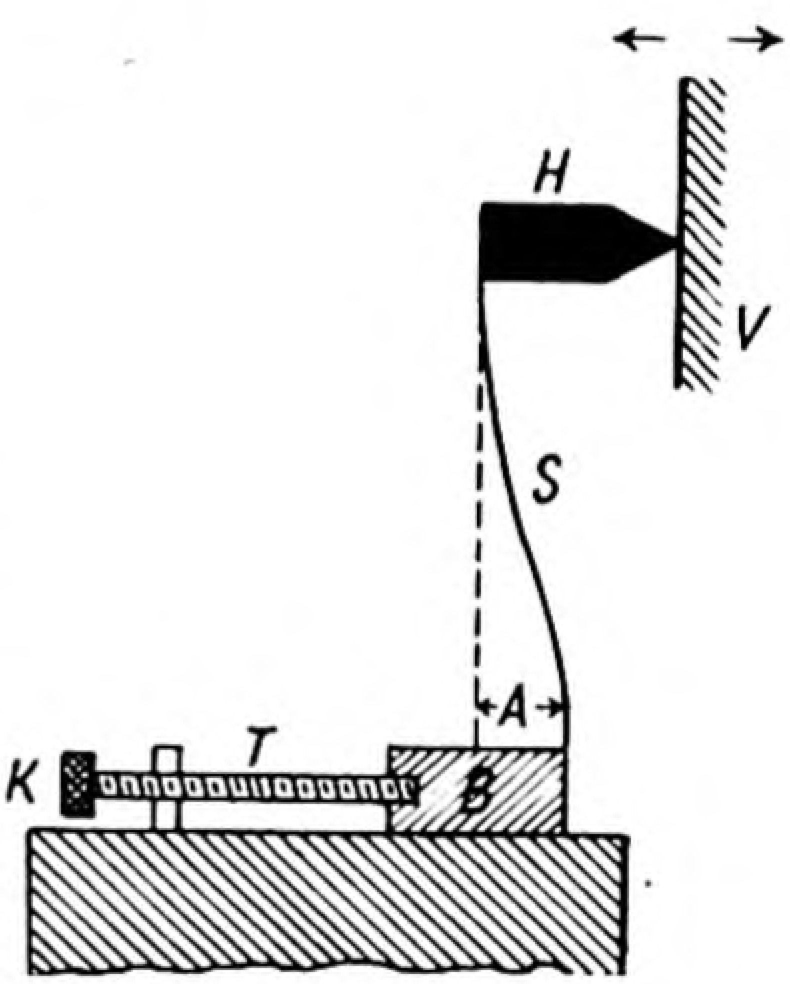


Fig. 13.19.

An interesting modification of the above experiment occurs with the chattering of small particles resting on a vibrating surface. Then, provided the size of the particles does not seriously modify the character of the motion of the surface, it is easily seen that chattering will cease when the maximum acceleration of the surface is equal to g, the acceleration due to gravity. If the frequency of an oscillating quartz crystal is 20 Mc.p.s., *i.e.* 2×10^7 c.p.s., then the chattering of a particle on its surface will just occur when its amplitude of vibration a is given by $a(2\pi n_H)^2=981$, i.e. $a=6\times10^{-14}$ cm.

It is interesting to note how sensitive as detectors of low frequency vibrations are the human sense organs; the finger tips, for instance, are capable of detecting vibrations of 50 c.p.s., having an amplitude as small as $0{\cdot}5\times10^{-4}$ cm.

If the area of a vibrating or radiating surface is large compared with the wave-length of the resultant vibrations in the surrounding air, then the acoustical energy radiated will be expressed by the product of (air-particle velocity)^2 and the resistive component of the air-load (p. 237). Assuming the air-particles in contact with the vibrating surface to acquire the velocity of the latter at every instant, then it follows that for many investigations into problems of noise the important vibration measurement is that of the velocity of the vibrating surface.

At low frequencies, however, when the surface area of the vibrating member may be *small* compared with the wave-length, as will

undoubtedly be the case with most types of machinery, for the predominant frequencies are usually low, then the reactive air impedance (p. 432) will be very low. Consequently only a small amount of acoustical energy will be radiated, and its estimation will be best furnished by measuring the acceleration of the surface. This follows from the fact that the reacting force is equal and opposite to the applied force, which by Newton's law of motion is proportional to the acceleration. Furthermore, since any strains or stresses set up in the vibrating body will be a function of the reacting force, it is evident that acceleration measurements are also important means of detecting possible causes of mechanical failure in a machine or structure subjected to vibration.

Direct measurement of displacement amplitude of air vibrations

The actual magnitude of these movements in a sound wave is very small, *e.g.* at 1000 c.p.s. the amplitudes at the thresholds of hearing and pain are respectively $2{\cdot}2 \times 10^{-9}$ and $7{\cdot}0 \times 10^{-3}$ cms. The corresponding sound intensities are 10^{-10} and 10^{3} microwatts per sq. cm.,

[*Andrade and Parker.*

Fig. 13.20.

so that even with a very intense sound wave the amplitude is only of the order of one-tenth of a millimetre, which it is possible to measure, however, with a high-power microscope. In order to render the air movement visible a fine powder, *e.g.* smoke-particles of magnesium oxide, is injected into the sound-field, and it is found that the particles, if sufficiently small, take up the full amplitude of the aerial vibrations The photograph (Fig. 13.20) showing the motion of particles in an excited Kundt's tube is due to Prof. Andrade and Dr. Parker, and indicates the appearance of the particles as bright lines when viewed by scattered light. Indirect measurements of displacement amplitudes of sinusoidal waves are made from excess pressure and velocity amplitude determinations using, for example, the hot-wire microphone or Rayleigh disc.

Energy transfer by longitudinal vibrations in liquid column

As mentioned previously, the passage of a sound wave does not involve a bodily motion of the medium as a whole, but since particles at points distant from the source are disturbed, there must necessarily

be a streaming of energy in the direction of wave propagation. If this transference of energy can be localised within narrow confines instead of spreading in its passage outwards, then sound waves would seem to offer a means of transmitting energy from one place to another. The actual energy of a sound wave in air, however, is extremely small, as a simple calculation will show. From eqn. (23), p. 43, the energy per cubic centimetre of air due to the passage of a plane sound wave is given by $\frac{1}{2}\rho.\left(\frac{2\pi a}{T}\right)^2$, where a is the amplitude of displacement of the air-particles, ρ is the density of air and $f=\frac{1}{T}$ is the frequency of the waves. Hence, assuming the sound to be of average intensity, i.e. $a \simeq 10^{-4}$ cm., then energy per cubic centimetre for a frequency of 50 c.p.s. $=\frac{1}{2}\times(\cdot 0013)(2\pi\times 10^{-4})^2\times(50)^2$

$$=0{\cdot}000\,000\,6 \text{ erg per cc.}$$

The *rate* at which energy is streaming away from the source through an area perpendicular to the direction of propagation, known as the intensity of the wave, is therefore given (from eqn. (24), p. 43) by $(6\times 10^{-7})\times c=6\times 10^{-7}\times 3{\cdot}4\times 10^4=0{\cdot}02$ erg per sq. cm. per sec. It is evident from the above formula that, for a *given amplitude*, the intensity of the wave increases with the density of the medium, and is proportional to the *square* of the frequency of motion. An obvious possibility, therefore, is to utilise a liquid medium such as water, but the difficulty arises here of the greater elastic resistance of the medium to compression due to its smaller modulus of compressibility. Hence quite large pressures would have to be exerted to produce appreciable displacements; the idea, however, was pursued by M. Constantinescu to form the basis of a method for the transmission of energy.

It should be noted here that since the mean power transmitted across unit area of a plane wave is given by (p. 238).

$$\frac{1}{2}\rho_o V^2{}_{\max} c=\frac{1}{2}\frac{(\Delta p)^2}{\rho c}=\frac{1}{2}\frac{(\Delta p)^2}{R},$$

where Δp is the pressure amplitude of the waves, and R is the acoustical impedance of the medium, then it follows that for the *same power* to be transmitted in two media of impedances R_1 and R_2 respectively, the corresponding pressure amplitudes Δp_1 and Δp_2 will be given by $\frac{\Delta p_1}{\Delta p_2}=\sqrt{\frac{R_1}{R_2}}$. For air and water at 20° C., R is equal to 41·2 and $1{\cdot}46\times 10^5$ gm. per cm. per sec. respectively, therefore

$$(\Delta p)_{\text{water}}=\sqrt{\frac{1{\cdot}46\times 10^5}{41{\cdot}2}}\times(\Delta p)_{\text{air}}=60(\Delta p)_{\text{air}}.$$

According to this system of Constantinescu, the transmission of energy takes place by impressing a periodic variation of pressure (or tension) upon a liquid column enclosed in an iron pipe. Longitudinal vibrations are, therefore, propagated in the liquid which is thus regarded as compressible in contrast to the assumption of incompressibility ordinarily made in hydraulic engineering. The problem

will be better understood by the quotation of figures given by Constantinescu. He considered a wrought-iron pipe 150 m. long, 2·5 cm. diameter, and 0·5 cm. thick wall, closed at one end and filled with water. If a water-tight piston is forced into the pipe at the open end under a steady pressure of 35 Kg. per sq. cm., and the liquid is in the first instance assumed incompressible, then a piston movement of 1·5 cm. would result. On the other hand, if the material of the pipe and not the water is assumed incompressible, then the piston would move inwards a distance of 25 cm. It follows, therefore, that the changes in the volume of the system are chiefly controlled by the compressibility of the water.

Consider the simple system shown in Fig. 13.21, where P_1 is the driving piston operated by a rapidly rotating crank C_1. In this way a periodic variation of pressure is impressed on the water column contained within the pipe T, which is closed by a rigid plate F. If this pipe is not too short a series of zones of high pressure (and liquid compression), alternating with zones of low pressure (and liquid

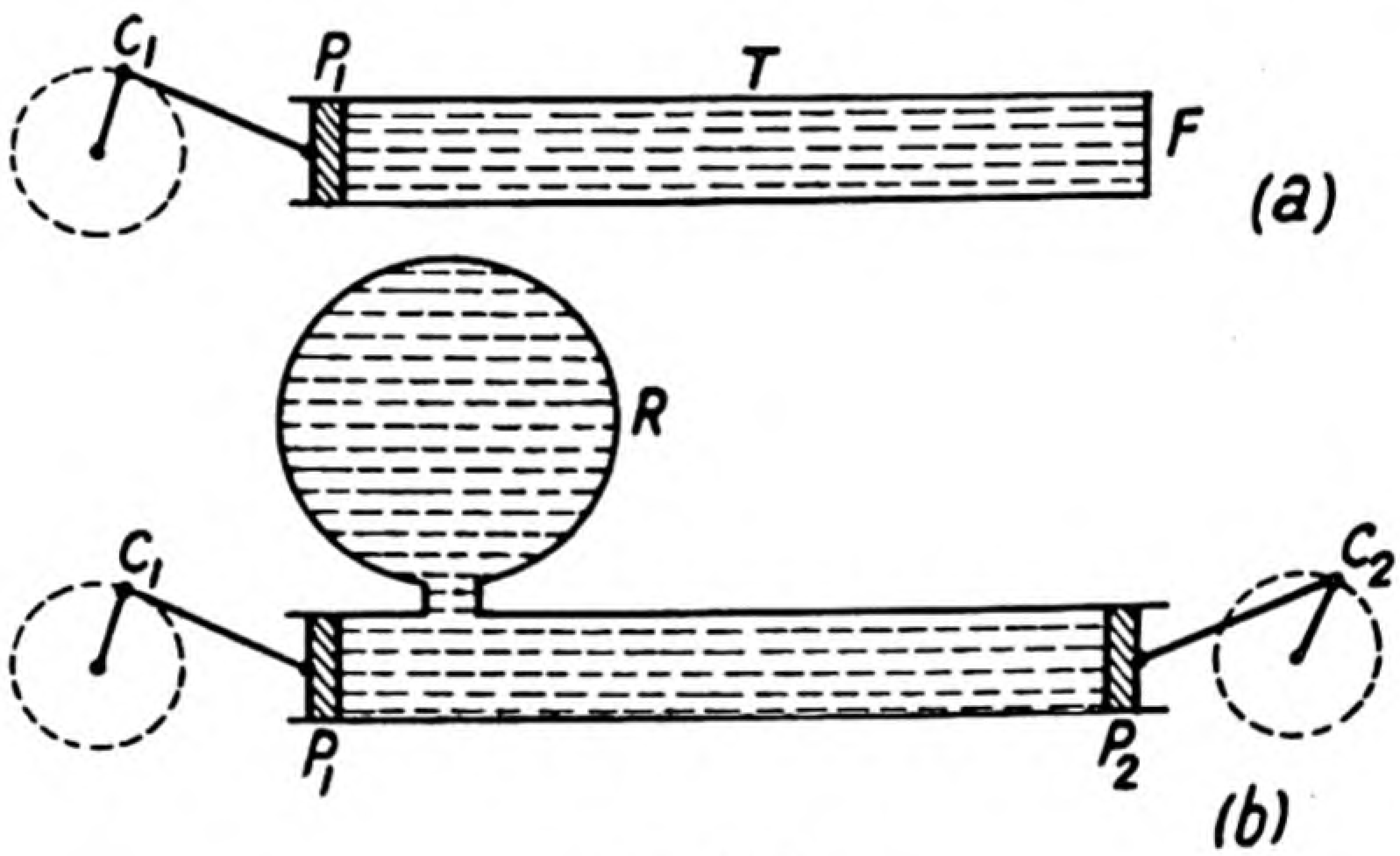


Fig. 13.21.

expansion) are produced, these zones travelling along the pipe. Furthermore, if the stroke of the piston is small compared with the length of these progressive waves, and the distance P_1F is a multiple of their length, then a standing wave-system will be set up. In consequence there will be a building up of the maximum pressure within the pipe and it may increase until the latter bursts. Suppose now, however, that the rigid plate F is replaced by a second piston P_2 connected to a crank C_2, and possessing the same period of reciprocation as P_1. If, furthermore, P_2 moves in synchronisation with P_1, then the whole energy of the waves propagated down the pipe *can* be taken up by it, and the maximum pressure within the pipe will then at no point exceed its value in the neighbourhood of the working piston P_1. However, should the piston P_2 not absorb all the energy of the waves, means must be adopted to absorb the residual energy of the reflected waves. The device adopted consists of a rigid vessel R (Fig. 13.21*b*) completely filled with liquid and placed near the piston P_1, and its function is analogous to that of a capacitance in an electrical

circuit (cf. p. 313). In other words, it will absorb the energy of the direct and reflected waves when the pressure is high, and return this energy when the pressure falls, and thus the rotating crank C_1 will only be required to perform work to the extent of the energy actually utilised by P_2 (neglecting frictional losses). It should be added that a piston may be placed at any point along the pipe-line, provided the period of reciprocation is the same as that of P_1, and that its movement is in phase with that of the liquid layer in contact with it.

In a particular case of a Constantinescu system, an engine of 10 h.p. transmitted its power by means of a reciprocating piston of $1\frac{1}{2}$ in. diameter with a stroke-length of 1 in. The frequency of the stroke, made as high as possible, was forty strokes per second, and the transmitting pipe-line was 80 yd. long. The system has been utilised in mining operations and was applied during the first World War in a device permitting a pilot to fire machine-gun bullets through his propeller without hitting the blades. This objective was attained by transmitting the engine vibrations to the gun by a tube of liquid,

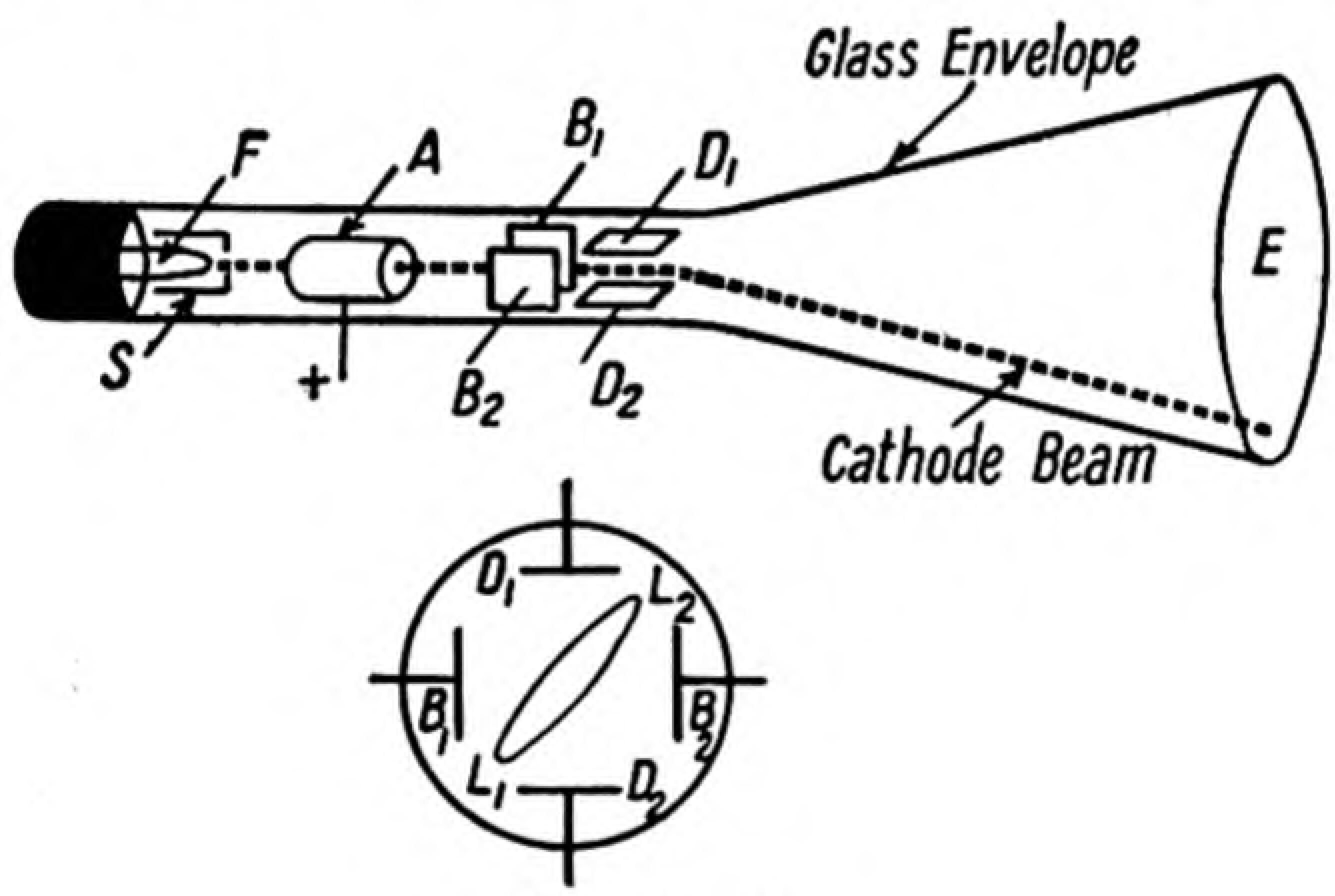


Fig. 13.22.

so that the firing could be adjusted to allow the bullet to be released always at the correct instant.

The oscillograph

An oscillograph or oscilloscope is a device for giving a visual indication of the variations of an electric current or voltage. In the older type of instrument the movement of an electromagnetic vibrator, due to the current or voltage variations, was exhibited by a spot of light focused on the mirror of the system and reflected on to a scale or screen (moving or stationary). Only comparatively slow variations of current could be recorded by this means owing to the appreciable inertia of the system. The advent of the cathode-ray oscillograph overcame this difficulty, however, by making use of a stream of electrons, which have very small inertia and so are able to respond to extremely rapid fluctuations. The source of electrons is a suitably heated filament F (Fig. 13.22) which is partly surrounded by a metallic shield S. This shield is usually at a small negative potential with respect to F, and its function is to restrict any tendency of the electron

beam to diverge or spread before reaching the deflector plates B_1 and B_2. In order that the electrons should acquire sufficient energy to produce a bright spot on striking the fluorescent screen E, the anode A (which may be a cylinder or in the form of a metal disc with a hole at its centre) is maintained at about 1000 volts positive with respect to F. The deflecting system comprises two pairs of parallel plates B_1, B_2 and D_1, D_2, which, after connection to the appropriate electrical

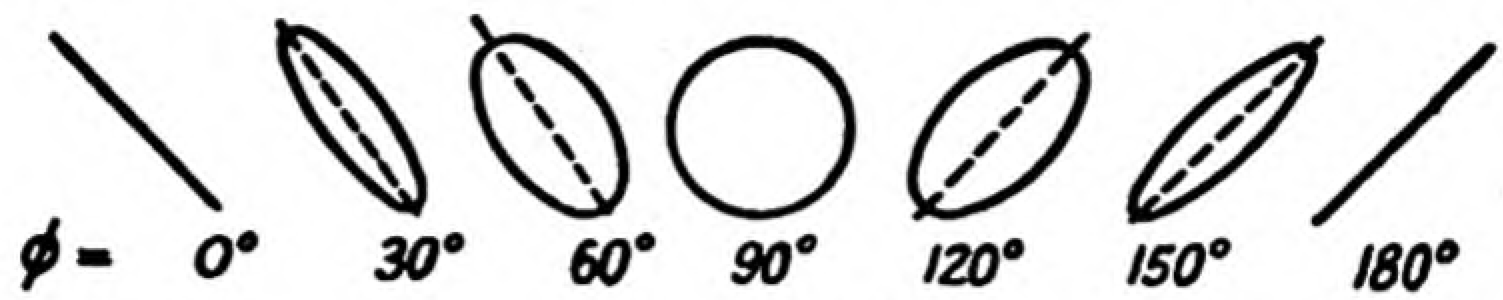


Fig. 13.23.

potentials, will respectively deflect the electron beam *perpendicular to*, and *in*, the plane of the paper. The smaller diagram in Fig. 13.22 represents an end-on view of the screen E and the plate-system with a typical elliptical figure L_1L_2, which is being traced out by the cathode-beam on the screen, when two A.C. voltages of the same frequency are each connected to one pair of plates. The form of the trace on the screen will depend upon the relative amplitudes and the phase difference (ϕ) between the applied voltages. If ϕ is zero and the amplitudes are equal, then the resultant trace will be a straight line inclined at 45° to the coordinate axes, and a similar trace, but crossing the other at 90° would be obtained when $\phi=180°$. For intermediate values of the phase angle the trace is an ellipse as shown in Fig. 13.23, and it will be noted that it becomes a circle when the voltages are in quadrature, *i.e.* 90° out of phase. If, however, the two voltages are

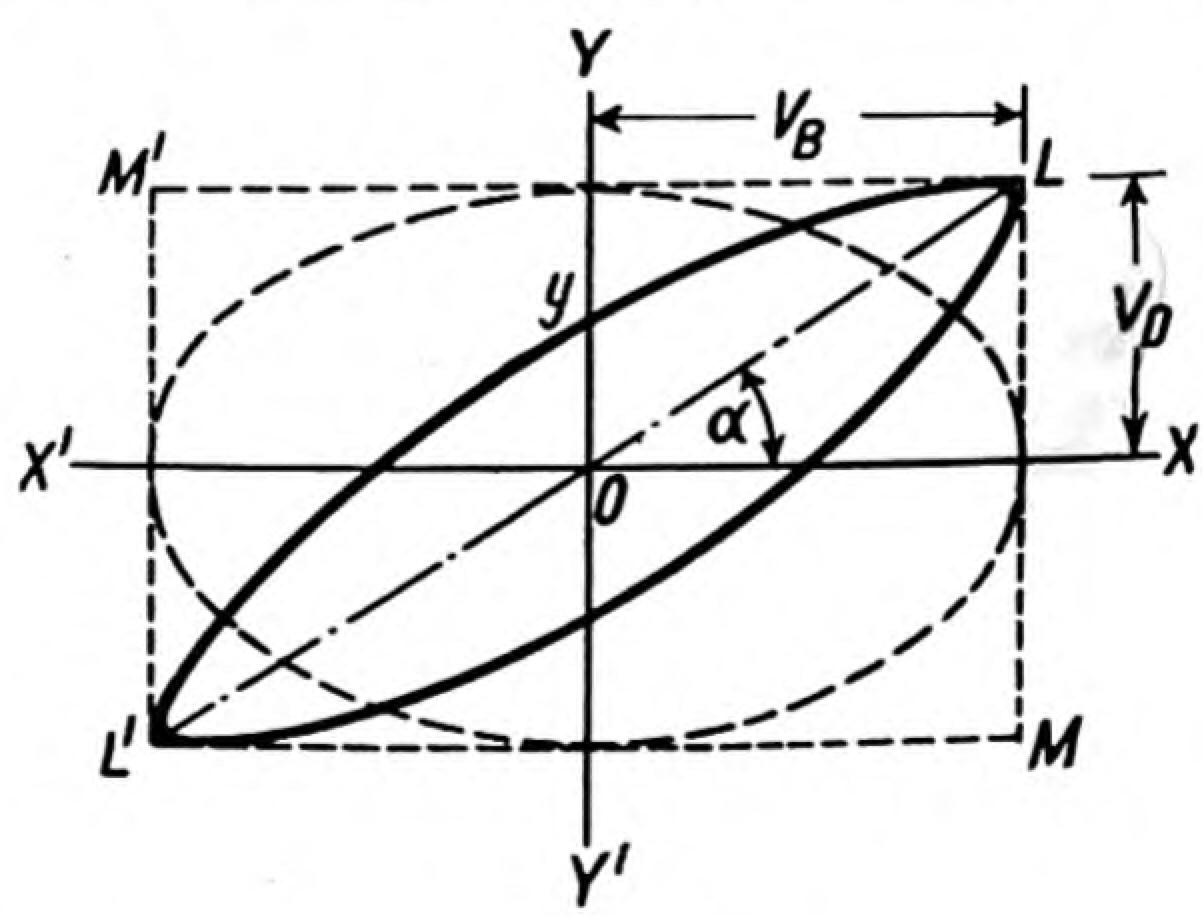


Fig. 13.24.

in quadrature but *unequal*, then an elliptical trace will be obtained (see dotted curve in Fig. 13.24).

To determine the phase angle between two alternating voltages applied to the two pairs of plates of a cathode-ray oscillograph.

Let $$x=V_B \sin \omega t \qquad (64)$$

and $$y=V_D \sin (\omega t+\phi) \qquad (65)$$

be the two alternating voltages of frequency $\frac{\omega}{2\pi}$ applied respectively to the B and D pairs of plates, where ϕ is the angle of phase difference.

Expanding (65) and substituting for $\sin \omega t$ and $\cos \omega t$ from equation (64), it follows that

$$V_B{}^2y^2+V_D{}^2x^2-2V_BV_Dxy\cos\phi=V_B{}^2V_D{}^2\sin^2\phi \quad . \quad . \quad (66)$$

which is the general equation for an ellipse.

Consider the intercept of this curve on the y-axis, *i.e.* put $x=0$ in equation (66).

The value of this ordinate Oy is given by $\pm V_D \sin\phi$ or $\sin\phi=\pm\frac{Oy}{V_D}$, *i.e.* $\phi=\sin^{-1}\frac{Oy}{V_D}$, which is directly calculable from the observed trace on the screen.

The ratio of the amplitudes of the applied voltages, $\frac{V_D}{V_B}$, is obviously equal to the tangent of the angle of slope (α) of the major axis LL' to the X-axis, due allowance being made for any difference in sensitivities of the oscillograph in the X and Y directions. It should be noted that the trace does not reveal which is the leading or lagging voltage, but in practice this may be ascertained by the introduction of a small inductance into *one* of the circuits, thus causing a small lag in the corresponding voltage. The ellipse will become respectively slightly broader or thinner, according as to whether this voltage was lagging or leading.

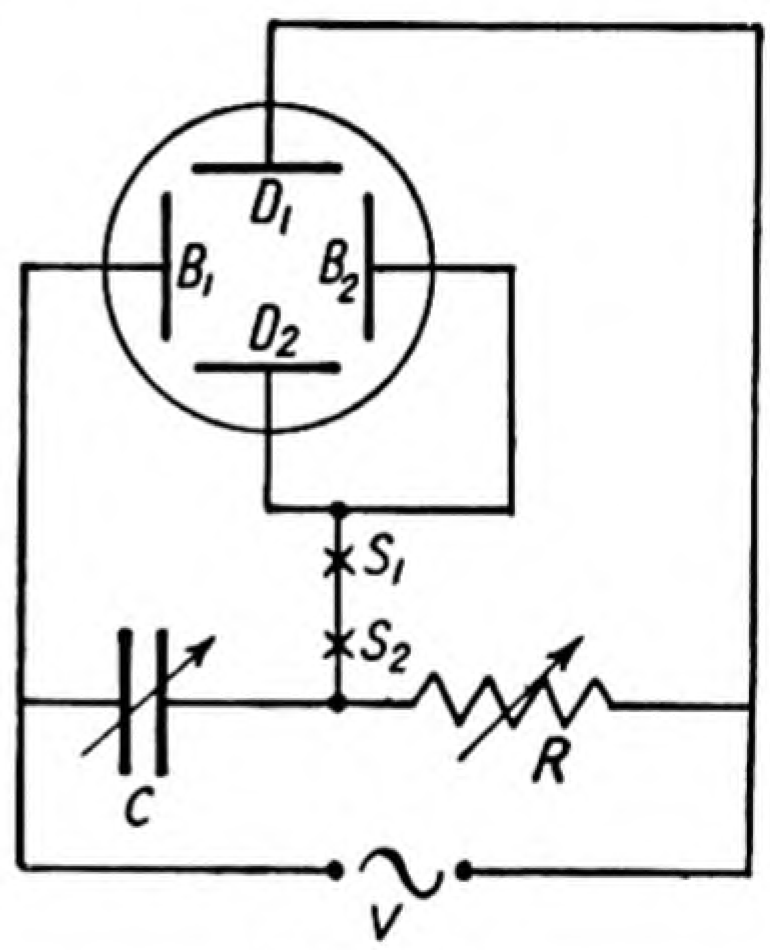


Fig. 13.25.

To obtain a circular trace and to use it for frequency measurements. A convenient form of circuit is shown in Fig. 13.25, where C is a fixed or variable capacitance of the order of 0·1 μF, and R is a variable 50,000-ohm resistance. V is the applied alternating voltage which may be conveniently obtained by tapping a fraction of the 230-volt mains supply by means of a potentiometer. S_1S_2 is a switch gap which may be opened for the insertion of an alternating supply of unknown frequency (see later). The resistance R is varied until the condition is attained of equality of the quadrature voltages V_R and V_C respectively across R and C, which will be revealed by a circular trace on the oscillograph screen (of course, if the x and y sensitivities are different, then $V_R \neq V_C$ when the circle is obtained). Such an experimental trace is shown in left-hand side of Fig. 13.26.

In order to determine the frequency f of a given sound the corresponding test voltage V_S is inserted between S_1 and S_2 (Fig. 13.25), and the supply voltage V of known frequency (F), *e.g.* the 50-cycle mains supply, is employed to produce the circular time trace. If $\frac{f}{F}=x$,

where x is an integer, then a stationary ripple of x waves will be superposed on the circular trace, thus enabling f to be calculated since F is known. In the trace on right-hand side of Fig. 13.26, $x=18$ so that $f=900$ c.p.s. if $F=50$ c.p.s. A slight departure of x from an integral value will result in a slow rotation of the ripple system in a clockwise or anti-clockwise direction, depending whether x is just smaller or greater than the integer. The arrangement is made more flexible by the substitution, for the mains supply, of a variable frequency calibrated source, such as a beat-frequency oscillator. A source of this type is especially valuable for sounds of high frequencies when the number of ripples becomes difficult to count if a low frequency, as the 50-cycle supply, is the comparison standard.

When the ratio of the frequencies to be compared is less than about 10 to 1, then the technique of Lissajou's figures may be employed; one voltage being connected to one pair of oscillograph plates and the other supply to the second pair of plates.

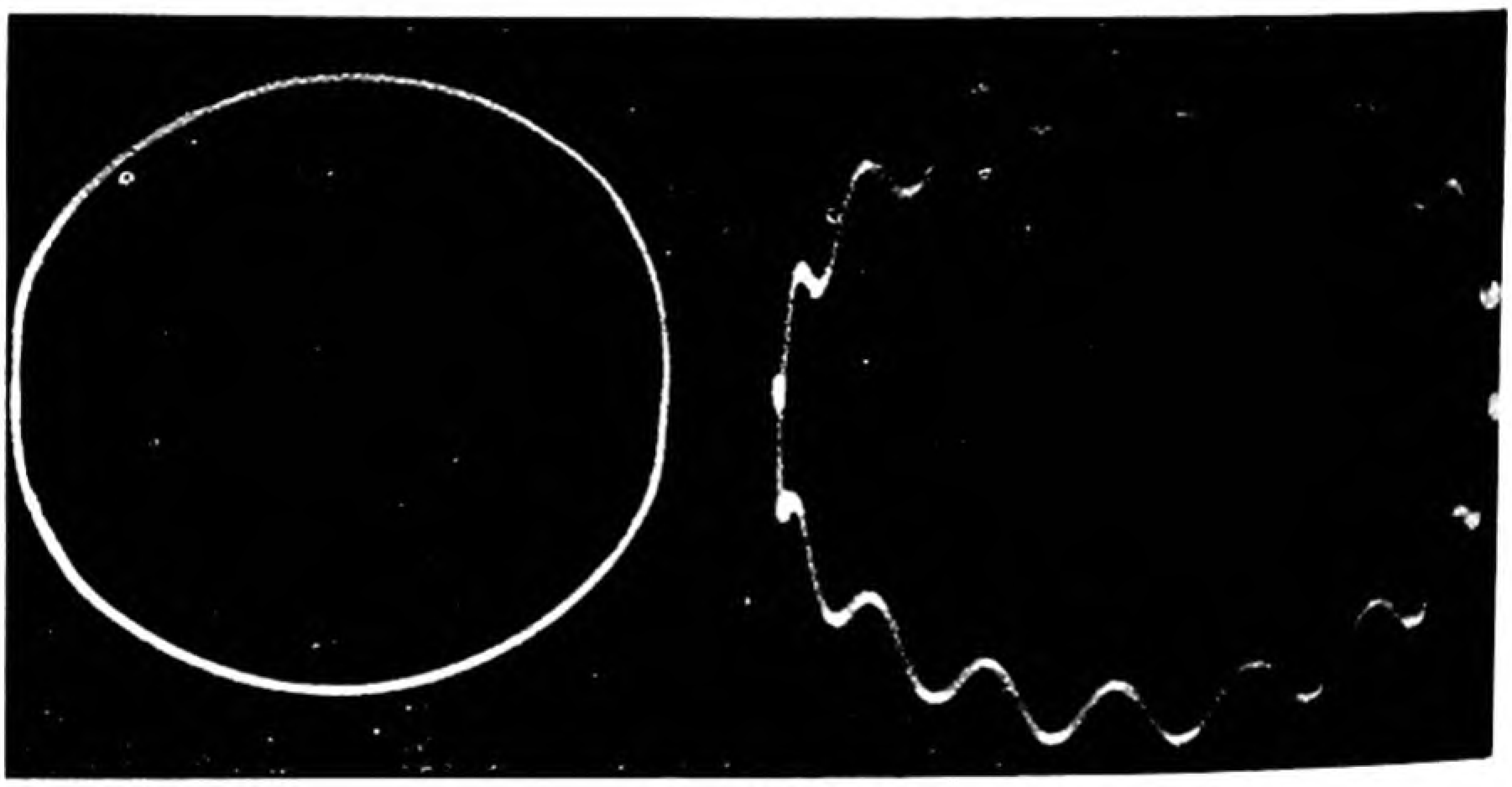


Fig. 13.26.

Cathode-ray oscilloscopes are also largely employed in the investigation of the wave-form of various sources of sound, a procedure which involves the determination of the time-variations of the equivalent voltages. For this purpose a voltage must be applied to the "B" pair of plates, which varies in such a way that the cathode beam moves across the screen at a constant rate, but returns to its starting point in a very short interval of time. Consequently, if the period of traverse of the beam is the same as, or an exact multiple of, the period of recurrence of the phenomenon under investigation, then the trace of the beam will be repeated exactly during every cycle. Furthermore, since the screen usually possesses an appreciable after-glow, an apparently stationary pattern will be observed and can be photographed.

Space permits only a very brief mention here of the general principle of the simplest type of linear time-base. Referring to the circuit diagram of Fig. 13.27, the capacitance C is charged from a direct current supply through a suitable resistance R, and when the potential difference reaches a critical value V_C the neon lamp N commences to glow and continues so for a brief interval until the voltage falls

to the extinction voltage V_E. Again the condenser voltage builds up again, and so the process is repeated (Fig. 13.28).

The ideal objectives are that the charging portions AB, CD, etc., of the cycles should be linear, while the duration of the return sweep,

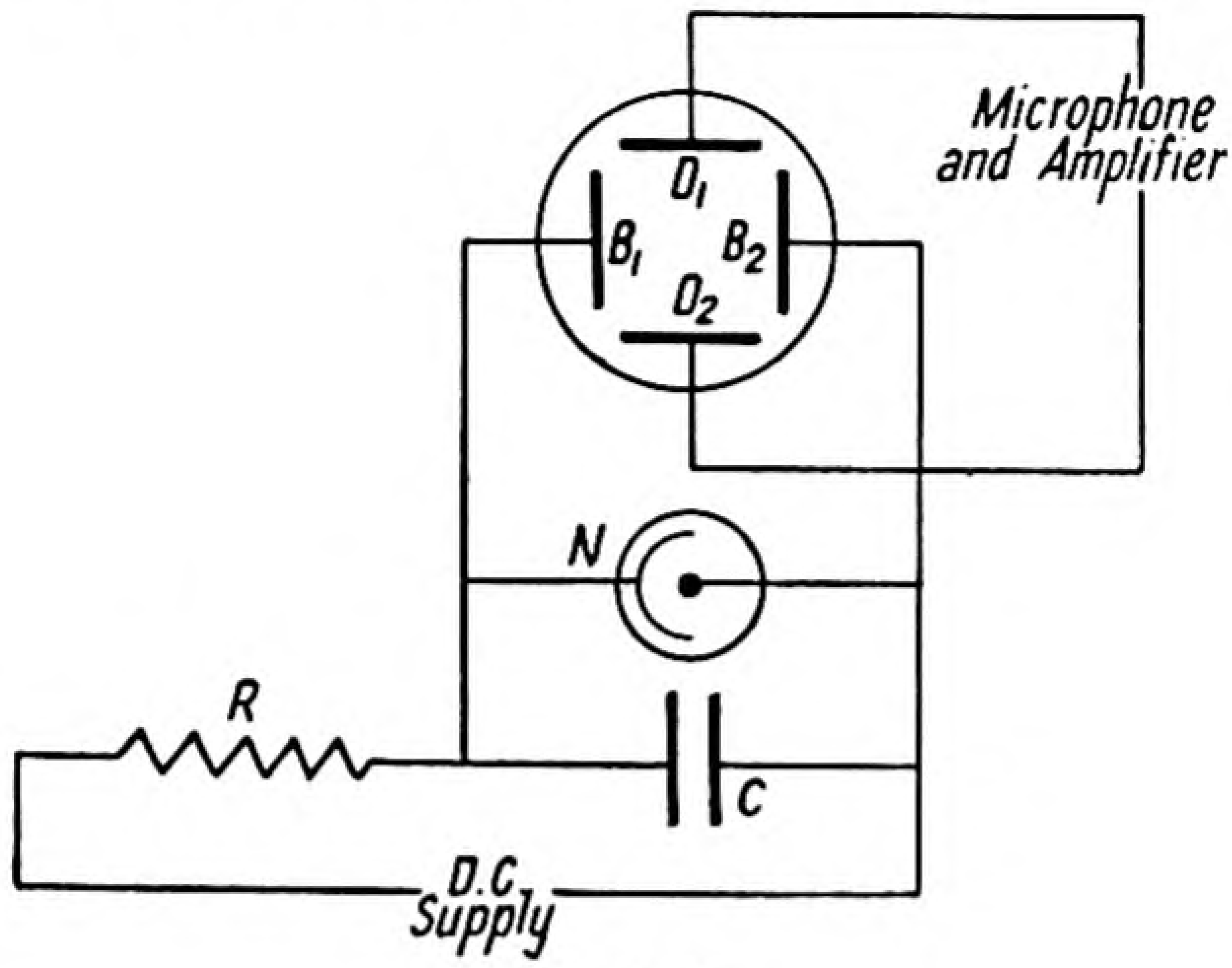


Fig. 13.27.

i.e. the fly-back time t_f, should be extremely small compared with the sweep period t_s. The charge and discharge of a neon lamp is a typical example of a relaxation oscillation (see Appendix 6).

Radiation pressure (see also Appendix 24)

When radiant energy falls on a reflecting surface it gives rise to a static pressure, which in the case of sound waves becomes equal to

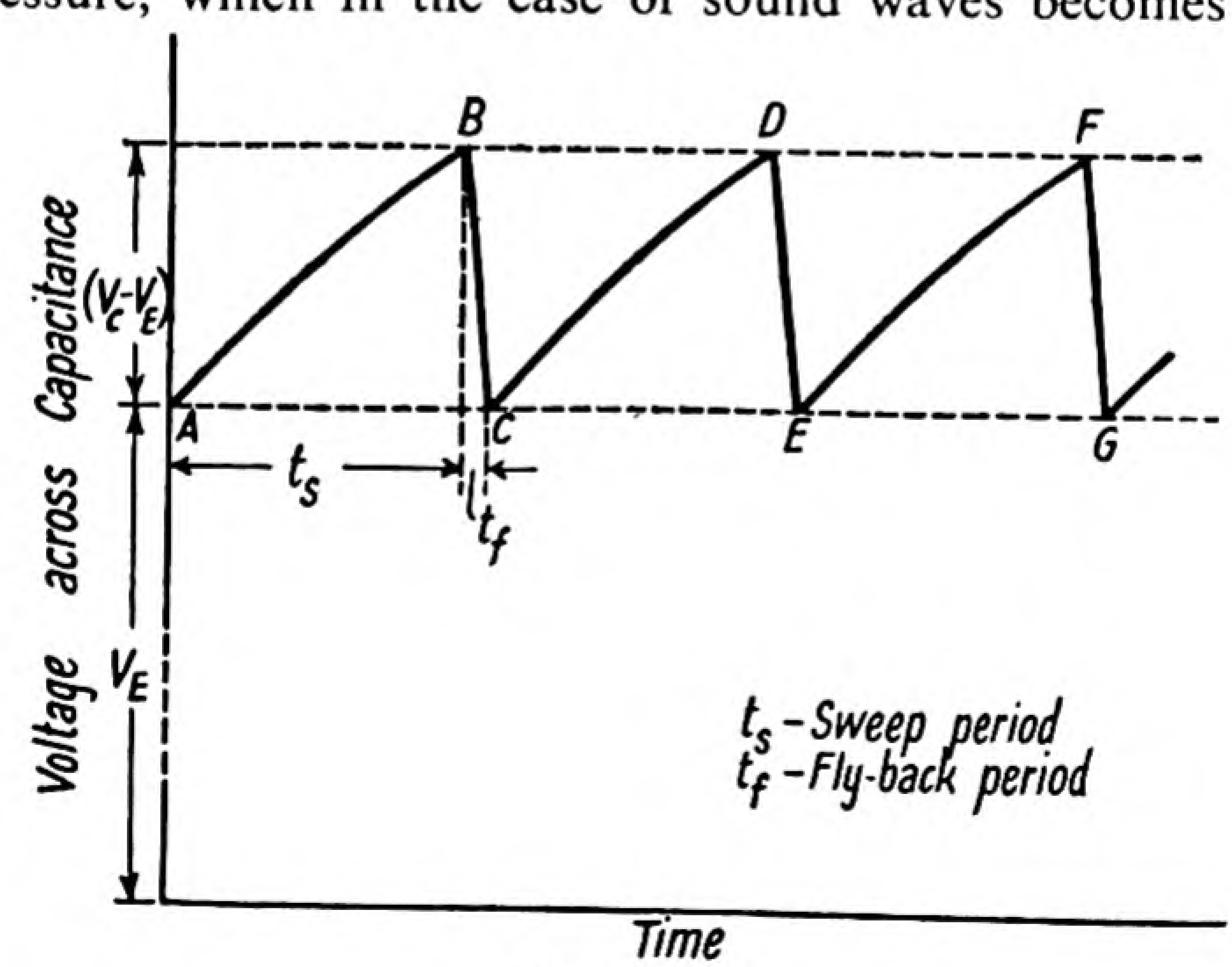


Fig. 13.28.

$\frac{(\gamma+1)E_f}{c}$, E_f (page 43) being the intensity, γ the ratio of specific heats of the gas. If the reflecting surface is in the form of a disc D

attached to the horizontal arm H of a torsion balance (Fig. 13.29), then the pressure due to the incident sound waves will exert a torque on the phosphor-bronze suspension S which will be indicated on a scale by the deflection of a beam of light reflected from the mirror M. Alternatively, the torsion head T may be employed to bring the mirror into its original position, and the angle turned through by T would then be a measure of the pressure exerted by the sound waves. C is a counterbalance to the aluminium disc at the other end of the arm H, while V is a damping vane. The torsion balance is suitably enclosed with a glass window to permit the use of the mirror, and there is a short-side tube opposite D to permit the entrance of the sound waves.

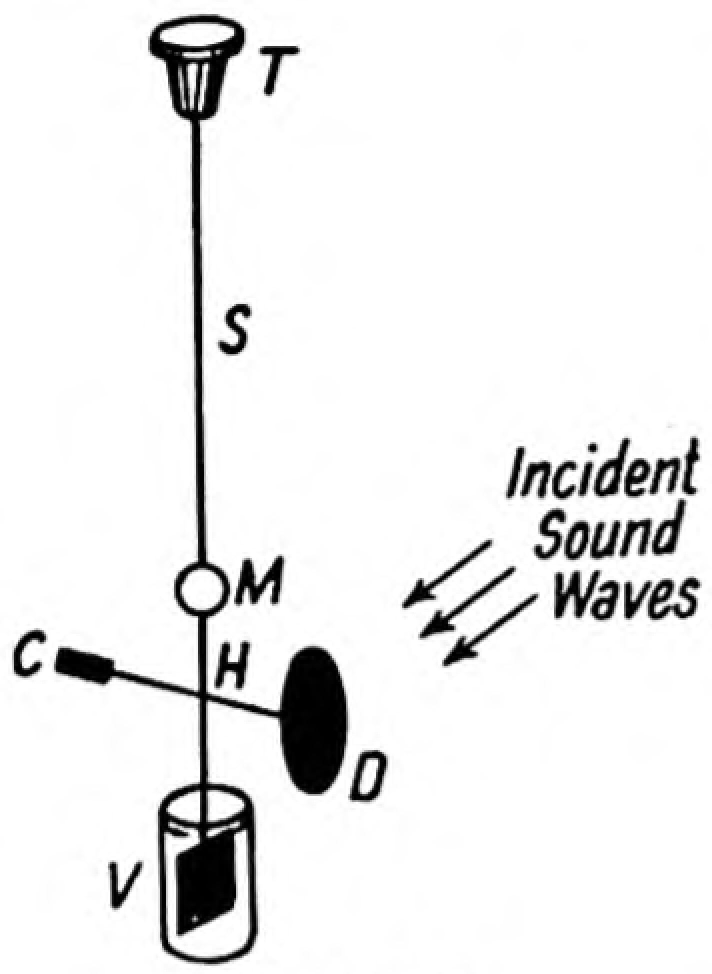


Fig. 13.29.

If p_s is the radiation pressure, A is the area of the disc, d is its effective distance from the axis of the suspension, τ is the torsional constant of the suspension, and θ is the angle turned through by T to restore the mirror M to its original position, then

$$Ad.p_s=\tau\theta \quad \text{or} \quad p_s=\frac{\tau.\theta}{Ad} \quad . \quad . \quad . \quad . \quad . \quad (69)$$

But the sound intensity E_f is given by $E_f=\dfrac{p_s.c}{\gamma+1}$,

hence
$$E_f=\frac{\tau c}{Ad(\gamma+1)}.\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

The instrument mentioned above is particularly suited to experiments with ultrasonic waves when the disc diameter is a large fraction of the sound wave-length. When used at the lower audio-frequencies, owing to the longer wave-length, it is necessary to use a baffle closely surrounding, but not touching, the disc, in order to avoid troublesome corrections for diffraction effects.

The existence of a radiation pressure suggests a departure from a linear law of force, for otherwise the decrease of air pressure in the troughs of the progressive wave would exactly counterbalance the pressure increase in the wave-crests. The conditions prevailing near a plane surface upon which sound waves impinge have been very simply described by Poynting, who considers the motion of a layer AA' of gas which is parallel to the surface PP' (Fig. 13.30). Then, for simplicity, assuming that Boyle's law holds and that the waves are simple harmonic in type, it is evident that a movement of AA' half-way towards PP' will result in a *doubling* of the pressure of the gas between AA' and PP', while a movement of similar amplitude in the reverse direction, *i.e.* to the

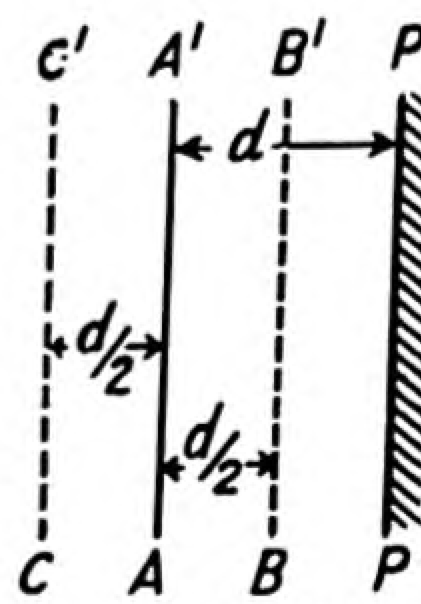


Fig. 13.30.

plane CC', will result in the pressure falling, but only to two-thirds of its original value. In other words, there is a net increase of pressure, although only a second order quantity, during a cycle, and this is the origin of the radiation pressure.

Hot-wire microphone

This microphone was developed by Tucker and Paris during the First World War, and depends for its action upon the cooling effect of a *transverse* flow of air upon a *very fine* platinum wire (of the order of 6×10^{-4} cm. diameter), which is heated electrically. Tucker placed the wire in the neck of a Helmholtz resonator (see also p. 163), and observed the maximum change of electrical resistance occurring when the resonator was in resonance with the sound source. The cooling of the wire is a function of the particle-velocity within the neck of the resonator, but experiment has shown that this is proportional to the pressure variation at the resonator mouth, and so it becomes possible to interpret the experimental observations in terms

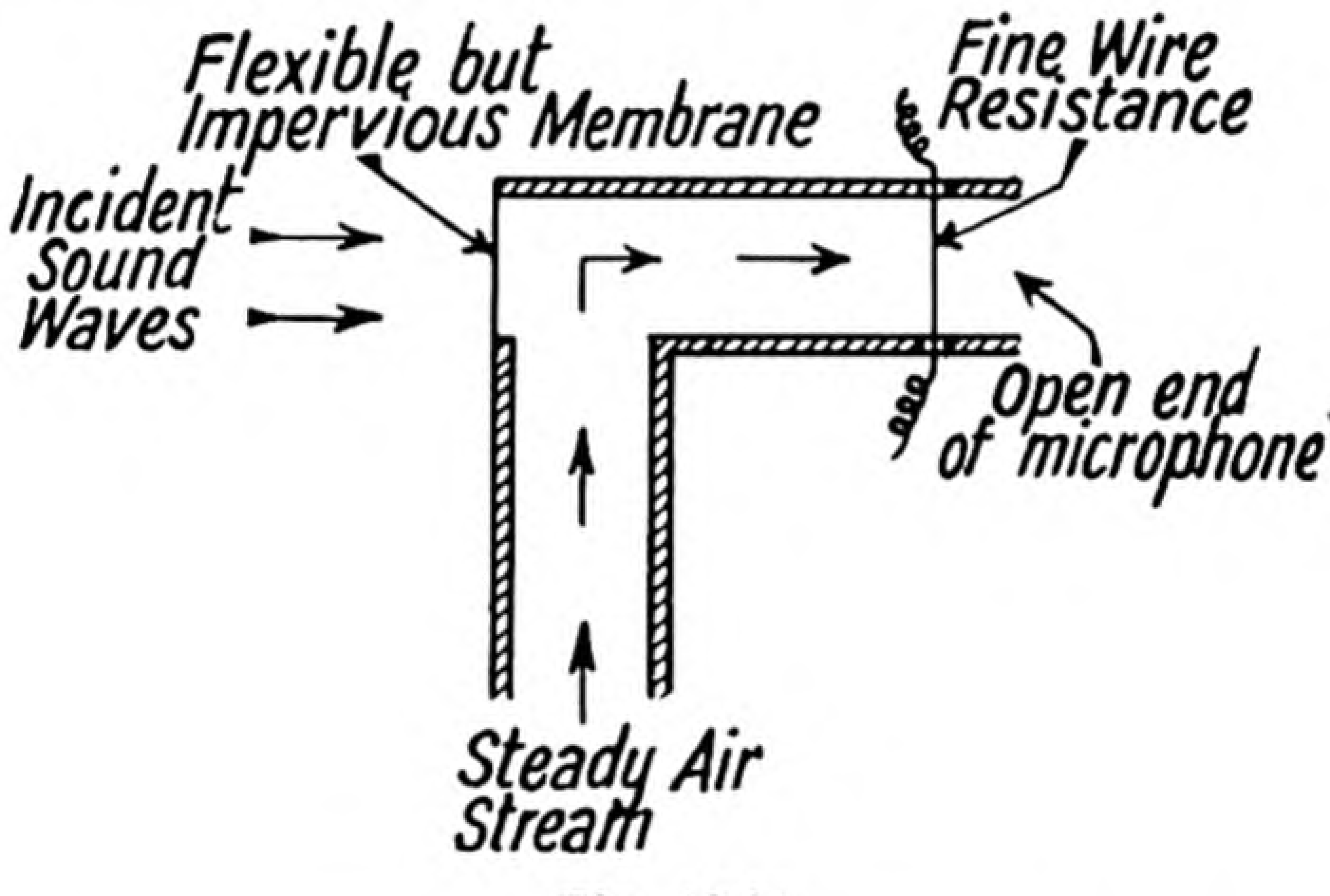


Fig. 13.31.

of the actual intensity of the sound waves. Besides employing the apparatus for gun location, Tucker and Paris applied it to explore the sound fields of trumpets, etc. The sensitivity of the hot-wire microphone as a sound detector is increased, and also the distortion is reduced, by passing a steady stream of gas across the wire* as indicated in Fig. 13.31. At a given frequency the resistance change is approximately proportional to the product of the steady stream velocity, the particle-velocity and the cosine of the included angle between their directions.

Theory of instrument and method of experiment. The cooling effect upon the wire of a *steady* air stream of velocity U may be assumed to be given by

$$\delta R = \delta R_0 + a(U - V_0)^2 + b(U - V_0)^4 + \quad . \quad . \quad . \quad (71)$$

where V_0 is the velocity of the *free convection* current from the heated wire grid, δR is the *change* in resistance of the wire grid due to the steady air stream, and δR_0 is the *maximum change* in resistance, *i.e.* when $U = V_0$; a and b are empirical constants.

* Cf. use of polarising voltage, p. 204.

It suffices to consider only the square term, so that in the case of the *alternating* particle-velocity resulting from the incident sound waves of frequency $\frac{p}{2\pi}$, equation (71) becomes

$$\begin{aligned}\delta R &= \delta R_0 + a(U_0 \sin pt - V_0)^2 \\ &= \delta R_0 + aV_0^2 + aU_0^2 \sin^2 pt - 2aU_0V_0 \sin pt \\ &= \delta R_0 + aV_0^2 + \tfrac{1}{2}aU_0^2 - \tfrac{1}{2}aU_0^2 \cos 2pt - 2aU_0V_0 \sin pt \quad . \quad (72)\end{aligned}$$

Hence the cooling effects have three main components:

(1) The steady change in resistance given by $\frac{1}{2}aU_0^2$, which will be proportional to the *intensity* of the sound, and may be simply and conveniently measured by including the grid of hot wires in one arm of a Wheatstone network.

(2) A periodic change of resistance due to the octave of the fundamental which is negligible for small amplitudes compared with—

(3) The oscillatory variation $2aU_0V_0 \sin pt$, due to the fundamental. This alternating component of resistance change may be measured by means of a suitably designed low-frequency valve-amplifier, and will be a function of the *amplitude* of the sound wave. The hot-wire microphone is very sensitive to air draughts, and its response falls off rapidly at higher frequencies.

Condenser microphone

The condenser microphone is, in principle, one of the simplest of all types, and consists essentially of a metal diaphragm tightly stretched and mounted parallel to and about 0·001 in. away from a fixed metal plate. The cushion of air enclosed between the plate and diaphragm *loads* the latter and effectively adds to its stiffness. In this way the resonant frequency of the diaphragm is raised and in consequence is usually outside the range of normal acoustic frequencies. The diaphragm and metal plate are connected in series with a high resistance and a battery of about 200 volts. When the diaphragm vibrates under the action of the incident sound waves the electrical capacitance between it and the plate will vary, and a corresponding alternating potential difference will be developed across the resistance. This P.D. is applied to the grid of the first valve of an amplifier, and is finally measured on a suitable meter in the output circuit. The capacitance of the microphone is extremely small, of the order of 50 to 300 pF, so great care has to be taken in making the connecting leads to the first valve of the amplifier as short as possible, for the capacitance of the leads would otherwise be comparable with that of the microphone, and in consequence would considerably reduce the voltage applied to the amplifier. With a well-designed amplifier, having a uniform response up to, say, 10,000 c.p.s., a P.D. of the order of 20 micro-volts may be measured which may correspond to such a small movement of the microphone diaphragm as 10^{-10} cm. This limit of voltage is imposed by the general noise level of the circuit. For use at the higher audio-frequencies the condenser microphone, as usually constructed, would distort the sound-field, and so smaller instruments have been designed, but with a consequent loss of sensitivity.

The condenser microphone is a pressure type of instrument, and although possessing a poor sensitivity, it has a very uniform response over a wide range of frequencies, thus forming a useful standard instrument which may be calibrated in a free sound-field by reference to a Rayleigh disc. If, however, the microphone is required for use in small enclosures a different method of calibration should be followed in which a *thermophone* may be employed. This instrument consists of two very thin gold-leaf strips, suitably disposed within an enclosed chamber which can be sealed tightly against the face of the microphone to be calibrated. A small alternating current of the desired frequency is now superposed on a larger direct current passed through the gold strips, and the consequent alternations of temperature will give rise to corresponding pressure changes, and hence sound waves of calculable excess pressure. In order that the dimensions of the chamber should be small compared with the wave-length of the sound the chamber is usually filled with hydrogen, thereby taking advantage of the high propagation velocity in that gas.

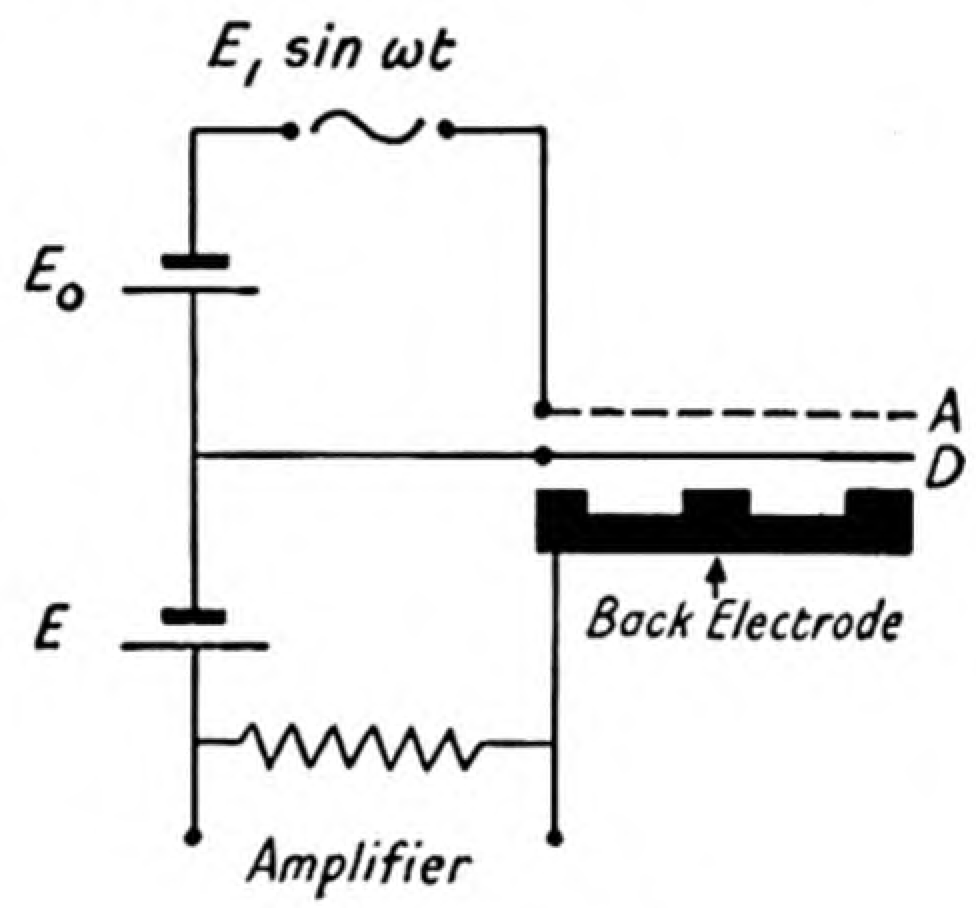


Fig. 13.32.

The thermophone method has, however, been largely superseded by one which involves the use of an auxiliary electrode A (Fig. .) in front of the diaphragm D of the condenser microphone. When an alternating E.M.F. $E_1 \sin \omega t$ is applied between A and D an electrostatic force will be exerted upon the diaphragm which will be given per unit area by $P=K(E_0+E_1 \sin \omega t)^2$, where E_0 is an applied D.C. polarising voltage (see Fig. 13.32) and K is a constant depending upon the dimensions of the instrument. The above expression may be rewritten as

$$P=K\left(E_0^2+\frac{E_1^2}{2}+2E_0E_1 \sin \omega t-\frac{E_1^2}{2} \cos 2\omega t\right) \quad . \quad . \quad (73)$$

which indicates that the polarising voltage E_0 should be chosen large enough so that the proportion of the first harmonic to the fundamental, viz.: $\frac{E_1^2/2}{2E_0E_1}=\frac{E_1}{4E_0}$ is negligible. The *alternating* pressure is then given by

$$P_a=2KE_0E_1 \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

The constant K is evaluated by applying a D.C. voltage $E_0{}^1$ between the grid A and the diaphragm D to balance a small pressure P_0 in excess of the atmospheric pressure applied to the diaphragm whence $K = P_0/(E_0{}^1)^2$. The balance is detected by using a sensitive capacitance meter to indicate any movement of the diaphragm.

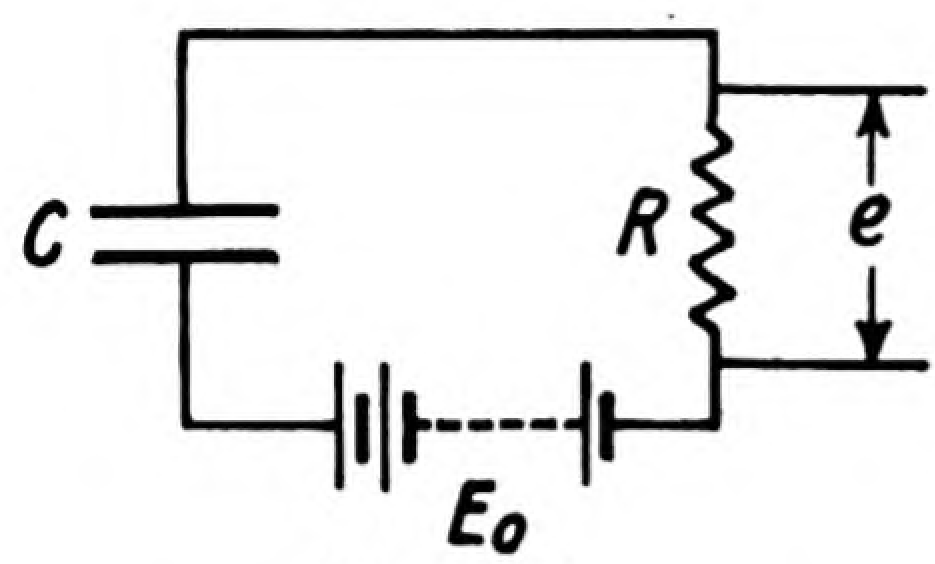


Fig. 13.33.

Theory of the instrument. The theory of the microphone may be developed by considering the conventional circuit of Fig. 13.33. E_0 is the polarising voltage, C represents the capacitance of the microphone, and R is a high resistance which must be sufficiently large so that the charge on the condenser remains approximately constant when the diaphragm vibrates. Let Q_0 be the value of this electric charge and C_0 the *equilibrium* capacitance of the microphone, *i.e.* $C_0 = \frac{\pi a^2}{4\pi\delta}$, where $2a$ is the effective diameter of the diaphragm and δ is the equilibrium distance between the diaphragm and the back plate (g in Fig. 13.34).

The restriction on the value of R implies that ωCR must not be less than unity, hence if 30 c.p.s. is the lowest frequency to be recorded and $C = 100$ pF., *i.e.* 10^{-4} μF., then it follows that R must be of the order of 10^8 ohms, viz. 100 megohms. It is also advantageous for R to be large from the point of the effective reduction of the noise voltage occurring in the resistance.

It is assumed that the excess pressure of the sound wave can be considered to be uniform over the surface of the diaphragm, which

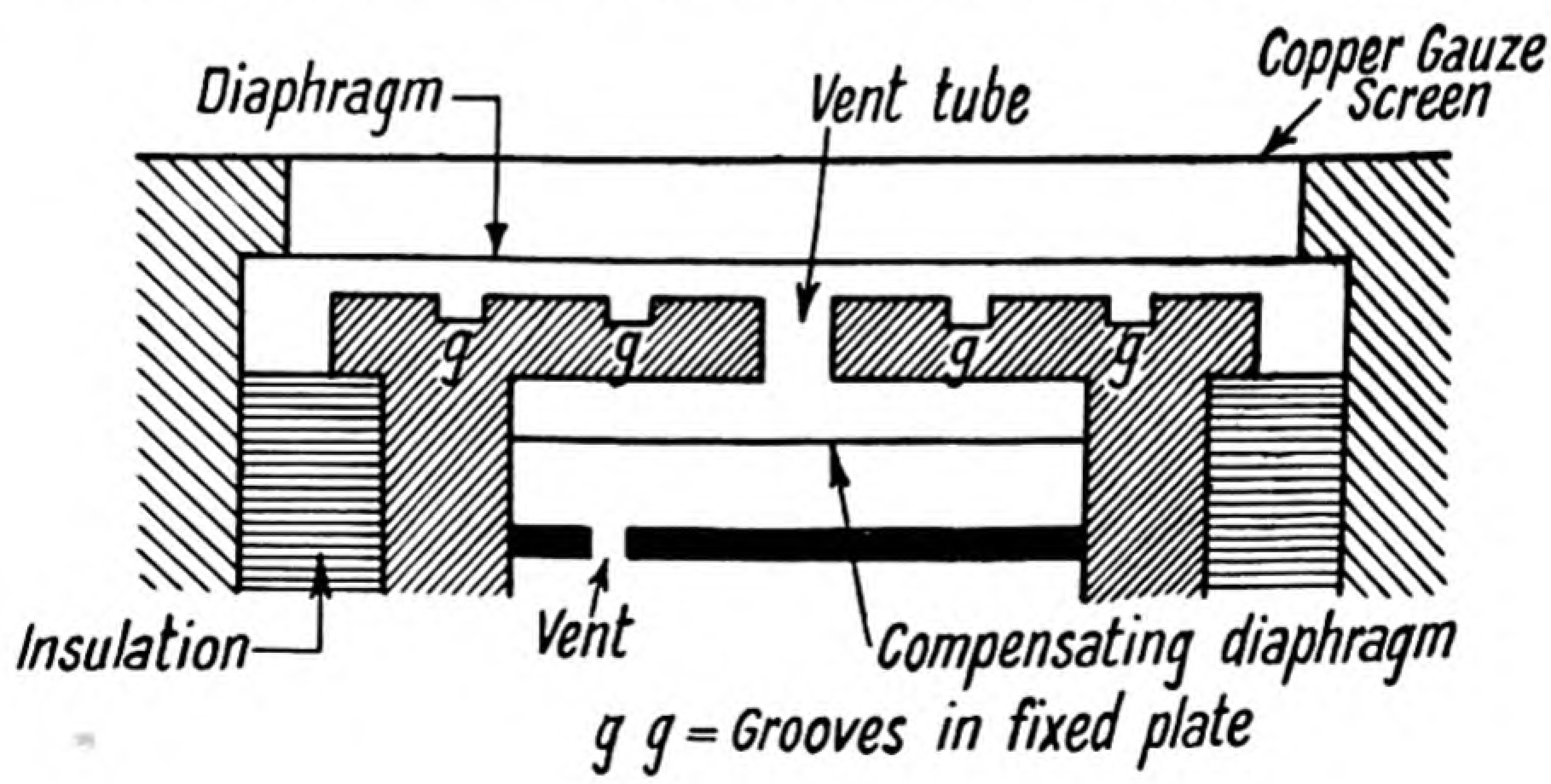


Fig. 13.34.

will be valid unless the wavelength of the sound is less than the dimensions of the microphone. This condition will not occur for most microphones until a frequency of 5 Kc.p.s. or so has been reached.

Consider firstly that R is infinite. Now $Q_0 = C_0 E_0$ will be the assumed constant electric charge of the microphone condenser and

suppose that, due to the impact of the sound waves, the capacitance at any instant has a value (C_0-c). It follows that the corresponding change in potential difference (e) between the plates of the capacitance will be given by

$$Q_0=(C_0-c)(E_0+e)$$
$$=Q_0+C_0e-cE_0-ce$$

or

$$e=\frac{c}{(C_0-c)}E_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (75a)$$

whence

$$e=\frac{c}{C_0}E_0 \text{ (approximately)} \quad . \quad . \quad . \quad . \quad . \qquad (75b)$$

If the capacitance varies simple harmonically with a sound wave of frequency $\omega/2\pi$, *i.e.* $c=C_0 \sin \omega t$, it follows from (75*b*) that the alternating component of the potential difference will vary in a similar manner. Actually, as follows from the more accurate expression (75*a*), a series of harmonics will exist in the alternating potential difference but their amplitudes are negligible compared with that of the exciting frequency if a large polarising voltage is employed (cf. equation 73).

When the microphone circuit contains the load resistance R the alternating potential difference appearing across R will be given by

$$e'=\frac{R}{\sqrt{R^2+1/(\omega C_0)^2}}e=\frac{e}{\sqrt{1+\left(\frac{1}{\omega RC_0}\right)^2}} \quad . \quad . \quad . \qquad (76)$$

It follows that

$$e'=\frac{E_0}{C_0}\,.\,c\frac{1}{\sqrt{1+\left(\frac{1}{\omega RC_0}\right)^2}}$$

$$=\frac{\left(\frac{E_0c}{C_0}\right)R}{\sqrt{\left(R^2+\frac{1}{\omega^2C_0^2}\right)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (77)$$

If e' refers to the peak voltage of the alternating potential difference across R, the latter equation suggests that the response of the condenser microphone in a sound field is equivalent to the introduction into the circuit of a generator of peak voltage $\frac{E_0c}{C_0}$, and having an internal impedance $\frac{1}{\omega C_0}$.

It is desirable that the change in capacitance corresponding to a *given* excess sound pressure should be independent of the frequency of the sound. This condition demands that the microphone should be stiffness controlled (p. 236), but Wente has shown that the air trapped between the plate and the diaphragm contributes to this

objective, so that the diaphragm need not be stretched so close to the elastic limit of its material as would otherwise be the case. Unfortunately the impedance of this air cushion is a function of frequency, but Crandall has shown how this effect can be controlled by cutting grooves or holes in the face of the metal plate (see Fig. 13.34). Fig. 13.35 (*a*) and (*b*) show typical circuit connections between the microphone and the first valve of the amplifier, the former being the usual form of coupling; a cathode follower connection is also employed.

An alternative method of using the condenser microphone, suggested by Riegger, consists in making the microphone capacitance a part of the variable capacitance of a high frequency electrical oscillator. The varying sound pressure acting on the diaphragm will modulate the *frequency* of the oscillatory current, and if this is passed through a circuit with suitable characteristics, the output current will correspond to the sound pressure.

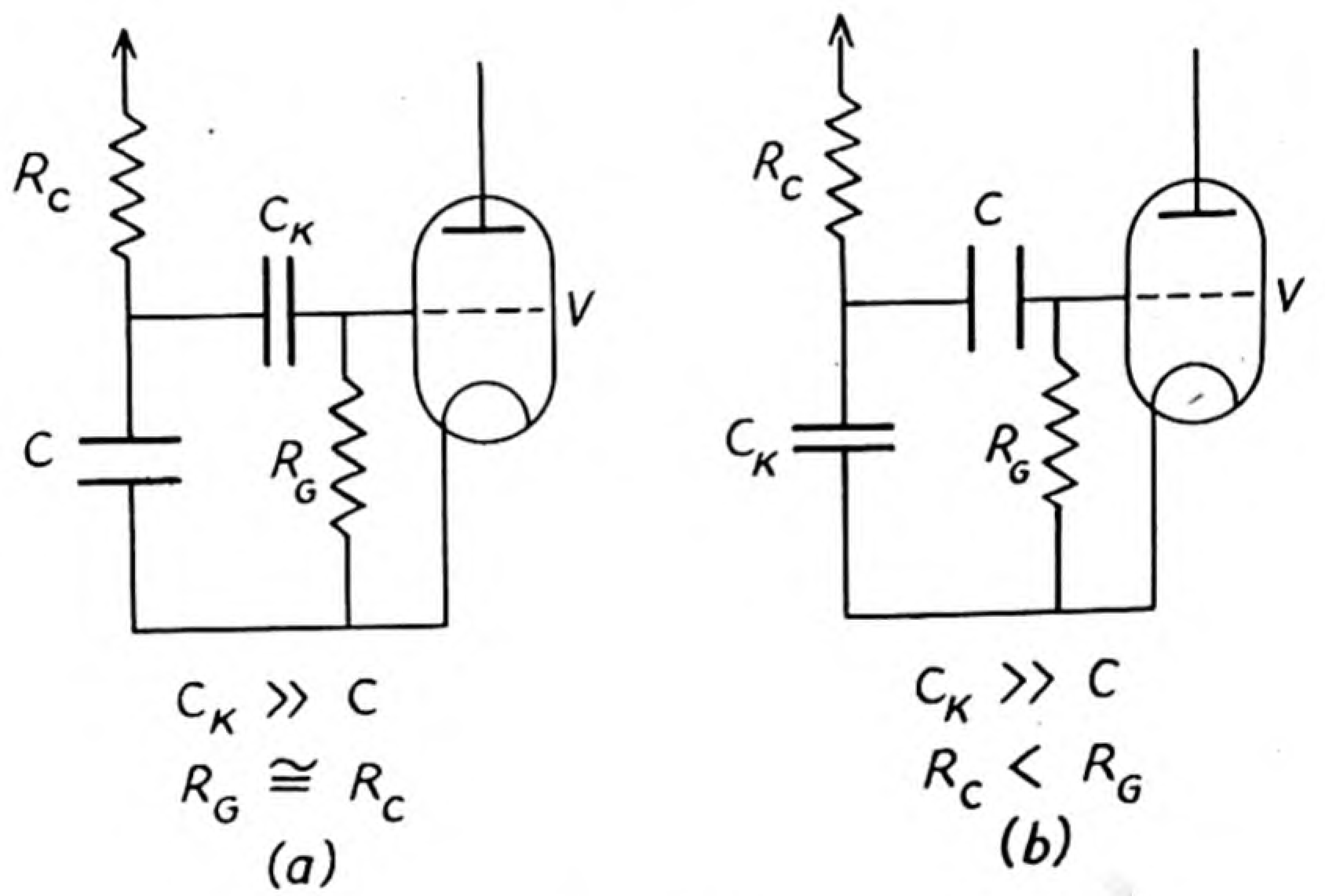


Fig. 13.35.

Velocity (or pressure-gradient) microphones

As their title suggests, these are instruments in which the indications are a function of the particle-velocities of the sound waves. If the distance between the two sides, for example, of a ribbon microphone (see later) is small compared with the wave-length of the sound, then the *pressure gradient* between points on each side may be taken to correspond to the particle-velocity. Actually it is easily deducible from previous fundamental theory that the pressure gradient is proportional to ω (particle-velocity), so in general the two quantities are not identical; both are vectors, however, having the same direction. These microphones are highly directional, and advantage is taken of this property in sound film "shooting" when, in order to eliminate camera noise, the microphone is placed overhead and tilted so that the camera lies in the plane of zero reception.

The ribbon microphone

The ribbon microphone consists of a very thin strip A (Fig. 13.36) of crinkled aluminium foil (thickness $\simeq 2{\cdot}5 \times 10^{-4}$ cm.) held at each end in terminal clamps C_1 and C_2, so that the mean plane of the strip coincides with the lines of force due to a permanent magnet NS. A movement of this conducting strip or ribbon in the magnetic field as a result of an impinging sound wave gives rise to the development of an electrical P.D. between C_1 and C_2. If *both* sides of the ribbon are freely exposed to the air, then at low frequencies, at least, the instrument will function as a velocity microphone. At higher frequencies, however, when the dimensions of the apparatus are comparable with the wave-length of the incident sound, the system will act as a baffle to shield the remote side of the strip from sound striking the front face normally. The microphone will then operate essentially as a pressure and not a velocity microphone, except that it will not respond to sound waves travelling parallel to the plane of the ribbon.

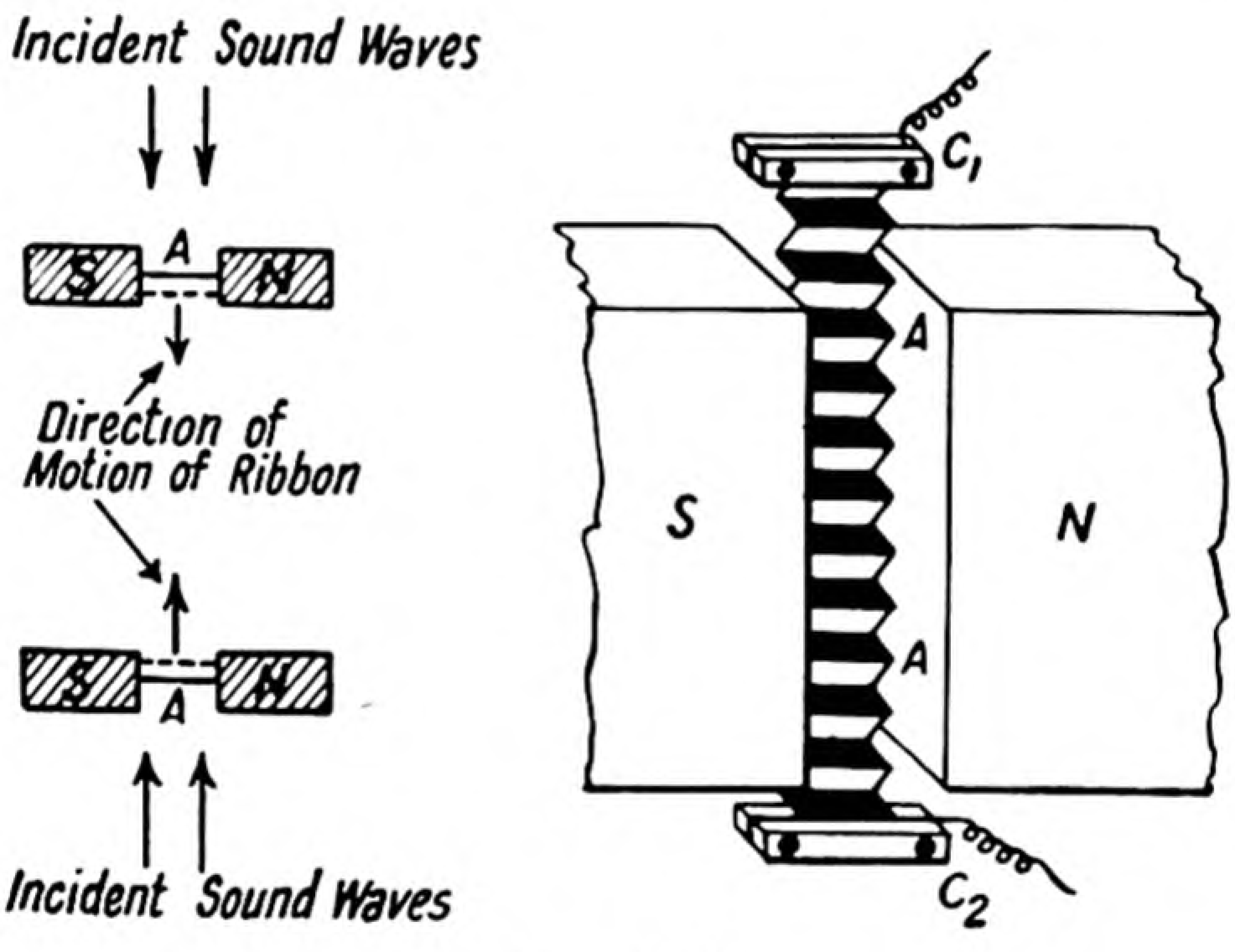


Fig. 13.36.

Theory. The following theory will have general application to any type of velocity microphone. Let $\Delta p = \Delta p_0 . \sin k(x-ct)$ represent, at a particular instant, the excess pressure distribution in a sinusoidal sound wave incident normally upon a ribbon of *equivalent* thickness d (Fig. 13.37). This equivalent thickness is to be interpreted as the shortest air-path between the front and the back of the ribbon *by way of the magnet structure*; the gaps between the ribbon and the pole pieces effectively prevent the transmission of sound energy by their narrowness. Then, assuming that the resultant excess pressure Δp acting on the ribbon in the direction of its thickness is the same as the pressure difference in a sound field between two points in space separated by a distance d, $\Delta p = \Delta p_1 - \Delta p_2$, where $\Delta p_1 = \Delta p_0 \sin k(d_1 - ct)$ and $\Delta_2 = \Delta p_0 \sin k(d_2 - ct)$.

Hence $$\Delta p = 2\Delta p_0\left[\sin \frac{k(d_1-d_2)}{2} . \cos k\left\{\left(\frac{d_1+d_2}{2}\right) - ct\right\}\right] \quad . \quad . \quad (78)$$

Suppose that the zero of the distance measurement is chosen to be midway between points at d_1 and d_2, then $(d_1-d_2)=d$, but $\left(\frac{d_1+d_2}{2}\right)=0$,

and so (78) becomes $\Delta p=2\Delta p_0 . \sin\frac{kd}{2} . \cos kct$ . . . . . (79)

If l and b are respectively the effective length and breadth of the ribbon it follows from (79) that the total instantaneous *force f* on the ribbon is

$$f=2(lb . \Delta p_0)\sin\frac{kd}{2} . \cos kct \quad . \quad . \quad . \quad . \quad . \quad (80)$$

But the instantaneous voltage e generated between the ends of the ribbon when it is moving in a magnetic field of strength H, is given by $e=Hl\dot{x}$, where $\dot{x}$ is the instantaneous velocity. Hence, assuming that the mechanical system is *mass* controlled, it follows (from p. 236) that the velocity is proportional to $\frac{\text{driving force}}{\text{frequency}}$. Now equation (80)

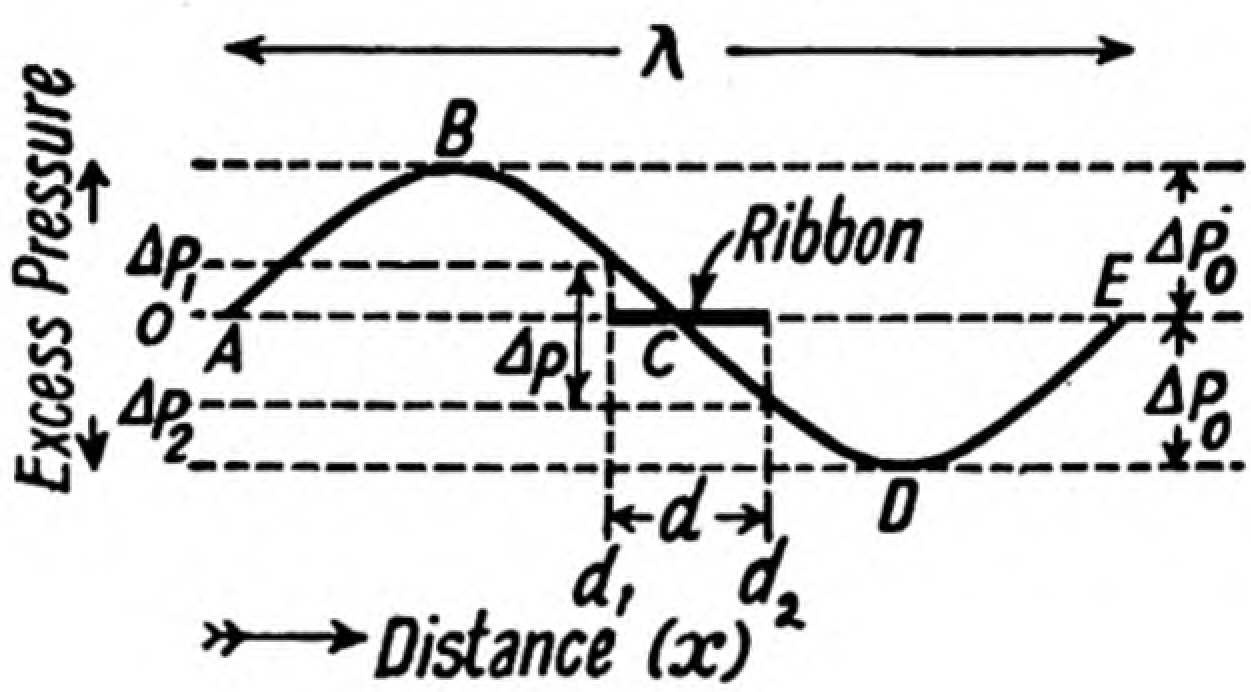


Fig. 13.37.

may be written, when d is small compared with $\frac{k}{2}$, *i.e.* with $\frac{\pi}{\lambda}$, as

$$f=2(lb . \Delta p_0)\frac{kd}{2}\cos kct=(lb . \Delta p_0)\frac{2\pi d}{c} . N\cos kct \quad . \quad (81)$$

where N is the frequency and c is the velocity of the sound waves.

Hence the E.M.F. generated e is $\propto\dot{x}\propto\frac{\text{driving force}}{\text{frequency}}\propto\frac{f}{N}$, and it therefore follows from (81) that it is independent of frequency provided d is small compared with $1/\lambda$.

Again from equation (80) it is evident that the force is a maximum when $\sin\frac{kd}{2}=1$, i.e. $d=\frac{\pi}{k}=\frac{\lambda}{2}$, and is zero when $d=\lambda=\frac{2\pi}{k}$. For a *given applied pressure*, therefore, the response of the microphone is governed by the variation with frequency of the function $\frac{\sin\frac{kd}{2}}{N}=\frac{\sin\frac{\pi N}{c}}{N}$, so that the general form of the response curve will be that shown in Fig. 13.38.

The moving-coil (or electrodynamic) microphone is essentially a diaphragm rigidly attached to a movable coil, which is located in the circular gap of a permanent magnet. The impact of the sound waves on the diaphragm causes a movement of the coil in the magnetic field with the consequent induction of small electric currents in the coil circuit. The coil usually comprises only a small number of turns, and is coupled to a valve amplifier by means of a step-up transformer.

The three types of microphone just considered, namely the condenser, the ribbon, and the moving-coil, illustrate how the uniformity of response with frequency is sought by using a diaphragm under different conditions. In the electrostatic type the diaphragm is tightly stretched until its fundamental frequency is at the upper limit of the frequency range desired, whereas the use of a very light and limp diaphragm, viz. the ribbon, virtually permits it to assume the particle-velocity of the incident waves. In the other type of microphone

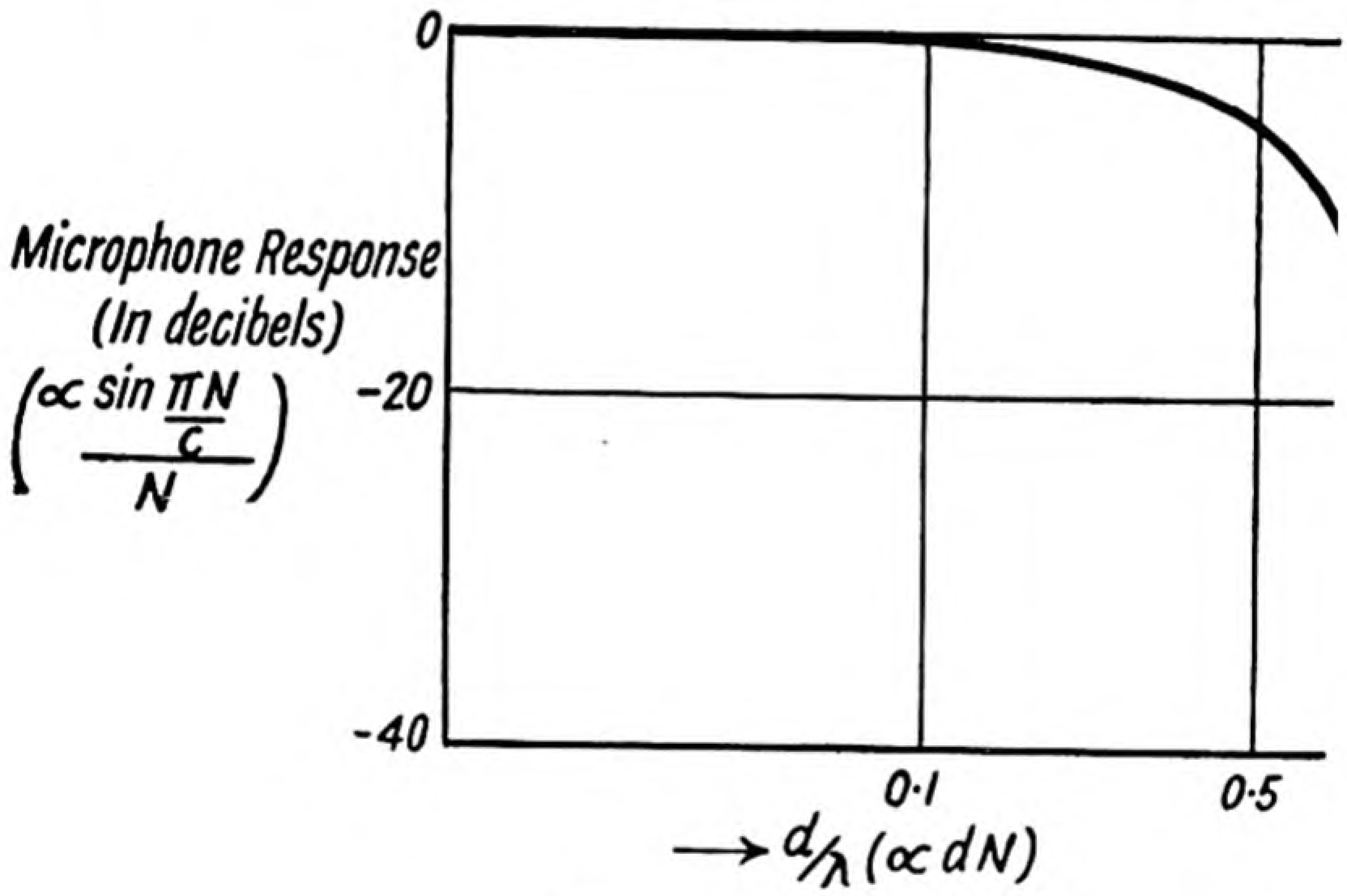


Fig. 13.38.

mentioned the diaphragm is rigid but under such slight restraint that its natural frequency is less than the lowest frequency to be recorded.

The carbon microphone is dependent upon the change of resistance between carbon surfaces when a varying pressure is applied. The carbon, in the form of granules to obtain a large number of contacts, is contained in a cylindrical metal cup. The granules also make contact with the metallic diaphragm which is suitably insulated by washers from the rim of the cup. Any displacement of the diaphragm will result in a change of the carbon resistance between cup and diaphragm, and if this resistance forms part of a closed electrical circuit containing a steady source of E.M.F. there will be a consequent variation of current. A suitable transformer included in this circuit enables the current variations to be amplified by a valve amplifier.

A reduction in the background noise is achieved by the use of a double-button construction, one carbon button being used on each side of the diaphragm. By employing push-pull electrical connections

it is possible to eliminate the even harmonic distortion obtained with the single button type of microphone. In order to obviate the use of the diaphragm as an electrode the transverse current type of instrument was designed. This microphone is constructed so that the current flows between two fixed electrodes in a direction parallel to the diaphragm, and the mass of the latter may now be reduced by making it of mica or other insulating material. In the Reisz carbon microphone the diaphragm is very highly damped by suitable choice of the particle size of the carbon powder, and adjustment of the layer thickness so that there is negligible reflection from the back surface of the cavity. As compared with the ordinary carbon telephone transmitter the efficiency of the Reisz is much lower, but the noise has been so much reduced that it may be "addressed" at much larger distances.

In concluding this brief discussion of various types of microphone it would seem that the ideal instrument is one in which the motion of the air particles produces a direct electrical effect, without any intermediate mechanical system. This idea has been successfully tried out, although not commercially developed, in a device by which the movement of the air particles directly modulated the electrical discharge in air between a point and a heated oxide-coated cathode.

Acoustic "wattmeter"

If p and v are the *instantaneous* values of pressure and particle-velocity in a sound field then the sound energy flow will be given by $\frac{1}{T}\int_0^T pv.dt$ where $\frac{1}{T}$ is the frequency of the waves. Hence if simultaneous measurements can be made of these quantities a means is available of calculating the energy flow. The electrical instrument for measuring power is the "wattmeter" and this comprises both "voltage" and "current" elements. The corresponding acoustic instrument consists of a pressure detector, a crystal microphone, and a velocity detector, a ribbon microphone. The alternating voltage outputs from these two microphones are amplified and any phase shift suitably corrected before application to the appropriate terminals of a wattmeter of the thermocouple type.

Directional properties of microphones

When a microphone is employed as a part of the acoustical equipment used to amplify speech or music in an auditorium, an advantage is often gained by choosing an instrument which possesses a prominent directional characteristic, *i.e.* it shows a much enhanced response to sound waves travelling from the direction of the speaker as compared with its average response to sound waves over all directions of incidence.

Now a pressure-operated microphone, provided its dimensions are smaller than the length of the sound waves, shows an equal response to sound waves impinging in any direction (Fig. 13.39, curve *a*). On the other hand, a microphone whose operation is dependent upon the existence of a pressure-gradient at the diaphragm normal to its

surface, shows a wide variation in response with direction of incidence (Fig. 13.39, curve *b*). In fact, if this direction makes an angle ϕ with the normal to the plane of the diaphragm, then the corresponding response is proportional to $\cos \phi$, since the component of differential pressure along the plane (*i.e.* $\propto \sin \phi$) excites no response.

By suitably combining a pressure-gradient with a pressure microphone, both possessing identical maximum responses, their combined response will be proportional to $(1 + \cos \phi)$, and the characteristic directional curve becomes a cardioid (Fig. 13.39, curve *t*).

Cavity resonance

In addition to the diffraction effect due to the presence of the microphone in the sound field there is another disturbing effect which results from the presence of a cavity in front of the microphone diaphragm, as usually constructed. The resonance of this cavity will give rise to an increased pressure on the diaphragm above that of the free field, but for most purposes it is possible to apply an approximate correction.

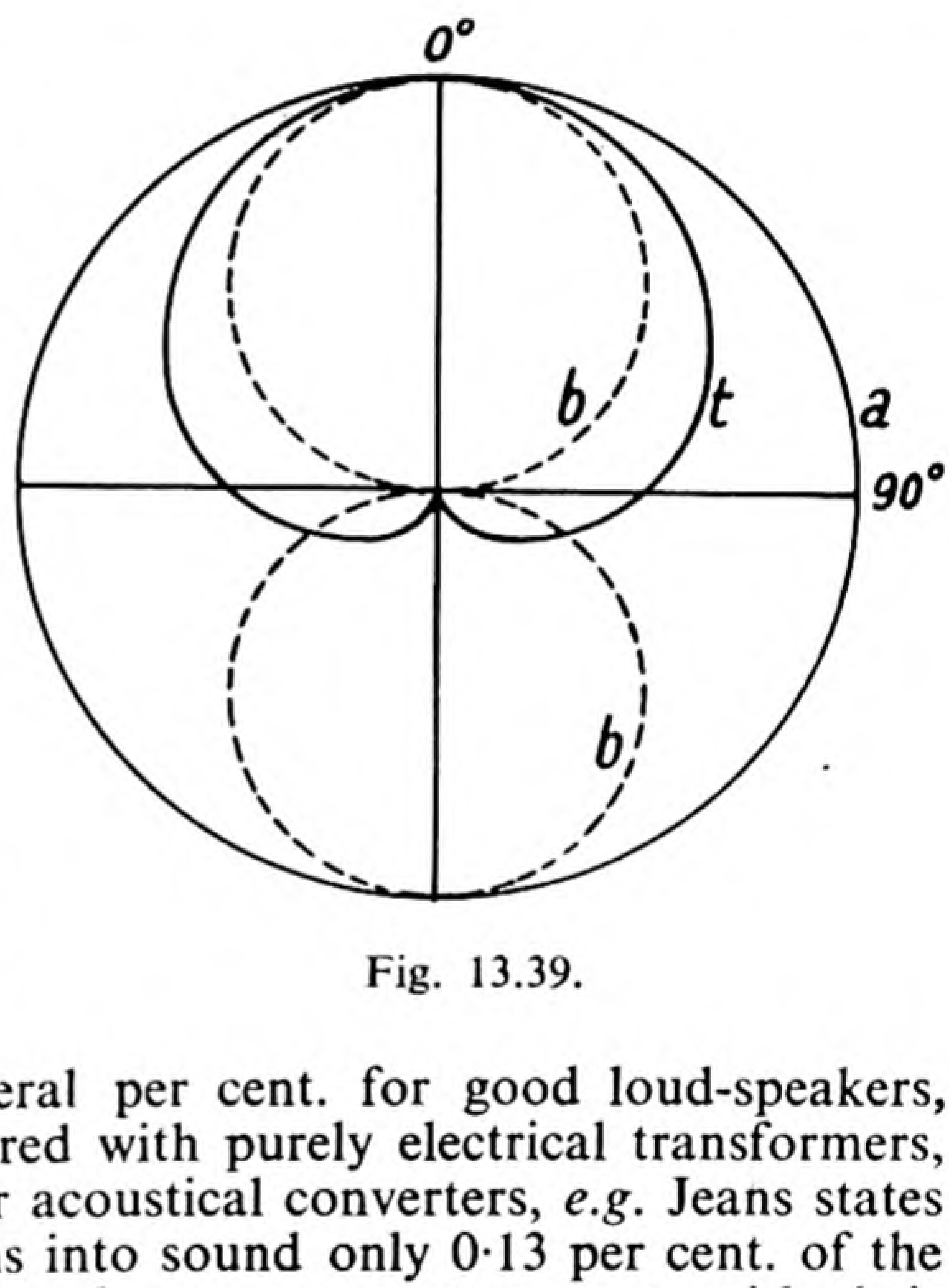


Fig. 13.39.

Loud-speakers (or reproducers) are essentially microphones used in reverse, with a device, *e.g.* horn or baffle, for efficiently transferring the vibrations of the moving part to the external medium, usually air. The various forms of microphone described are typical of the basic mechanisms by which the input electrical energy is transformed into radiated acoustical energy. The efficiency of this energy conversion is of the order of several per cent. for good loud-speakers, which is a poor figure compared with purely electrical transformers, but is high in relation to other acoustical converters, *e.g.* Jeans states that a church organ transforms into sound only 0·13 per cent. of the energy supplied. The use of modern permanent magnets with their high field-strengths is a means of increasing the efficiency of the electrodynamic loud-speaker.

The two main characteristics of a loud-speaker are those concerned with "distortion" and with "frequency." The latter gives the relative sound intensity produced by the transducer when equal alternating voltages of different frequencies are applied to it, while the other characteristic gives a measure of the non-linear distortion produced in the loud-speaker itself. Other factors require to be considered, however, in assessing the qualities of a transducer, notably its directional properties (see p. 123) and the conditions under which the characteristics were determined. The characteristics as measured in an

SUMMARY OF MICROPHONE CHARACTERISTICS (AFTER GREENLEES)

Type of Microphone	*Electrical Characteristics* Impedance at 1000 c.p.s.	Sensitivity expressed as the Open Circuit Voltage (in db. below 1 volt) per dyne per sq. cm.	*Advantages*	*Disadvantages*
Carbon	200 ohms	−38 db.	Cheap. Sensitivity high. Reliable and robust.	Background noise (hiss). Requires external battery for polarising current. Granules liable to pack. Frequency response irregular. Directional effects with frequency discrimination.
Condenser	0·5 megohm	−50 db.	Frequency response good. No background noise.	Requires carefully constructed amplifier. Affected by moisture and is fragile.
Moving Coil	20 ohms	−63 db.	Frequency response good. No background noise. Sensitivity good. Reliable and robust.	The directional types have frequency discrimination. Liability of diaphragm to flutter in wind.
Ribbon	200 ohms (including associated transformer)	−73 db.	Frequency response very good. No background noise. Good directional effect with little frequency discrimination. Reliable and robust.	Liability of ribbon to flutter in wind. Output relatively low.
Crystal	0·05 megohm	−72 db.	Frequency response very good. No background noise.	Requires carefully constructed amplifier. The diaphragm (directional) type shows frequency discrimination.

Note.—The sensitivity of a microphone is measured by the electrical output produced by sound of a specified intensity; the British Standard Specification No. 661 defines the sensitivity as the open circuit voltage generated when the sound pressure is one dyne per square centimetre, the direction of the sound with respect to the axis of the microphone being also stated. Hence a sensitivity of −38 db. below 1 volt per 1 dyne per square centimetre is equivalent to a generated voltage v given by $20 \log_{10}\left(\frac{1}{v}\right) = -38$, i.e. $v = 0{\cdot}0126$ volt.

ordinary-sized living-room would obviously involve the properties of the room as much as those of the loud-speaker. Hence open-air measurements were standardised under conditions in which the loud-speaker and microphone were disposed to avoid unwanted reflections, *e.g.* in a position projecting over the edge of a tall isolated building. The disadvantage of uncertain weather conditions, however, stimulated the erection of so-called "dead" rooms the walls of which are suitably covered with sound-absorbent materials to render them as complete absorbers as possible. Slag-wool (or glass-fibres), maintained in position with wire-netting of suitable mesh, is a typical wall-covering for such chambers. In one particular "dead" room this covering was supplemented by the addition of parallel strips of crepe paper, each about a foot wide with a half-inch separation, which were hung perpendicular to the faces of the walls. A "false" floor was used to support the apparatus and between this and the true floor a series of closely separated curtains were hung vertically. More recently the walls of "dead" rooms have been constructed in the form of narrow angle wedges of glass-fibre, held by wire-netting, so that the incident sound in general suffers considerable absorption by successive reflections within these angles.

The sound field

In an analogous manner to the conception of an electric or magnetic field, the sound field of a source is considered to be the surrounding space in which its influence can be detected. The acoustic intensity at any point in such a field may be measured by utilising any of the following physical properties: (*a*) the amplitude of the disturbance, (*b*) the excess pressure, (*c*) the particle- (or displacement-) velocity, (*d*) the density change (or corresponding change of refractive index), (*e*) the *steady* pressure on a surface due to the impact of sound waves (analogous to the corresponding phenomenon in light), (*f*) the thermal changes accompanying the alternating compressions and rarefactions, and (*g*) the power which may be absorbed from the sound waves. An obvious advantage accrues from the choice of a property which shows no dependence upon frequency, and for this reason the excess pressure is most generally measured, although the particle-velocity is often utilised by virtue of its control in the operation of the Rayleigh disc.

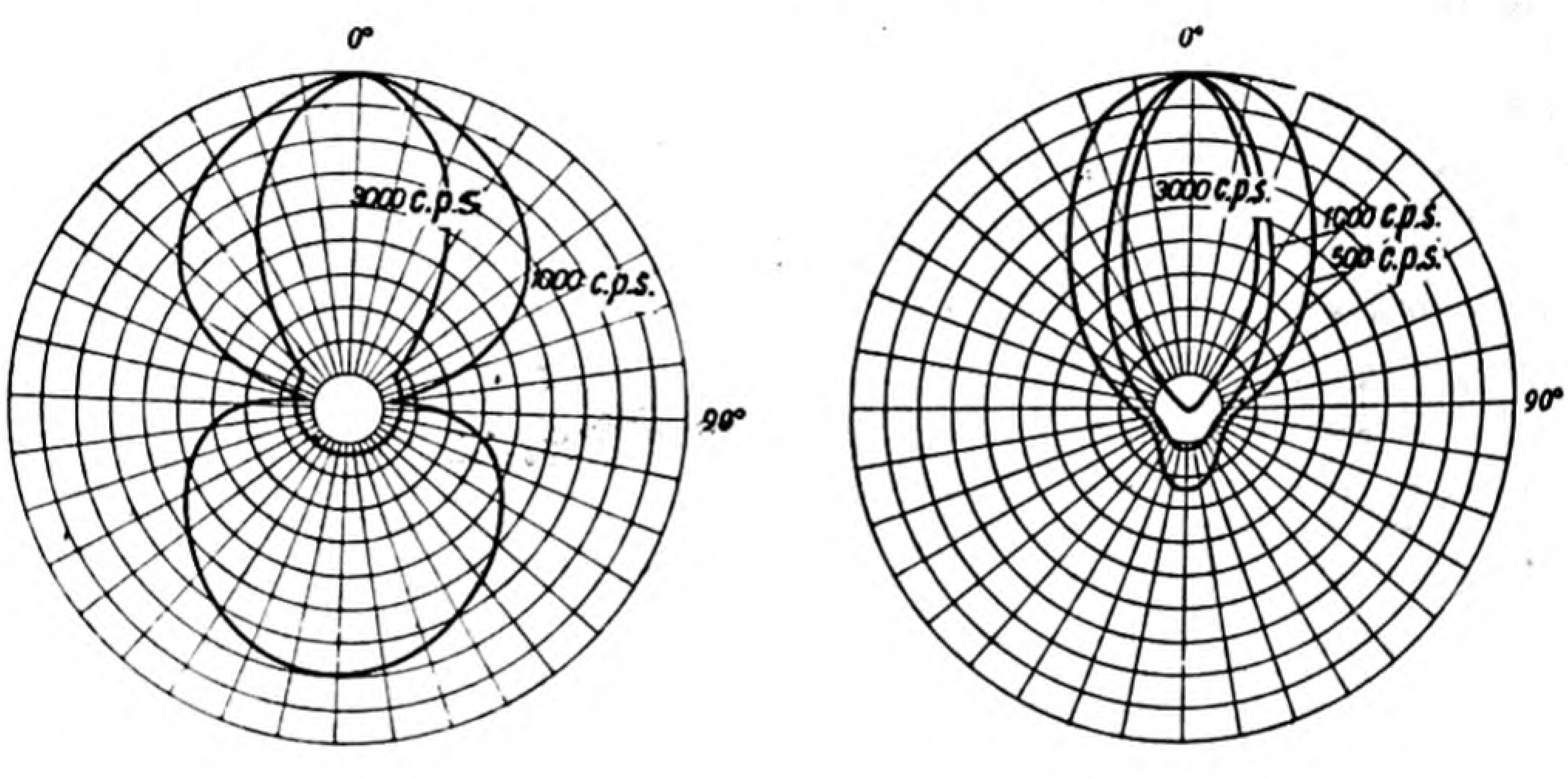
A typical representation of the distribution of excess pressure in the sound field due to a loud-speaker is shown in Fig. 13.40. The heavy curves join places of equal intensity and the radial lines locate the position of any point, in angular measure, with respect to the plane of the loud-speaker diaphragm (*i.e.* the 90° line). Fig. (*a*) refers to a loud-speaker located in a baffle and Fig. (*b*) to the location in a horn.

Sound location

It is an everyday experience that the ability to locate a source of sound is dependent upon the use of both ears, *i.e.* a binaural effect, for a person deaf in one ear would always localise a sound at his good ear, unless, of course, there are extra-auditory factors such as a moving car in the street which help him to fix the source of sound. An

individual with normal hearing can experience this effect by temporarily blocking up one ear. Now the possible physical differences in the sound waves reaching the ears of an auditor which enable him to determine the direction of a *simple* tone are intensity and phase.

Considering first the case of differential intensity, it is evident that the sound will be louder at the ear nearer the source, and that the location can be assisted by turning the head in different directions until one ear is towards the source. For positions remote from the latter, however, the distances to each ear will be approximately the same, but now the possible effect of sound shadows due to the presence of the listener's head must be considered. This effect will be apprecia-ble only for higher audible frequencies, when the wave-length becomes comparable with the distance between the ears, and it means that low-pitched sounds are difficult to locate, for they will bend round the head so that the intensity in the two ears is indistinguishable. When the



(*a*) (*b*) [*After de Boer.*

Fig. 13.40. Sound-field of a loud-speaker.

sound is complex, some judgment of direction may be permitted, for the higher pitched components will be weakened at the ear within the sound shadow.

In recent years it has been found that it is the difference of phase at the two ears which is the most important factor in the localising of sound, at any rate for frequencies below 1300 c.p.s. Stewart ("Physical Review," May, 1920) has shown that the apparent position of the origin of the sound is dependent only upon the difference in *time of arrival* of like phases at the two ears. Hence, if the source is emitting various sounds of different frequencies, all will have the same difference in time of arrival at the ears, and it follows that the listener will correctly judge them all to come from the same location.

In modern warfare the question of sound location is very import-ant. According to Tucker ("Discussion on Audition" (1931), The Physical Society (London), p. 115), aeroplane location without accessory apparatus or visual help, can be achieved by an average listener to about 10° if the aircraft is flying at low angular elevations.

If the plane is in an overhead position, however, the intensities of the sound in the two ears are the same and change very slowly at this point, and the listener can decide when it is overhead only by estimating when the *total* sound received is maximum. Tucker has shown that the impression of a maximum, when the aircraft is at 10,000 ft. or more, might persist for nearly a minute, although this time will be reduced if the head is suitably tilted. If the ears of the listener could be separated further apart, then the difference of phase at the ears for a given position of the head would be greater, and hence the accuracy of location increased. This effect is realised in sound locators by the use of listening trumpets 10 ft. or so apart, which are connected one to each ear of the listener by means of a stethoscope tube.

The measurement of noise

Noise has been defined as any unwanted sound in the judgment of the individual concerned, for it is to be remembered that "one man's noise is another man's music." The physical distinction often made between a musical sound and a noise, namely, that the latter is a *random* mixture of notes of definite frequencies, rather breaks down when it is considered that pure notes of a certain frequency and intensity can be highly objectionable.

The industrial development of this century has made the problem of noise a common concern for all so-called civilised countries, and a certain amount of legislation has been introduced in this country to cope with the menace. Exposure to constant loud noises will gradually impair the hearing (see p. 170), while the efficiency of a worker is seriously reduced by excessive noise. The reaction to noise is largely temperamental, and the degree of tolerance varies widely with the individual and also with the general "noise" background. It is interesting to note that during the last century the disagreeability of noise was recognised in the Navy by the payment of "noise-money" to the crew during the period of operation of fog-signals.

Any scientific investigation into the causes and reduction of noise obviously demands some quantitative estimation of its degree of annoyance, and experience agrees that its loudness would be the most important criterion. Hence it became necessary to correlate in some way the aural loudness with the physical energy involved, and a scale of *ratios* of energy, in which the chosen unit, called the *bel*, was expressive of a ten-fold increase in energy. The choice arose from the purely physical aspect of measuring intensity, but it was found to possess the advantage of forming an approximate fit with the aural scale of loudness sensation. The *zero* of the scale was taken as the *threshold of audibility* for the particular frequency concerned, but a smaller scale unit, the *decibel*, was found to be a more convenient unit for acoustical purposes. If W_0 is the acoustical power corresponding to the threshold intensity, and W_s that of the sound under test, then $x=10 \log_{10} \frac{W_s}{W_0}$ represents the intensity of the sound in *decibels* above the threshold of audibility for that frequency, and furthermore, x is taken as defining the sensation level of the sound. (See also p. 172).

Unfortunately, in general, two pure sounds of different frequencies do not produce the same aural sensation of loudness, even if (i) their *physical* intensities are identical, or (2) their *sensation* levels are equal. Hence for the comparison of the loudness of pure sounds of different frequencies an arbitrary scale has been adopted as a practical standard, viz. the sensation scale of a pure note of an agreed frequency. For medium audio-frequencies above about 700 c.p.s., there exists a fairly constant relationship between sensation level and loudness which permits a latitude of choice. The standard frequency adopted by U.S.A., Germany and this country was a pure tone of 1000 c.p.s., although much of the earlier work at the National Physical Laboratory (N.P.L.) was with reference to a frequency of 800 c.p.s. The British Standards Institution have decreed that the recognised procedure should be to listen with both ears alternately to the standard tone and to the test noise, and that the intensity of the standard be adjusted until it is judged to be as loud as the noise. This intensity of the standard is then measured by means of a microphone, and is expressed as, say, x decibels above the threshold audibility of the 1000 c.p.s. tone, which corresponds to an excess pressure in free air of 2×10^{-4} dyne

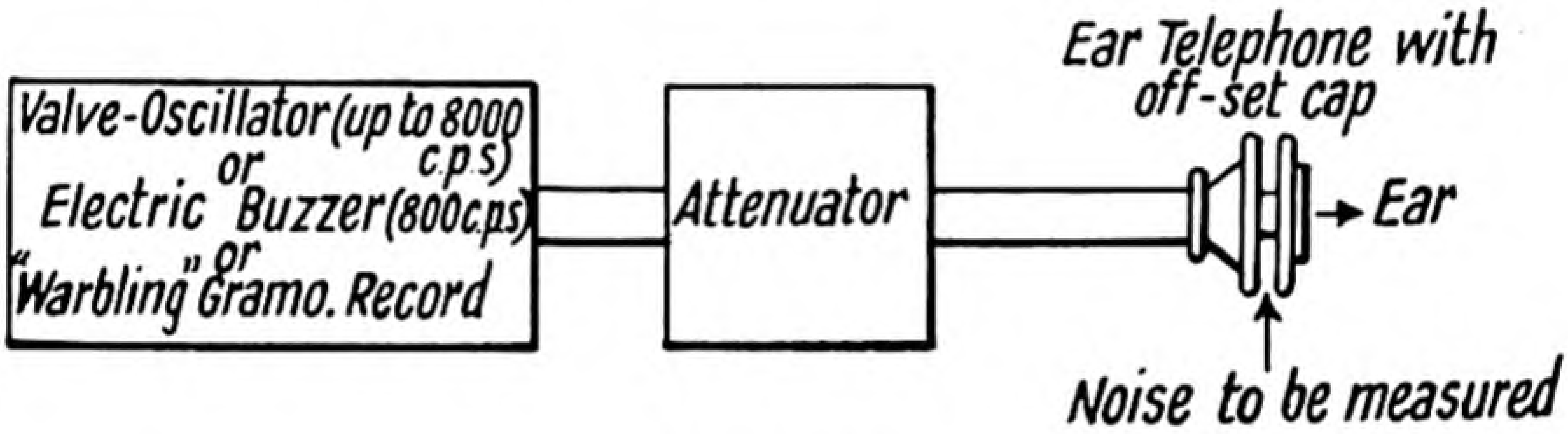


Fig. 13.41.

per sq. cm. The noise is then said to possess an *equivalent loudness* of x *phons*.

The standard technique outlined above is not particularly suitable for commercial practice, and various forms of noise meter or audiometer have been designed which are more convenient for such use. These instruments comprise a standard tone, which may be produced by an electric buzzer, a valve-oscillator or a "warbling" gramophone record (with electrical pick-up), together with a calibrated attenuator indicating the loudness of the tone. The acoustical output from the attenuator is fed into a telephone ear-piece which is applied to one ear of the observer, while the other ear is uncovered to listen to the external noise. An alternative procedure is for the standard tone and the noise to be heard by the same ear simultaneously, the telephone ear-piece being modified for this purpose by offsetting the cap (Fig. 13.41). Variations of procedure in which either or both ears listen alternatively to the noise and to the standard tone, are often followed, and sometimes the intensity of the standard tone is noted when it is just *masked* by the noise when both are heard simultaneously (see p. 173). In general, it is found that masking measurements are more easily obtained than matching of intensities, and are sometimes preferred as giving a measure of the "degree of deafening" produced

by the noise upon the ear, for the particular tone employed. All the above methods are said to be subjective since they depend upon aural matching, and in consequence, for greater accuracy, it is necessary to employ a number of observers, which, of course, extends the time required for observation. In order to meet this difficulty, objective noise meters have been designed in which a microphone and amplifier replace the ear, the amplifier being designed to simulate the response of the ear at each frequency for the approximate intensity level concerned. In practice it is found that it suffices to have adjustable response for levels between 40 db. and 90 db. only.

A very simple and portable apparatus for measuring noise has been used by Davis at the N.P.L. The apparatus consists of a tuning-fork which is struck in a standard manner, and then held as close to the ear as possible; the time t is noted between striking the fork and the moment its loudness falls to that of the observed noise. If the rate of decay of the fork has been found by means of some other type of audiometer, then its intensity I_t of vibration at any instant t sec. after striking will be given by $I_t=I_0e^{-\alpha t}$, where I_0 is the initial intensity and α=coefficient of decay. The difference of intensity in decibels will, therefore, be $10\left(\log\frac{I_0}{I_t}\right)=x$, say. The sensation level S_n is then given by $S_n=S_I-x$, where S_I is the initial sensation level of the fork.

In tracking down the source of noises, *e.g.* in machinery, it is very helpful to obtain the frequency spectrum of the noise, but this aspect of the problem is dealt with in Chapter 11.

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CHAPTER 14

ARCHITECTURAL ACOUSTICS

The hearing of speech and music is of fundamental importance, and when it is necessary to convey sound to many people the most favourable conditions are essential, particularly for speech. In the course of the evolution of auditoria, orators found that an elevated position with respect to the audience was necessary to extend the range of hearing, as this prevented those listeners nearest the speaker from absorbing all of the acoustical energy. Another advance was made by having a hard surface immediately behind the speaker to reflect sound which would otherwise be lost. This was followed by arranging seats in tiers to assist seeing and hearing, and finally, by roofing the amphitheatre, a more nearly complete conservation of the acoustic energy was attained. These improvements, however, gave rise to new problems which are discussed in the following sections of this chapter.

The subject of Architectural Acoustics owes its development as a science largely to the work of Wallace Sabine. Towards the end of the nineteenth century, Sabine commenced a systematic investigation into the acoustical properties of a hall at Harvard University, where he was Professor of Physics. These properties may be conveniently divided into two classes, the one concerned with the satisfactory hearing of speech and music, and the other with the elimination of noise or unwanted sound. The problems which arise in these two groups are largely concerned with the reflecting, absorbing, and transmitting properties of materials towards sound waves, but psychological factors also demand consideration.

Sabine sought to remedy the defects of an existing auditorium, but as a result of his work and that of later observers, troubles in new halls can be largely avoided by suitable design. Problems of sound insulation and of noise suppression occur in almost every inhabited building, and the type of remedy to be employed may be complicated by certain unexpected factors. For example, it is found that the output of work in a factory is diminished by excessive noise from a machine, but that the effect on the actual operator of the machine is less than on the other people in the room. It appears that the person controlling the cause of the noise suffers the least discomfort.

Sound insulation

Sound requires a material medium for its propagation, and when it enters a different medium, part is reflected at the surface, part is absorbed, and the rest is transmitted (see p. 106). The intensity of the reflected sound beam, and hence of the transmitted beam, will depend upon the angle of incidence, the nature of the interface and of the two media, and, in general, upon the frequency of the sound. The amount of acoustical energy absorbed within the second medium

will be dependent upon the nature and thickness of the material and also upon the sound frequency.

In the passage of light through a transparent medium, *e.g.* a block of glass, the fraction *transmitted* is large, but if the same medium is in powder form a considerable reduction is affected. This loss of transparency is the result of multiple reflections at the increased number of air-glass interfaces. In an analogous manner sound may be absorbed by breaking up a medium into granules or strata. On this principle a given thickness of wood panelling provides better sound insulation between rooms if split into two or more panels separated by air-gaps than when acting in one piece. Furthermore, the insulation may be improved by inserting lightly packed granulated cork between the partitions. If a partition is flexible it may become a source of aerial vibrations should its natural frequency coincide with a frequency present in the incident sound, but this effect may be avoided by suitably stiffening the section. Sounds communicated to and via floors and ceilings by running machinery, etc., may be minimised by mounting the machine on rubber insulation (see p. 324). The structure of a building, particularly if steel-framed, frequently acts as a good conducting path for sound waves; to reduce transmission, insulating layers of lead and asbestos cloth sheeting are employed to line the steel beams where they overlap. As a further precaution, connecting bolts are sheathed with asbestos and bitumen to prevent them from coming into direct contact with the beams. Where possible, however, the sound should be deadened at its source, *e.g.* typewriters are placed on rubber feet to reduce the impact transmitted to the table, and transparent covers, which leave only the keys exposed, are provided to minimise the air-borne sounds due to the key action. Traffic noises are minimised by using double doors and windows and by employing ventilation conduits lined with absorbent material.

Absorption

The problem of the attenuation of a wave in its passage through a material medium was first put on a mathematical basis by Lambert, a German physicist and mathematician, whose name is usually associated with light and heat radiation. Lambert's generalised statement is as follows: "layers of equal thickness of isotropic medium absorb equal *fractions* of the radiation intensity which is incident upon them." Suppose that the medium is divided into a series of parallel slices of unit thickness, then if I_0 is the intensity incident entering the boundary face the intensity I_1 entering the second slice will be some fraction, say β, of I_0, *i.e.* $I_1=\beta I_0$. This result follows from Lambert's law and the fact that no reflection will occur at these "layer boundaries" within the homogeneous medium. Similarly, it follows that $I_2=\beta I_1=\beta^2 I_0$, etc., and in general $I_x=\beta^x I_0$, where x is the distance travelled by the wave motion into the absorbing medium of *transmission* coefficient β. Also $(I_0-I_1)=(1-\beta)I_0=\alpha I_0$.*

Consider, now, two parallel planes within the medium Δx apart and let I and $I+\Delta I$ be the intensity of the radiation crossing the two planes at x and $x+\Delta x$ respectively. It follows from the foregoing

* α is the absorption coefficient per unit length.

that $-\frac{\Delta I}{I}=\alpha\Delta x$ or $\frac{dI}{I}=-\alpha dx$ as Δx becomes infinitesimal, where α is a constant known as the *absorption* coefficient per unit length. Integrating this expression to obtain the value I_x of the intensity after the wave has travelled a distance x,

$$\int_{I_0}^{I_x}\frac{dI}{I}=-\int_0^x \alpha dx$$

i.e.

$$I_x=I_0e^{-\alpha x}$$

whence

$$\alpha=\frac{1}{x}\log_e\frac{I_0}{I_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The *amplitude* coefficient of absorption per unit length will be given by $\alpha/2$, energy being proportional to (amplitude)^2. Hence $A_x=A_0e^{-\frac{\alpha x}{2}}$, where the A's refer to amplitudes.

Another absorption coefficient (μ) gives the absorption per wavelength of the radiation in the medium and it follows that $I_x=I_0e^{-\frac{\mu}{\lambda}x}$, since $\mu=\alpha\lambda$.

An alternative aspect of μ is seen if cT is substituted for λ, where c refers to the velocity and T to the period of the wave motion within the medium.

It is evident that

$$\frac{dI_x}{dt}=\frac{dI_x}{dx}\cdot\frac{dx}{dt}$$

$$=-\frac{\mu I_x}{\lambda}.c$$

$$=-\frac{\mu}{T}.I_x$$

So for a given particle, *i.e.* at *constant x*,

$$\int_{\text{cycle}} dI_x=-\frac{\mu I_x}{T}\int_0^T dt=-\mu I_x,$$

and hence μ has the significance of being equal to the fraction absorbed of the total vibrational energy per cycle of a vibrating particle.

In sound absorption the value of α is determined by two factors: (*a*) the internal friction of, and (*b*) the thermal conductivity of, the sound conducting medium. The latter effect arises from the fact that the compression regions of the sound wave will be at a higher temperature than the expansion regions and also the surroundings, and so there will be a tendency for heat to be conducted away from the regions of compression. The conduction effect in liquids is negligible compared with the absorption due to viscosity, but in gases it becomes of the same order of magnitude. The total absorption coefficient, according to classical theory, increases with the square of the frequency of the waves, so that propagation becomes more difficult as the ultrasonic region is approached. At these high frequencies water provides a much better medium than air for the propagation of signals, its absorption coefficient being 1/1700th of that for air at the same frequency.

Acoustics of auditoria

Desirable conditions in an auditorium depend on—

(*a*) The shape of the enclosure. This should enable the sound to be projected uniformly towards the audience without undue effort on the part of the performer or speaker, and yet remain audible.

(*b*) Absence of echoes. These depend on reflection from surfaces over about 60 ft. away. Such reflections must be reduced to negligible proportions by diminishing the reflecting power of the surfaces in question.

(*c*) The duration of reverberation. This depends on the size of the hall, and also on the purpose for which the hall is intended, a longer period being desirable for orchestral music than for speech, and longer still for choral music.

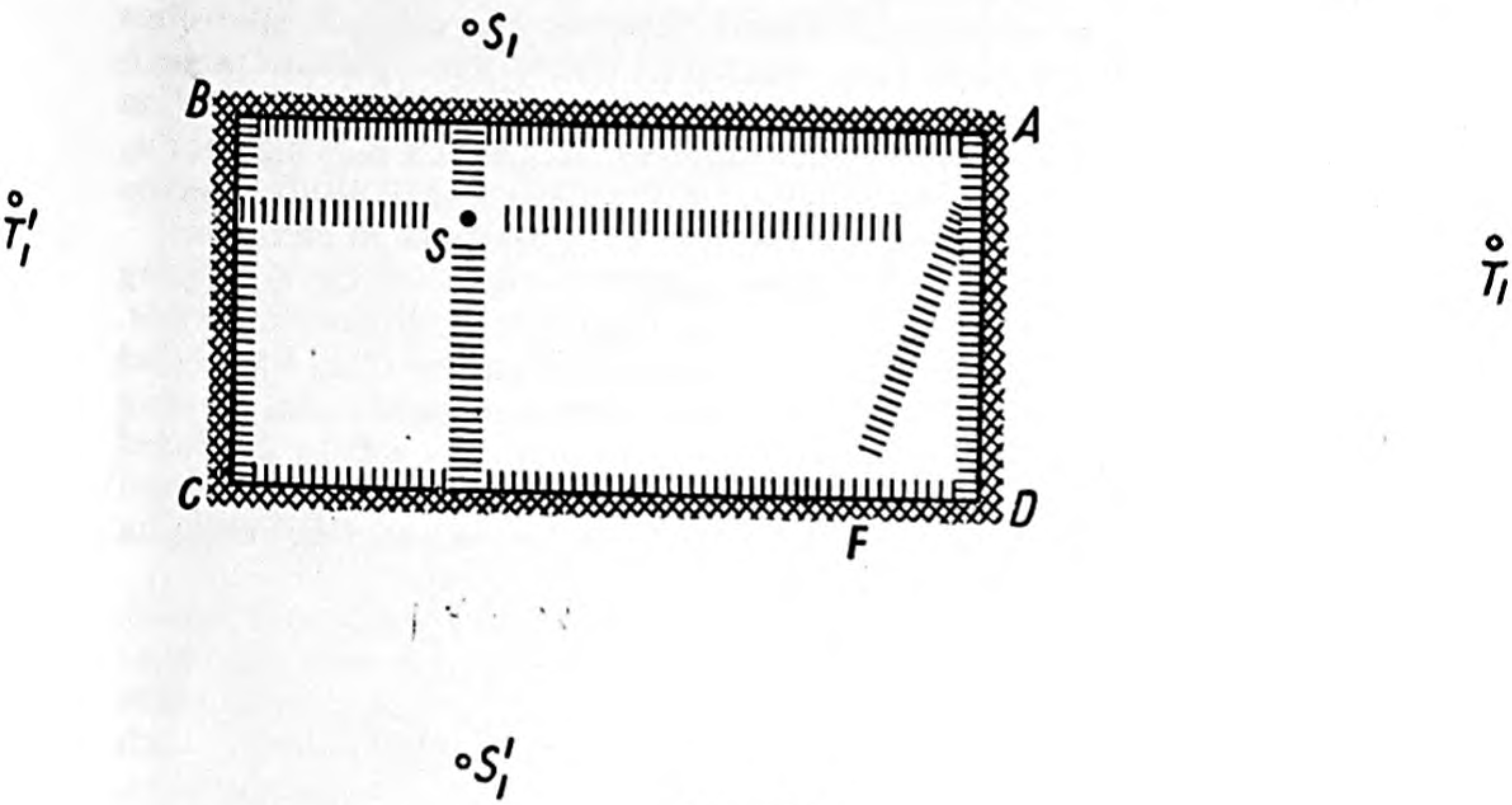


Fig. 14.1.

Behaviour of sound in a rectangular hall

For simplicity the hall will be assumed to have hard walls and to be unfurnished. Fig. 14.1 represents the horizontal section of the room at the level of the source of sound S, which is of constant intensity and frequency.

The hard walls act as plane reflectors, and primary images S_1, T_1, S_1', T_1' are formed by reflection in the faces AB, DA, CD and BC respectively. In turn, these primary images give rise to secondary images S_2, T_2, etc. Treating these as sources which are in phase, and considering them in pairs, S_1 and S give rise to (*a*) standing waves in the line S_1 S, and (*b*) progressive waves along the wall AB. S_1' and S act in the same way with respect to the wall CD. Further, S_2' and S_1' have a similar effect along AB. Images T_1 and T_1' behave in the same manner with respect to the other walls. Matters are further complicated by reflections at the floor and ceiling, and by the presence

of tertiary, etc., images. Yet another series of reinforcements in certain directions occur due to the joint effects of images such as S_1 and T_1. This pair is responsible for projecting sound in the direction AF; note that the angles BAS and DAF are equal and AF is on the perpendicular bisector of S_1AT_1. Fortunately there is no perfect reflector, subsequently the sound will be absorbed, the time taken depending on the nature of the walls. Ordinary 9-inch brick walls absorb about 2 per cent. and reflect about 98 per cent. of the incident sound, an infinitesimal amount being transmitted *through* the *wall.* Interference between the virtual sources causes zones of diminished intensity, so that in such a room of small absorption a sound pattern is built up and remains so long as the sound persists. A sound of a different frequency will cause a different pattern, so that, if sounds of two different frequencies are generated simultaneously at a particular point, a listener in the hall will receive sound from two superimposed sound patterns, the relative intensities depending on his position with respect to the source. Music in such circumstances is unsatisfactory. It should be noted that the position of the source determines the actual patterns, and that a moving source, *e.g.* an opera singer, produces moving patterns which may result, at given points in the hall, in intelligibility which is intermittent in character.

Suppose now that the source of sound is replaced by a tapping device which can be adjusted to give a loud tap at distinct intervals. If the hall is large and the walls are good reflectors, then the sound of each tap is reflected back as a series of distinct echoes, and if tapping occurs two or three times in a second the actual taps will be confused by the echoes. Similarly, the words of a speaker may be reflected back sufficiently late to mask words which follow, thereby creating conditions of unintelligibility.

Clearly, such a hall is unsatisfactory both for speech and music, and the reflections must be modified. The reflection from a particular wall can be eliminated by removing the wall, which is impracticable and undesirable from an acoustical standpoint. In practice such walls are covered by sound absorbing material, *e.g.* heavy curtains, which absorb some 30 per cent. of the incident sound, reflect 5 per cent. and transmit the remaining 65 per cent. As stated above, some 98 per cent. of the latter is reflected at the wall, and is again diminished by the curtains before reaching the audience. The decreased intensity of the reflected sound may cause it to be almost inaudible; in any case, subsequent reflections are probably inaudible. Concave walls and domed ceilings tend to concentrate sound at certain places in a hall, and this reflected concentration may exceed the direct sound at these points, a condition which is highly undesirable in an auditorium. Such walls and ceilings are to be avoided in design when possible. The effect is eliminated in an existing building by covering the walls with sound absorbent materials (Fig. 14.2), and by screening the dome with a horizontal curtain or velarium.

Reverberation

The intensity of sound in the open air at a particular point depends on the amplitude of the sound waves at that point. This amplitude

diminishes with distance, so that at a particular distance from the source the sound becomes inaudible. Thus, a monosyllabic word of command, *e.g.* "Halt!" becomes inaudible beyond a certain range. If, however, such a command is given in an enclosure, a portion of the sound is reflected by the walls. Superposition of such reflections causes the sound to be of such an amplitude, and therefore of such an intensity, as to prolong the audibility; this gives rise to the effect termed *reverberation*. The actual duration in the case of a source of given intensity depends on the absorption at the walls, but it is necessary to give this period of time a qualitative significance, and, for a reason given below, it is defined as the time required, after the source has been silenced, for the intensity of the sound to drop to one-millionth of its original value, *i.e.* by 60 db. It is termed the *reverberation time*.

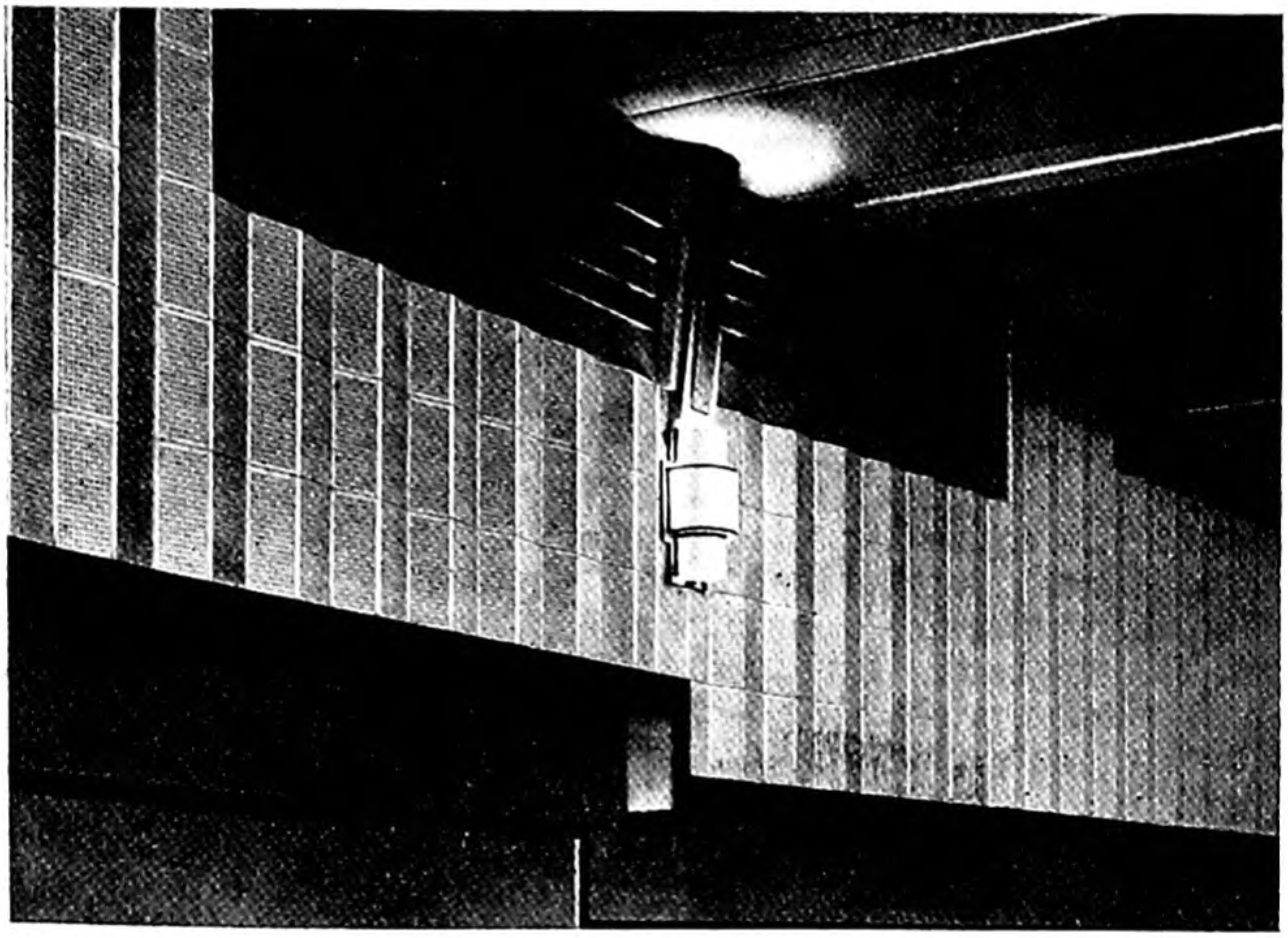


Fig. 14.2. Acousti-celotex (fibrous composition) tiles on walls of a cinema.

Reverberation is desirable for reasons to be discussed later, but its duration must not be excessive. The reverberation times for halls judged by trained musicians to be good have been plotted against the cube-roots of the volumes. It was found that the points lay in a region bounded by two lines which were practically straight, and further, that speech required a shorter reverberation time than orchestral music, whereas choral music required a longer time. A formula, adapted from these results but hitherto unpublished, and which gives the actual time T in seconds, to within one or two per cent., is

$$T=(0{\cdot}0036V^{\frac{1}{3}}+{\cdot}107)r,$$

in which V is the volume of the auditorium in cubic feet, and r assumes the values 4, 5 and 6 for speech, orchestra and choir respectively.

Sabine, in the course of his earlier work, used an organ pipe as a source of sound, and measured the time for the sound to become inaudible after the pipe had been silenced. This was repeated many times, but with different numbers of identical cushions placed on the floor, and it was found that the time of reverberation depended on the number of cushions used. In this way it was possible to find the number of cushions required to diminish the reverberation of the room by a known amount, hence the effect of the walls, ceiling, etc., of the *empty* room could be stated in terms of exposed area of cushion. This led to the expression

$$AT = \text{constant},$$

which expresses the relationship between the equivalent absorbing area A of the room in terms of cushion area, including the exposed area of any cushions present, and T the reverberation time. This expression, however, is applicable only to this particular room with particular cushions, and for a note of fixed frequency (512 c.p.s.) from a particular organ pipe. It was generalised thus: the organ pipe was tested and found to produce an intensity which was approximately 10^6 times that of minimum audibility, and, as stated above, this ratio has since been accepted as the basis for defining reverberation time. The effect of opening windows on the value of T was compared with the effect of the cushions, and enabled the latter to be expressed in terms of open window area, the latter being perfectly absorbent *for all frequencies* if its dimensions are sufficiently large compared with the wave-length of the sound. This reservation is necessary as otherwise diffraction effects will render a small window, by comparison, a more efficient absorber than a large window.

The absorption due to an audience was measured and reduced to open-window units. By experimenting in different rooms it was found that the product AT, A now denoting the total equivalent area in open-window units (O.W.U.), was proportional to the volume, i.e. $\frac{AT}{V}$ is constant, and is equal to 0·05 when A is in square feet of open window, and V is in cubic feet. This constant is 0·16 when the unit of length is the metre; *i.e.* the reverberation time in seconds, T equals $0{\cdot}05\frac{V}{A}$ in foot units, and $0{\cdot}16\frac{V}{A}$ in metre units. It has been noted already that the acoustic absorption of unit area of most materials depends upon the frequency of the incident sound, hence, from this fact alone it is to be expected that the reverberation time will depend upon frequency.

Experimental investigation of reverberation

The original method of Sabine, described previously, although much improved in sensitivity by automatic timing, did not give an indication of the time variation of sound intensity. An improvement on this procedure involved the use of a microphone to pick up the sound, and the alternating voltage developed was applied to an amplifier. The latter incorporated a potentiometer which was automatically regulated to maintain a constant output voltage from the amplifier, for varying

input voltages. The mechanical recording apparatus at the same time operated a tracing needle on a recording drum. The sensitivity of this arrangement to brief fluctuations of intensity was limited by the mechanical inertia of the moving parts of the recording apparatus. With this type of apparatus, decay rates up to 600 db. per sec. have been recorded, and it has the advantage of the speed being adjustable so that any desired degree of smoothing of the decay curve can be realised and in this way only the major fluctuations may be shown. The cathode-ray oscillograph is a recorder of extremely small inertia and consequently is ideal for exhibiting brief and small fluctuations in intensity and frequency. A particular method in which it was employed by Van Urk involved the use of an exponential amplifier, *i.e.* an amplifier whose output voltage, for a given applied input voltage,

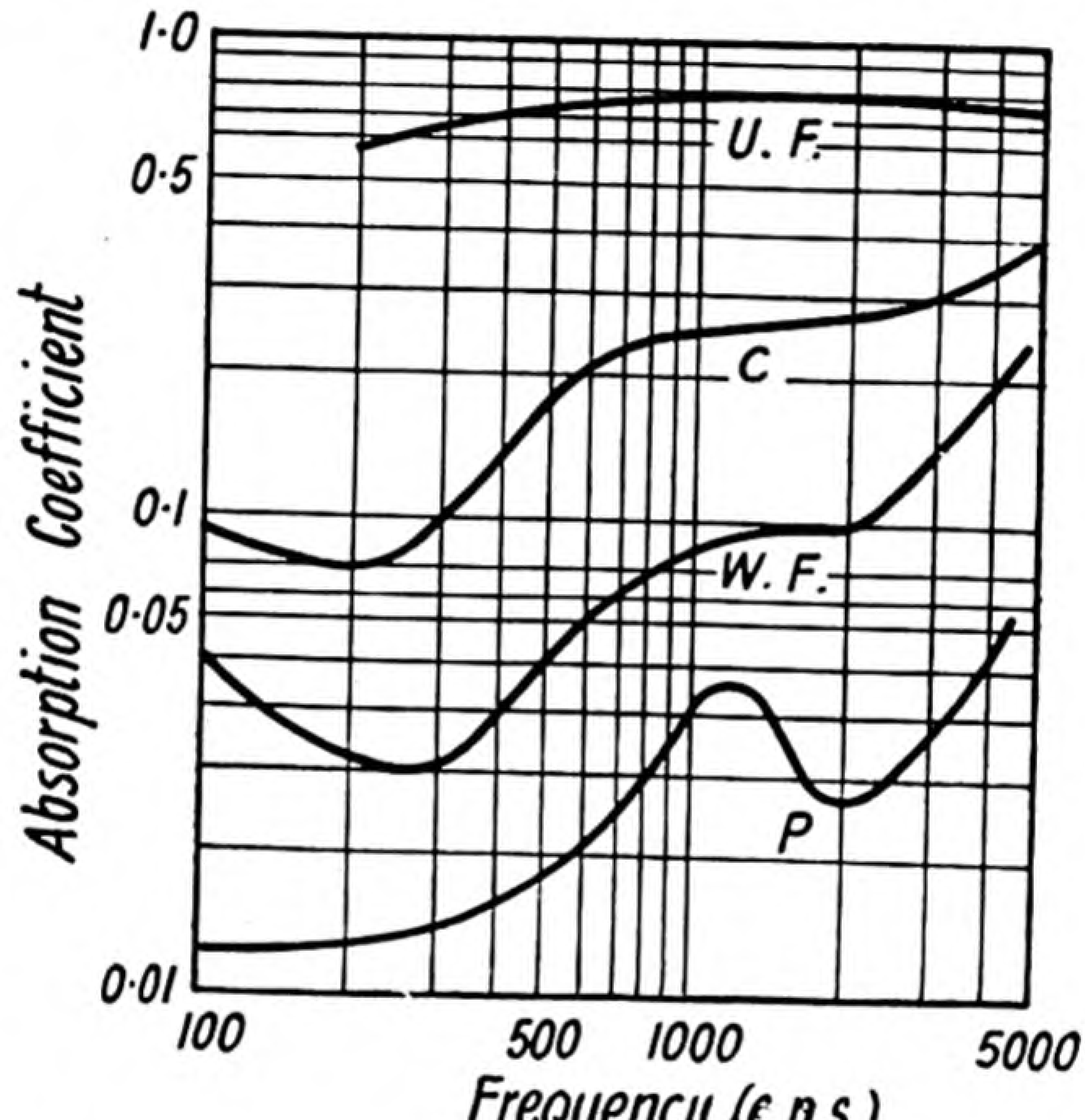


Fig. 14.3. *U.F.*: Upholstered Furniture. *C*: Carpet. *W.F.*: Wooden Furniture. *P*: Plaster.

increased exponentially with time according to the law $e^{+\beta t}$. If α is the exponential decay constant of the sound in the auditorium due to reverberation and β is adjusted to be equal to α, then it follows that on shutting-off the source of sound and simultaneously applying the output voltage of a microphone to the amplifier, the latter should give a constant output voltage during the time of decay. This output voltage is applied to the vertical deflection plates of the oscillograph, and if the above condition has been attained a small fluctuation about a horizontal trace should be observed on the screen. The screen image will appear steady and permanent if the time-base is synchronised with the apparatus regulating the continuously repeated switching on and off of the sound source. In more complicated cases of decay, where more than one decay constant is involved, it will be necessary to

vary β over a range of values, at the same time noting whenever the screen exhibits a horizontal line over a restricted region. It is evident that, by using Sabine's or Eyring's formula, the measurement of reverberation time affords a means of measuring the absorption coefficients of acoustic materials.

Coefficient of sound absorption of a substance

This may be defined as the area of open window which has the same absorption for sound as unit area of the substance. If it is large compared with the wave-length of the sound an open window is 100 per cent. absorbent for all frequencies, so the unit of absorption is taken as that of 1 sq. ft. of open window, and this has been given the name of one Sabine. The statement that a thick carpet has an absorption coefficient of 0·25, means it is equivalent to an area of open window one-fourth of its area. As the coefficient varies with frequency (see Fig. 14.3), the value when unspecified is usually that obtained at 500 c.p.s.

The following table shows the values for a few substances at different frequencies:—

Material	250 c.p.s.	500 c.p.s.	1000-2000 c.p.s.
Brick	0·03	0·03	0·05
Axminster carpet ¼″ thick	0·05	0·10	0·35
Do., on felt ¼″ thick ..	0·05	0·40	0·65
Turkey carpet ½″ thick ..	0·1	0·25	0·30
Do., on felt ¼″ thick ..	0·3	0·5	0·65
Sprayed Asbestos 1″ thick	0·35	0·7	0·7
Thick hair-filled cushion ..	0·4	0·7	0·55
One Adult in audience is, in square feet of O.W.U., or Sabines, equal to ..	4·3	4·7	5·0

By suitably curtaining and carpeting a room, the reverberation time can be modified as required. Where this is impracticable, *e.g.* in a vaulted roof or on pillars, asbestos is often sprayed in the form of a fibrous powder from one spray gun, while a second gun simultaneously sprays an adhesive to cause the asbestos to adhere to the surface. Such coatings can be made to any thickness without perceptibly changing the architectural features. Moreover, it is possible to decorate the surface of such an absorbent without altering the absorption to any marked extent.

A further point which is dealt with on p. 298, but which must be noted here, is that on the average one instrument is required to provide the sound energy for 200 units of absorption; thus there is, in a particular room, an optimum number of instruments in the orchestra. A smaller number causes the music to lack "body," while a greater number tends to make it overpowering.

The following example may serve to illustrate several of the points mentioned in this chapter:—

A rectangular hall has a height of 20 ft. and a floor measuring 68 ft. $\times$ 100 ft. The hall is full at one person per 200 c. ft. Calculate (*a*) the

most suitable reverberation time for orchestral music; (*b*) the absorption in O.W.U.; (*c*) the absorption due to the audience; (*d*) the mean coefficient of absorption of the walls, etc.; (*e*) the optimum number of instruments.

Solution—

(*a*) The hall has a volume of 136,000 c. ft. Substituting this in the empirical formula $t=[0{\cdot}0036\sqrt[3]{V}+0{\cdot}107]5$, $t=\mathbf{1{\cdot}5}$ **sec.**

(*b*) Substituting this in Sabine's formula, $t=0{\cdot}05\frac{V}{A}$, gives $A=\mathbf{4700}$ sq. ft. of open window.

(*c*) Number of people$=\frac{136000}{200}=680$;

$\therefore$ absorption$=680\times 4{\cdot}7=\mathbf{3200}$ sq. ft. of open window.

(*d*) Absorption due to walls, etc.$=4700-3200=1500$ sq. ft. O.W.

$$\therefore \text{ Mean coefficient of absorption}=\frac{\text{absorption due to walls, etc.}}{\text{area of walls, etc.}}$$

$$=\frac{1500}{20300}=\mathbf{0{\cdot}07}.$$

(*e*) Average absorption per instrument$=200$ O.W.U.

$$\text{Optimum number}=\frac{4700}{200}=\mathbf{23}.$$

It should be remembered that the value of A in Sabine's formula is based on a frequency of 512 c.p.s., so the coefficients selected should be at or near this value. Further, as the reverberation time varies with frequency, calculated values are to be regarded as approximate. In fact, Eyring has shown that, for surfaces with large absorption coefficients, the expression

$$T=\frac{0{\cdot}05V}{(-S\log_e\overline{1-a})}$$

is more satisfactory, S being the total area in square feet of the exposed surfaces of the room, and a the average absorption coefficient, a being $=\frac{A}{S}$. When a is small (less than 0·2) the expression may be replaced by Sabine's formula (see p. 296).

Articulation and intelligibility

A factor of supreme importance in speech in a room is intelligibility, and this is governed, in addition to the positions of the speaker and listener as already indicated, by the rapidity of speech, the intensity of the sound and its pitch, noise both inside and outside, and by reverberation characteristic of the room. Recorded speech and telephone conversation also tend to become unintelligible due to distortion by microphones and other causes. The subject has been investigated in what are known as articulation tests by Harvey Fletcher and others, much of the work being carried out at the Bell Telephone Laboratories.

In these tests a listener is required to record words which are spoken at a fixed rate into a telephone mouthpiece. Earlier tests had shown that, although certain words in a sentence may be inarticulate, the meaning of the sentence can often be inferred from the remaining words, and so become intelligible. This possibility is eliminated by using

meaningless monosyllabic words such as zut, wa, mis, kev. These are spoken at a fixed rate, usually three per second. The percentage of words correctly recorded is termed the articulation, and a mean value is obtained for several speakers and listeners. Such words are arranged to reproduce the vowel and other sounds in the proportion that they occur in the language in which intelligibility tests are to be made.

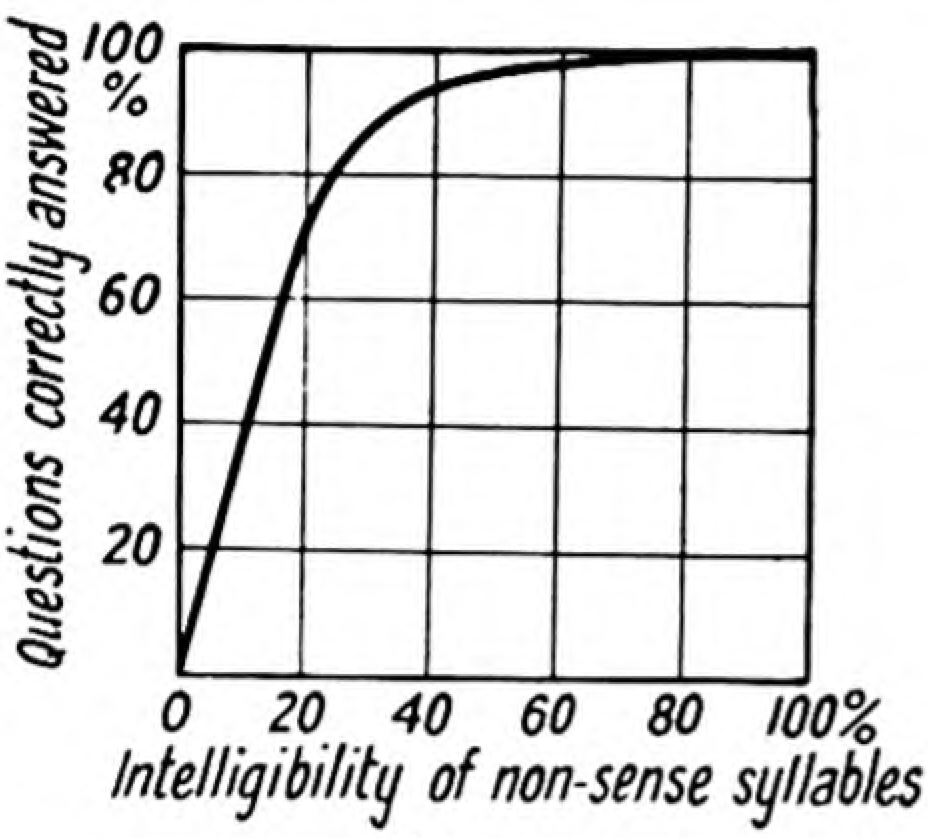


Fig. 14.4.

The non-sense syllables of the articulation tests are replaced by questions in the intelligibility tests, otherwise the procedure is similar. Intelligibility is judged by the percentage of questions correctly answered. These questions are formulated to use a variety of representative words, examples being:

Name some use to which electricity is put.

Explain why a corked bottle floats.

Why are books bound in stiff covers?

The results of these tests indicate that the percentage intelligibility is about four times the corresponding value of the articulation when the latter is poor, but that the ratio diminishes to unity as the intelligibility approaches 100 per cent. The curve (Fig. 14.4) shows that 50 per cent. articulation is sufficient to carry on an intelligent conversation, although 70 per cent. articulation is regarded by telephone engineers as the lower limit of satisfactory transmission. Articulation can be varied by altering the intensity of the sound. It is not sufficiently recognised that articulation is also dependent on the language employed.

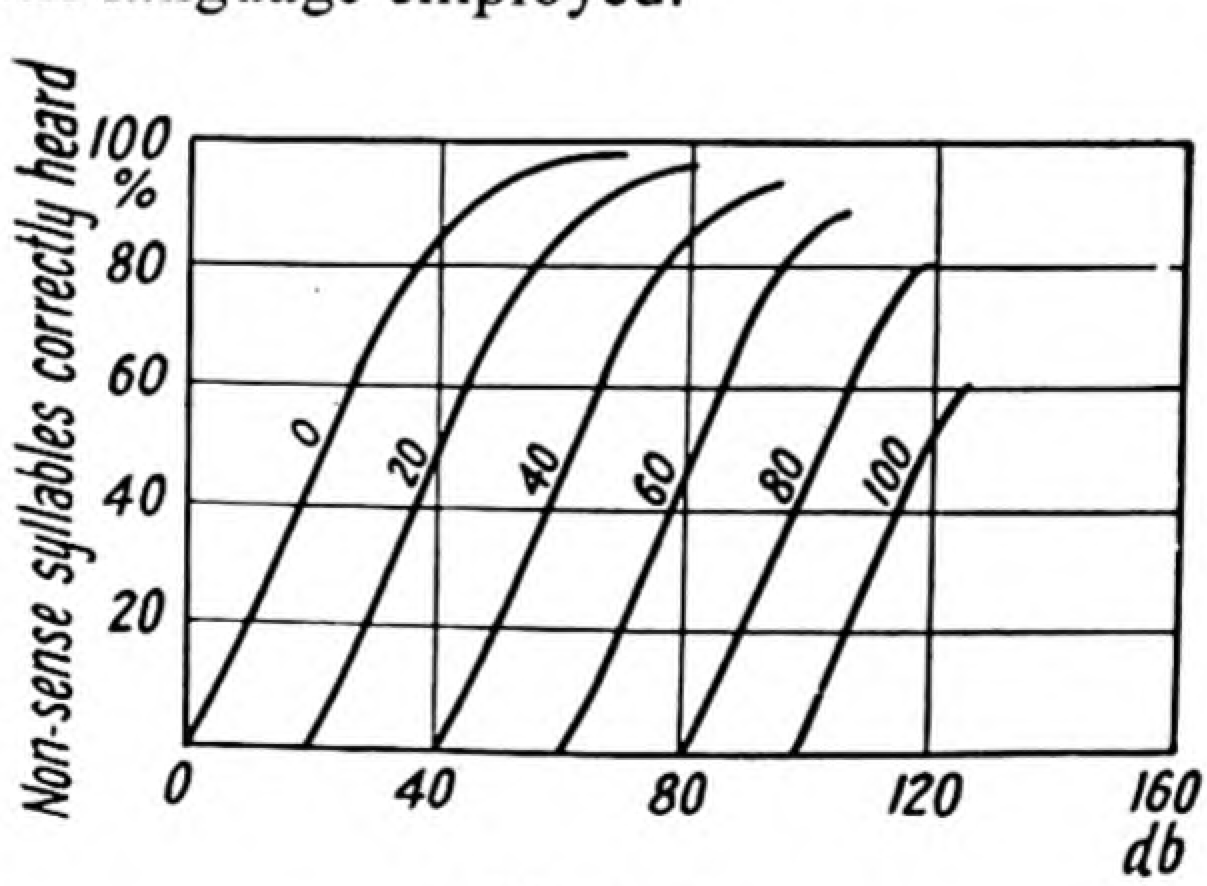


Fig. 14.5.

When background noises are imposed on the non-sense syllables some interesting results are obtained from the graphs connecting articulation and intensity for different background intensities. Fig. 14.5, which is after Fletcher, shows that the curves are practically identical, but are separated by distances corresponding to the differences in noise intensity. This is clearly seen on the 20 per cent., 40 per cent.,

60 per cent., and other articulation levels. The curve marked 0 was obtained from tests in a room without noise. When the test syllables are just audible (threshold of audibility), the articulation is zero. Background noise of an intensity 20 db. raises the intensity required for the threshold of audibility to 20 db., so that the effect of the noise is to neutralise the desired sound when their intensities are equal. The graph shows that this is also the case when the noise and sound intensities are unequal, up to very large intensities. For a 50 per cent. articulation a 22 db. difference between background noise and intensity of speech is necessary.

The effect of the presence of overtones on articulation has been tested by filtering out certain frequencies. In one series of experiments of this nature, first the high and then the low frequencies were progressively removed, and the articulation measured. The results, which are shown graphically in Fig. 14.6, indicated that the region 1000-2000 c.p.s. is most important for intelligibility. The fact that the

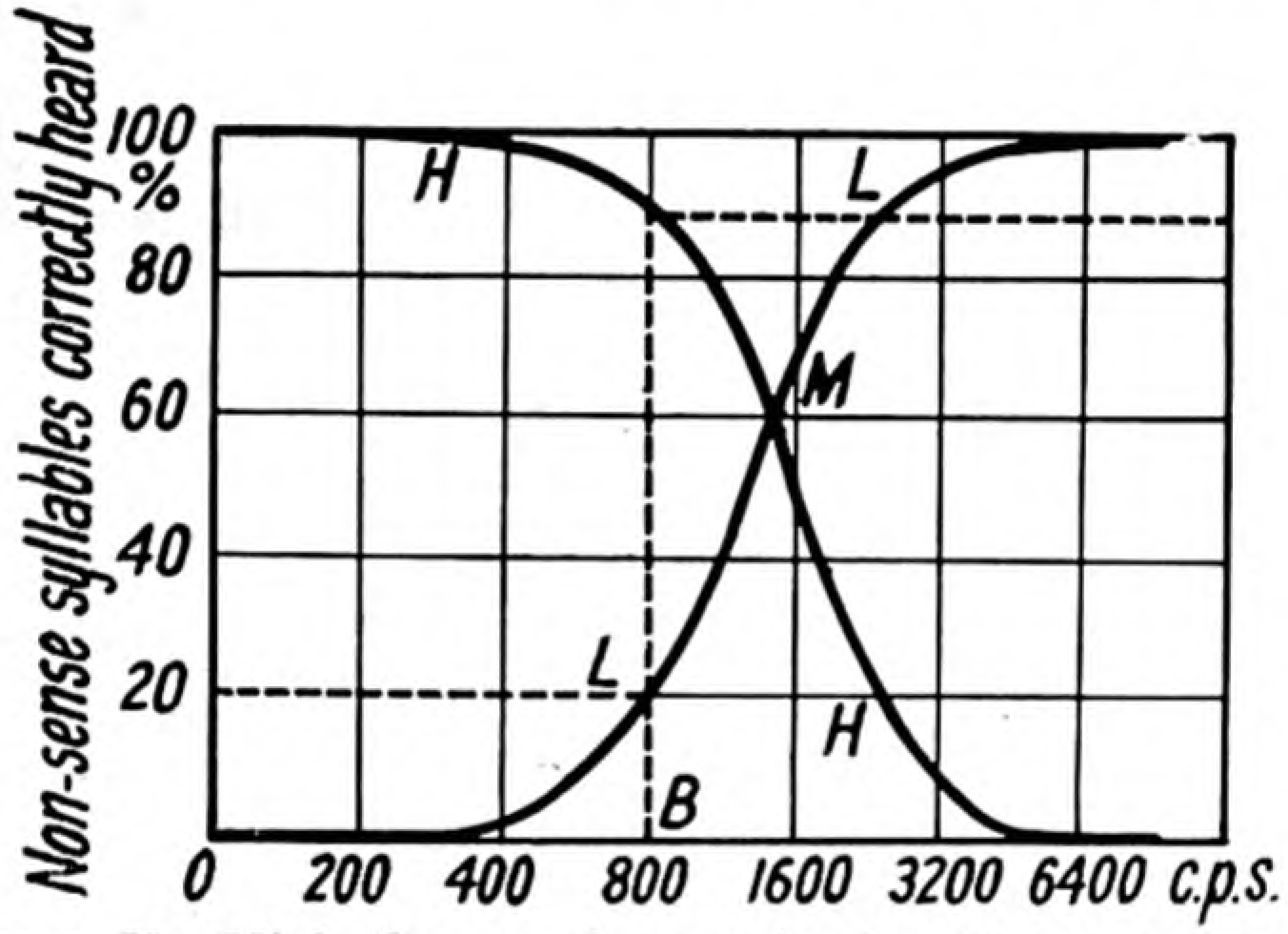


Fig. 14.6. Curve H: High frequencies retained. Curve L: Low frequencies retained. M: Point of equal intelligibility. The dotted horizontal lines indicate range of frequencies retained.

curves intersect at a frequency of about 1500 c.p.s. implies that the ranges of frequencies above and below this value are equally effective for articulation. Further, the articulation at this point is 60 per cent., and this means practically 100 per cent. intelligibility (see Fig. 14.4), in spite of the obvious change in the quality of the voice. Clearly, these facts must be taken into account in the design of loud-speakers for speech, deaf-aids and other such devices, with special attention to the 1000-2000 c.p.s. range of frequencies.

Attention is directed to the fact that, for a frequency limit of 800 c.p.s. the articulation is 20 per cent. when the lower frequencies remain, whereas retention of the higher frequencies gives an articulation of 95 per cent. Only by including all frequencies up to about 2300 c.p.s. is it possible to achieve this degree of articulation when the higher frequencies are extracted; by removing frequencies below this when the upper ones are present, the articulation is reduced to about 20 per cent.

In connection with these tests, experiments were made in which that portion of the sound with frequencies above 1500 c.p.s. was conducted into one ear, and that below that frequency into the other. The results showed that, with speech, the normal voice was heard, whereas music was heard as a jangle of sound. It therefore appears that for any other separating frequency than 1500 c.p.s. the two-filtered frequency ranges would not produce the effect of the original voice. The experiments confirm Paget's view that intelligibility depends on the presence of the high frequencies, the low frequencies (proper to the vocal cords) acting in the manner of a carrier wave.

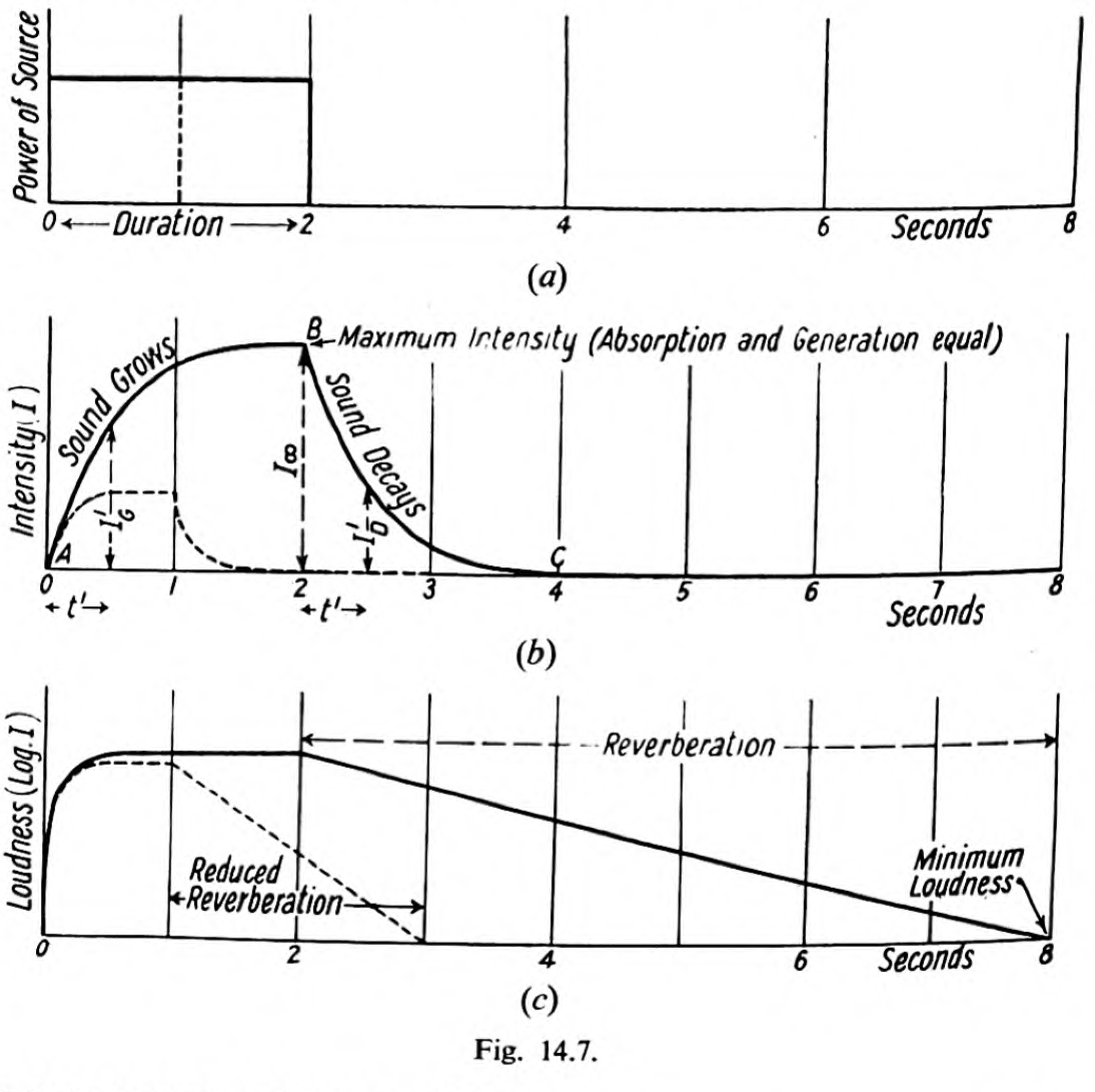


Fig. 14.7.

Effect of reverberation on intelligibility

The overlapping effect produced in an auditorium by a succession of discrete sounds or taps was discussed above, and will now be examined in greater detail.

When a source of sound of uniform power is switched on in a room it "builds up" until the acoustic energy is being absorbed at the rate at which it is generated. This is represented in Fig. 14.7 by the portion of the full curve *AB*. When the source is switched off the sound dies away, decaying as shown by *BC*. The formulae for growth and decay being respectively $I_G=I_\infty(1-e^{-at})$ and $I_D=I_\infty . e^{-at_1}$, in which I_∞ denotes the

steady intensity, I_G the intensity during growth after time t sec., and I_D the intensity t_1 sec. after switching off. The growth and decay curves are complementary for, after an interval of t' sec. from switching on, the ordinate is represented by I_G'. After the same interval from switching off, the ordinate is I_D'; $I_D'+I_G'=I_\infty$ for all values of t'. The complementary nature is readily appreciated by considering the switching off as the switching on of an identical "negative" source, *i.e.* the rate of growth, whether negative or positive, is constant.

The reverberation of the sound represented by the curve ABC has a duration of 6 sec. This is shown clearly by the full curve of 14·7c, which also indicates why the *loudness* approaches its maximum more quickly and dies away more slowly than the intensity-time curve appears to show.

The dotted curve of 14.7b shows the effect of reducing the reverberation to 2 sec. by increasing the absorption. The maximum intensity reached is some 4–5 db. less than before, although the source is of the same power as

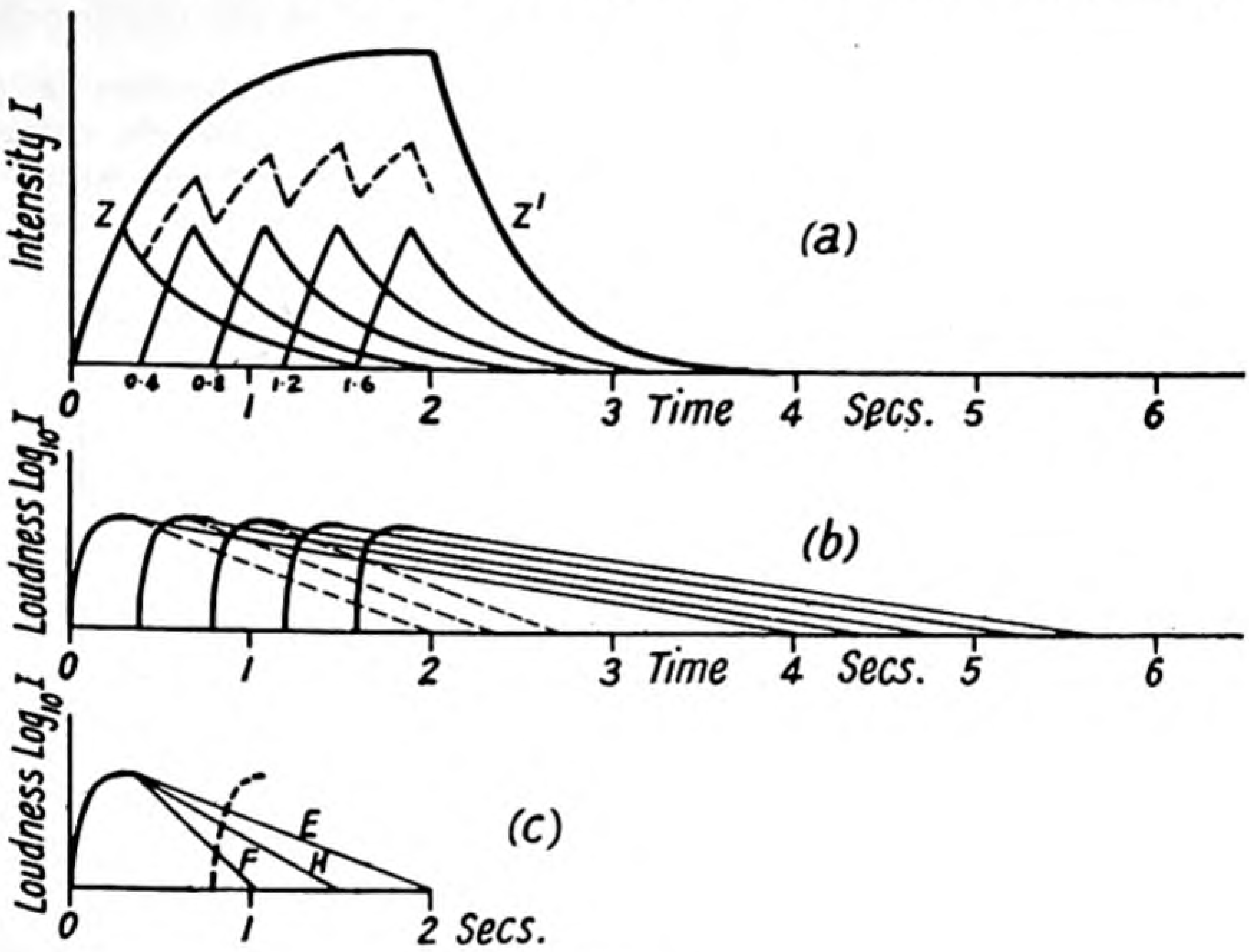


Fig. 14.8.

at first, but the dotted curve of 14.7c shows that the loudness is nearly the same in both cases.

The next diagram, Fig. 14.8a, indicates the effect of speaking monosyllabic words at intervals of 0·4 sec., the duration of each sound being 0·3 sec. followed by a 0·1 sec. interval; the time of reverberation is 4 sec. The intensity curve follows the complete one to the point Z, and then falls away as the complete curve does after the point Z'. This is repeated every 0·4 sec. The total intensity follows the dotted path, whose peaks indicate the extent to which the individual syllables stand out. The corresponding loudness curves of the syllables are shown in Fig. 14.8b, in which the dotted curves show the effect on loudness when the reverberation is reduced to 2 sec. Fig. 14·8c shows the effect of the presence of an audience on the reverberation of the first word. The loudness of a later syllable—the third for convenience—which is shown dotted, is included to indicate the effect of the full (F), half-full (H) and empty hall (E), on the articulation. Clearly, the interference is reduced by the presence of the audience.

Intelligibility may be increased, then, by: (*a*) Reducing the reverberation: this method has its limitations, for as open-air conditions are approached the auditorium becomes "dead."

(*b*) Speaking more slowly: this allows the peaks of Fig. 14·8*a* to stand out well above the background noise.

It appears that intelligibility could be improved by increasing the intensity until the direct sound and the background noise due to echoes differ by 22 db., as mentioned on p. 291. Actually, however, the intensity of each of the echoes would increase proportionally, and as the decibel depends on the *ratio* of intensities, no improvement would result. Probably conditions would become worse, as the duration of reverberation and also the rates of growth and decay would increase.

"Dead-spots," however, are still likely to exist, and can be treated by using directional loud-speakers. These should be of low intensity, near to but higher than the dead-spot concerned, and directed towards it. The absorption of the sound by the audience would tend to eliminate reflection.

Influence of shape on intelligibility. Consider an auditorium in section (Fig. 14.9) in which S is the position of the source. A listener situated at B will receive direct sound along the path SB, and reflected sound along paths such as SA, SB. If the path difference $SA+AB-SB$ is large, as in

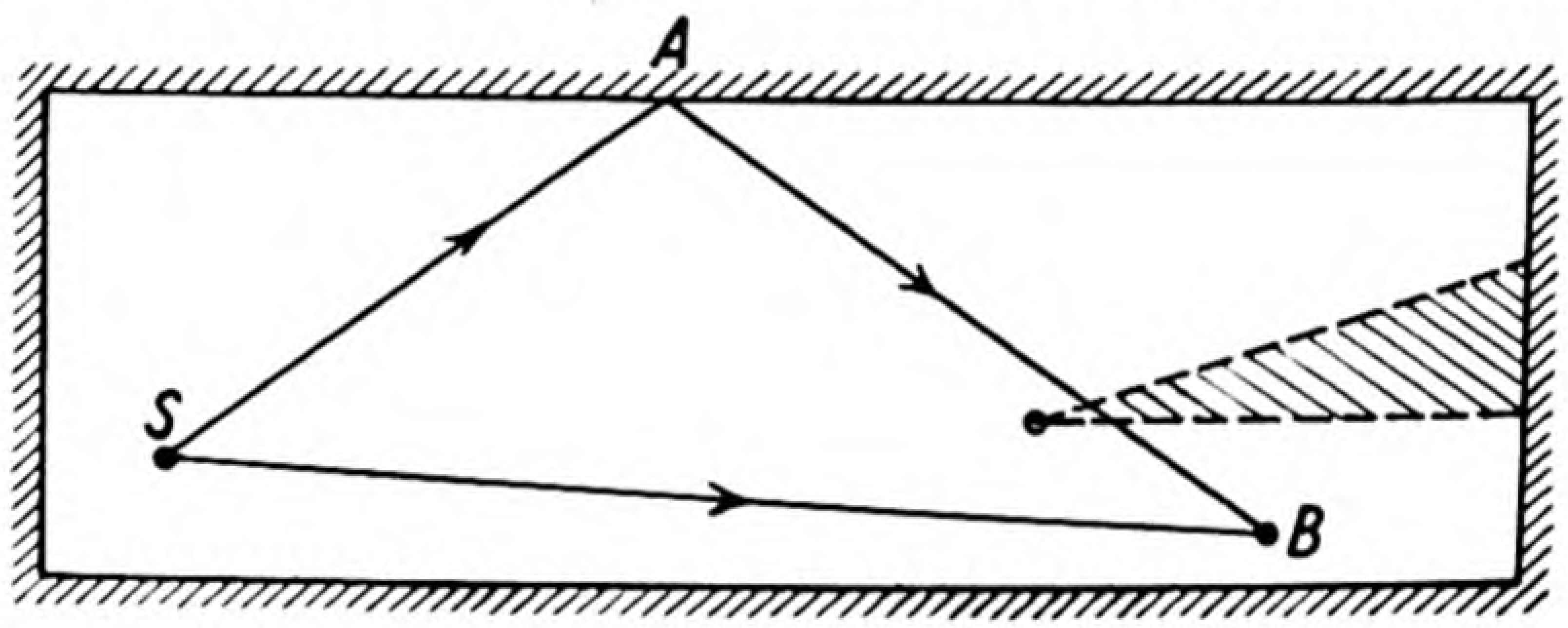


Fig. 14.9.

a lofty building, direct and reflected sound will arrive at B at appreciably different times, but with a lower ceiling the direct and reflected sounds will arrive at B practically simultaneously, so that intelligibility is maintained. Alternatively, the reflected ray may be intercepted by a gallery at the position shown dotted in the figure. Further reference to this subject is on p. 303.

Theoretical derivation of Sabine's formula

It is first necessary to obtain an expression for the rate at which sound energy is dissipated in a room by absorption at the walls and to use it to determine the rates of growth and of decay of sound.

Assumptions

(1) The energy density of sound is uniform throughout the room; it is defined as the average vibrational energy associated with the air particles in unit volume of the enclosure, due to the passage of sound waves.

(2) Energy is transmitted equally in all directions.

(3) The source maintains a constant supply of energy during emission.

(4) Effects of superposition may be neglected.

(5) Dissipation of energy is confined to the bounding surfaces, *i.e.* the attenuation due to the viscosity of the air is negligible.

(6) The coefficients of absorption are independent of frequency.

(*a*) *Energy flux per unit area.* Fig. 14.10 represents an element of area δS of the surface of a wall of the room in which the sound energy is uniformly dense, perfectly diffuse, and of magnitude I' per unit volume. The quantity of energy at any instant in a small volume δV is $I'.\delta V$, and that portion which will ultimately arrive at δS from an element of δV will be moving within the solid angle $\delta\omega$ given by $\delta\omega = \dfrac{\delta S \cos\theta}{r^2}$.

Assuming energy to flow from δV uniformly and in all directions, the quantity which will arrive at δS is

$$I'.\delta V.\frac{\frac{\delta S\cos\theta}{r^2}}{4\pi} = \frac{I'.\delta V.\delta S\cos\theta}{4\pi r^2}.$$

The other elements of volume similarly situated with respect to δS constitute a torus of cross-sectional area $r.\delta\theta.\delta r$ and of circumference $2\pi r\sin\theta$, hence the total volume of the elements is that of the torus, which may be written $\delta V = 2\pi r^2 \sin\theta . dr . d\theta$.

The total energy received by δS in an interval δt comes from a hemisphere of radius $c.\delta t$, c being the velocity of sound in air; $\therefore$ the total energy arriving in time δt on the area δS

$$= \Sigma \frac{I'\delta V.\delta S\cos\theta}{4\pi r^2}$$

$$= \iint \frac{I' 2\pi r^2 \sin\theta . dr . d\theta . \delta S . \cos\theta}{4\pi r^2}$$

$$= \frac{I'\delta S}{2}\int_0^{c.\delta t} dr \int_0^{\frac{\pi}{2}} \sin\theta . \cos\theta . d\theta$$

$$= \frac{I'\delta S}{2}.c\,\delta t.\frac{1}{2}$$

$$= \frac{I'c}{4}.\delta S\,\delta t;$$

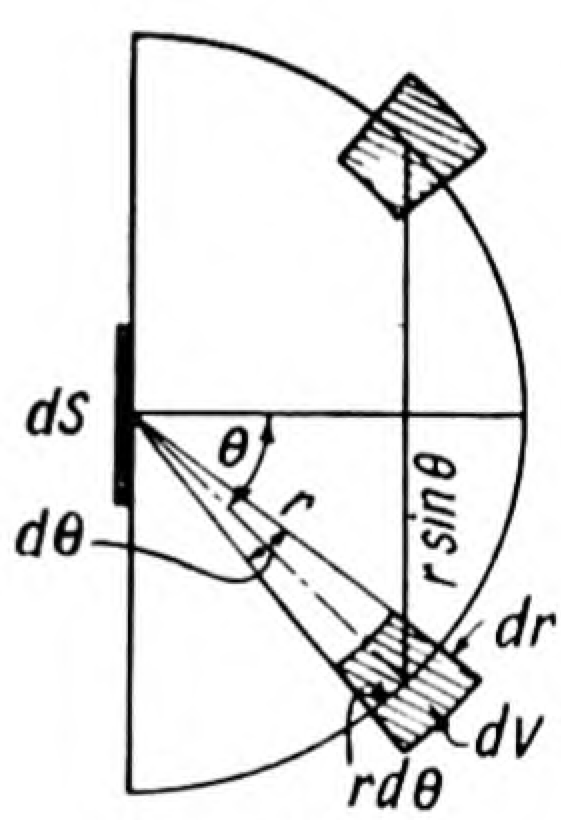


Fig. 14.10.

$\therefore$ The total energy falling *on unit area* of the walls *in one second* is

$$\frac{I'.c}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(*b*) *Decay of sound in a room.* For simplicity the walls are assumed to be perfect reflectors except for an open window of area A. Suppose the source to act uniformly for a time sufficient to allow the energy density to attain a uniform maximum value I_M. After the source is cut off at time $t=0$, say, the energy density within will diminish and finally become zero. The rate of disappearance of energy will be $\dfrac{I.c}{4}.A$, since all of the energy incident upon the open window is lost to the enclosure. Now the energy remaining in the room at the time t is $V.I$, where I stands for the energy density at the instant considered, and V represents the total volume. Clearly, therefore,

$$-\frac{d(VI)}{dt} = \frac{I.c.A}{4}$$

or

$$-\frac{dI}{dt} = \frac{I.cA}{4V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Remembering that $I=I_M$ at $t=0$ and $I=0$ at $t=\infty$, integration of equation (2) shows that at time t,

$$I_t=I_M e^{-\frac{cA}{4V}\cdot t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(*c*) *Growth of sound in a room.* Let W be the uniform rate at which the source emits sound energy. Then the rate of increase of energy within the enclosure is given by

$$W-\frac{I.c.A}{4}=\frac{d(VI)}{dt}=\frac{V.dI}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting k for $\frac{cA}{4V}$, this becomes

$$kdt=\frac{kV}{(W-kIV)}.dI \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which, on integration, gives

$$\frac{W-kVI_t}{W}=e^{-kt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This may be rewritten as

$$I_t=\frac{W}{kV}(1-e^{-kt})=\frac{4W}{cA}\left(1-e^{-\frac{cA}{4V}\cdot t}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which I_t is the intensity at any time t after the source has commenced.

In the steady state as much acoustical energy is absorbed as is created per second, i.e. $\frac{dI}{dt}=0$, hence from (5) it follows that the uniform maximum intensity attained is

$$I_M=\frac{4W}{cA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equation (8) may now be rewritten as

$$I_t=I_M\left(1-e^{-\frac{cA}{4V}\cdot t}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

[Note: This equation may be deduced alternatively from (8) by substituting the particular value I_M for I_t, for

$$I_M=(I_t)_{t\to\infty}=\frac{4W}{cA}(1-0)=\frac{4W}{cA}.\Big]$$

(*d*) *Sabine's formula.* Reverberation time is defined as the time T taken for sound in an auditorium to fall to one-millionth of its original intensity, so the energy density will fall to I_T such that $I_t=I_M 10^{-6}$, whence, from (4), $e^{kT}=10^6$, and therefore $T=\frac{\log_e 10^6}{k}=55{\cdot}3\frac{V}{cA}$.

The velocity of sound (c) in air at ordinary temperatures lies between 330 and 340 m. per sec.

$$\therefore\ T=\frac{0{\cdot}16V}{A}, \text{ when the unit of length is the metre} \quad . \quad (11a)$$

$$\text{and} \quad \frac{0{\cdot}05V}{A} \quad ,, \quad ,, \quad ,, \quad \text{foot} \quad . \quad (11b)$$

When the enclosure contains a number of different surfaces of areas S_1, S_2, S_3, etc., whose absorption coefficients are a_1, a_2, a_3, etc., respectively, then it follows that $A=a_1S_1+a_2S_2+a_3S_3+\ldots=\Sigma aS$.

(*e*) *Criticism of Sabine's formula.* When the walls of a room are perfect reflectors, the absorption, and therefore A, is zero, and T, the reverberation time, is infinity. This is to be expected. Where, however, the walls are

perfect absorbers, open air conditions prevail, and the reverberation time is zero. This is not in accordance with the formula, for the maximum value of A is equal to the area of the walls, etc., which is finite, hence the calculated value of T cannot have the infinitely small value which theory demands.

Theoretical derivation of Eyring's formula

Sabine's formula was derived on the assumption that sound energy travelling along any given path in a room dies out in a continuous manner. Eyring, however, takes account of the fact that the decay is not continuous, but occurs in consecutive drops, each one occurring as the sound strikes one of the surfaces of the room at which it is partly absorbed and partly reflected. On this basis it is shown that sound energy dies out more rapidly than if continuous decay takes place.

When a "ray" of sound strikes a wall it suffers reflection and absorption. If the coefficient of absorption and the mean free path (M.F.P.) of the ray between reflections are known, it is possible to calculate the time of reverberation by finding the number of reflections, etc. It is shown in the Appendix that if S is the *total* surface area the mean free path of sound in the room is $\frac{4V}{S}$ when *all* rays are considered. This is the statistical value. Hence, if N reflections occur in 1 sec., NT will take place in the reverberation time T. It follows that

$$\frac{\text{velocity of sound}}{\text{M.F.P.}}=N=\frac{c}{\frac{4V}{S}}=\frac{c.S}{4V}.$$

Let I_0 be the intensity of the original sound at time $t=0$ and suppose $\bar{a}$ is the average absorption coefficient of the reflecting surface. The intensity of the original sound beam after one reflection is

$$I_0-\bar{a}.I_0=(1-\bar{a}).I_0;$$

after two reflections $(1-\bar{a})(1-\bar{a}).I_0$, *i.e.* $(1-\bar{a})^2.I_0$ and therefore after NT reflections it is $(1-\bar{a})^{NT}.I_0$.

But by the definition of reverberation time the intensity will now have dropped to one-millionth of its original value, *i.e.* to $10^{-6}I_0$, hence

$$(1-\bar{a})^{NT}.I_0=10^{-6}I_0, \quad \text{or}$$

$$NT=\frac{-6}{\log_{10}(1-\bar{a})}=-\frac{-6\log_e 10}{\log_e(1-\bar{a})}$$

and

$$T=\left[\frac{-6\log_e 10}{\log_e(1-\bar{a})}\right]\frac{4V}{cS}; \quad \text{as } N=\frac{cS}{4V},$$

i.e.

$$T=\frac{0{\cdot}16V}{-S\log_e(1-\bar{a})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

As $\bar{a}<1$, the expression $\log_e(1-\bar{a})$ may be expanded and written as $-\bar{a}-\frac{\bar{a}^2}{2}-\frac{\bar{a}^3}{3}\ldots$, and for small values of $\bar{a}$, say $<0{\cdot}2$, the terms beyond the first may be neglected as a first approximation. Eyring's formula then becomes $T=\frac{0{\cdot}16V}{S\bar{a}}$, which is identical with Sabine's formula, as $S.\bar{a}$ will be equivalent to the area of open window denoted by A in equation (11). These formulae are quoted in metre units; when the foot is the unit,

$T=\dfrac{0{\cdot}05V}{-S\log(1-\bar{a})}$. By using the expanded form of $\log_e(1-a)$ for the first two terms only, *i.e.* assuming $\bar{a}$ to be small,

$$T=\frac{0{\cdot}05V}{S\left(\bar{a}+\dfrac{\bar{a}^2}{2}\right)}$$

$$=\frac{0{\cdot}05V}{S\bar{a}\left(1+\dfrac{\bar{a}}{2}\right)}$$

$$=\frac{0{\cdot}05V}{S\bar{a}}\left(1-\frac{\bar{a}}{2}\right)$$

$$=\frac{0{\cdot}05V}{S\bar{a}}-\frac{0{\cdot}05V\bar{a}}{2.S\bar{a}},$$

i.e.

$$T=\frac{0{\cdot}05V}{A}-\frac{0{\cdot}025V}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which S is the total surface area.

Optimum number of orchestral instruments in a room

As stated on p. 288, experience shows that an orchestra should be composed of such a number of instruments that each one, on the average, provides sufficient sound energy for every 200 sq. ft. of open window, or its equivalent absorbent, in the room. This gives a suitable "volume" of sound. The relationship can be established qualitatively as follows:—

In the steady state the acoustic energy density I_M was shown on p. 296, to be given by the expression

$$I_M=\frac{4W}{cA},$$

W being the total energy generated per second. Now if w be the average amount of sound energy given out per second by each of n instruments in the orchestra, $W=nw$. It follows that $I_M=\dfrac{4nw}{cA}=K\dfrac{n}{A}$, where K is a constant, and so, for a uniform value of I_M, the number of instruments, n, is proportional to A, the area of absorbent in open-window units which is exposed to the sound.

The geometrical and wave aspects of room acoustics

When a source of sound is started in a room there will be set up a steady state vibration of the same frequency as the source, together with a transient free vibration, having the frequencies of the normal modes, but which will decay away. The steady state vibration will, of course, differ from the radiation of a "free" source, *i.e.* one in the open and mathematical analysis shows how it may be synthesised from a suitable choice of standing waves. At low frequencies it is found that the values of the resonance frequencies of the various standing waves are widely spaced so that the intensity of the sound in the room will show large fluctuations as the frequency of the source is varied. These regions of good and bad transmission occur *below* a critical frequency given approximately by $f=10\sqrt{\dfrac{T}{4V}}$ Kc.p.s., where T is the reverberation time in seconds and V is the volume of the room in cubic

feet. Above this frequency, when the wave-length is small compared with the room dimensions, the resonances become closer together, so that the intensity may be reasonably assumed to be uniform and consequently it is justifiable to apply statistical mechanics to the analysis of the problem.

In the geometrical aspect of the problem, as exemplified by Sabine's work, all the sound in the room is regarded as acting together as a single simple oscillator. The derivation of Sabine's empirical formula, initially due to Jaegar, follows the geometrical classical kinetic theory in which the acoustic energy is considered to travel in rays. The underlying assumptions of the theory (p. 294) appeared to be satisfied by Sabine's experimental results, due to the fact that his experiments were chiefly restricted to moderate-sized halls which were fairly reverberant. The theory and experimental results suggest that the logarithmic decay curve should be a straight line, but it must be remembered that Sabine's use of the ear as detector would tend to "iron-out" any but very large space and time variations of intensity. The failure of Sabine's formula when applied to very dead rooms led Eyring to replace the continuous absorption assumed in the development of Sabine's formula by a process involving discontinuous drops in intensity during the sound decay, but he assumes that the acoustical energy rapidly reverts to a uniform distribution after each set of incidences.

The most serious criticism of the geometrical approach to the problem of room acoustics has been the discordance of measurements on acoustic absorption coefficients of materials. The effective absorption of any particular material has been found to depend on its area and its position in the room. It is only by a "wave" approach to the problem of room acoustics that a theoretical investigation in this so-called "pattern" effect has been possible.

In a rectangular room three types of standing waves, oblique, axial, and tangential (see App. 14) can occur, and each has a different decay rate. It follows that under such circumstances the log-decay curve cannot be linear. This variation of decay factor will be emphasised in a room of regular contour and smooth walls, for a tangential wave moving parallel to the most absorbent wall will damp out more rapidly than a normal one.

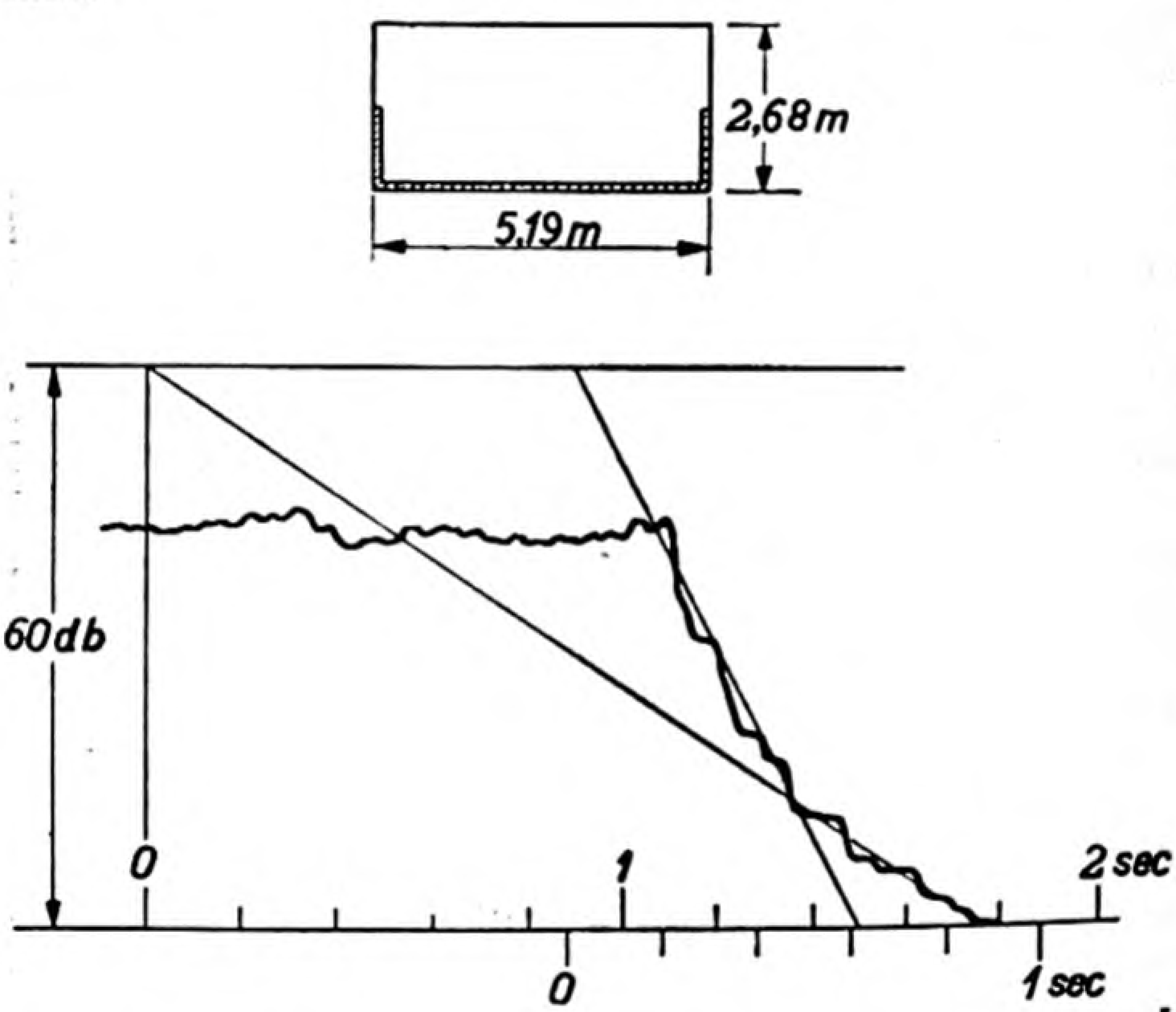
Summarising the position very briefly it may be said that the geometrical approach will usually suffice for the investigation of large auditoriums, but that the wave method must be employed in the case of small and regularly shaped rooms.

Note on distribution of absorbing material and absorption coefficient

If most of the absorbing material is on two opposite walls then the standing waves *not* reflected from these walls will have a longer decay time. The existence of these more highly damped waves can give rise to interference phenomena and if the waves are few in number the sound intensity will fluctuate rather than decrease uniformly. If there are a number of such waves excited, however, an averaging effect will occur and the intensity-time curve will be "smoothed-out," but will show two different slopes, a steep one to start with, followed by one of smaller slope corresponding to the decay of those waves not striking the strongly absorbent walls (see Fig. 14.11).

It should be noted that the energy absorption by the air in the enclosure only becomes important at higher frequencies, but at 4 Kc.p.s. may be much greater than the total absorption at the walls; the absorption is accentuated if the humidity is high. Eyring's formula may be modified to take into account air absorption as follows: $T=\frac{0{\cdot}05V}{-S\log_e(1-a)+4\alpha V}$, where the symbols have their previous significance and α is the attenuation or absorption coefficient as defined on p. 282 in the expression $I_x=I_0e^{-\alpha x}$.

The absorption coefficient as defined by Sabine in his reverberation formula is given by the average value of the ratio $\frac{\text{Absorbed sound intensity}}{\text{Incident sound intensity}}$ for the material exposed. The coefficient so defined is not satisfactory as a unique measure of the acoustic properties of a wall surface as indicated by the inconsistency of the values obtained by different types of measurement. The measured value of the coefficient changes when the material is placed in different rooms and in some cases varies with angle of incidence of sound. A more fundamental measure is the acoustic impedance Z of the absorbing surface, which is defined as the complex ratio of the sound pressure at the surface to the air velocity normal to and just outside the surface. This air velocity may be due to a motion of air into the surface, if porous, or to a movement of the wall itself. An absorption of energy at the surface is implied when the impedance has a real component, whereas the existence of a purely reactive impedance indicates that the reflection involves only a change of phase.



[*After Van Urk.*

Fig. 14.11. Recording of the sound intensity in a room in which the lower half is covered with absorbing material, as indicated in the cross-section. In the recorded curve, two reverberation times may be seen belonging to the two halves of the room.

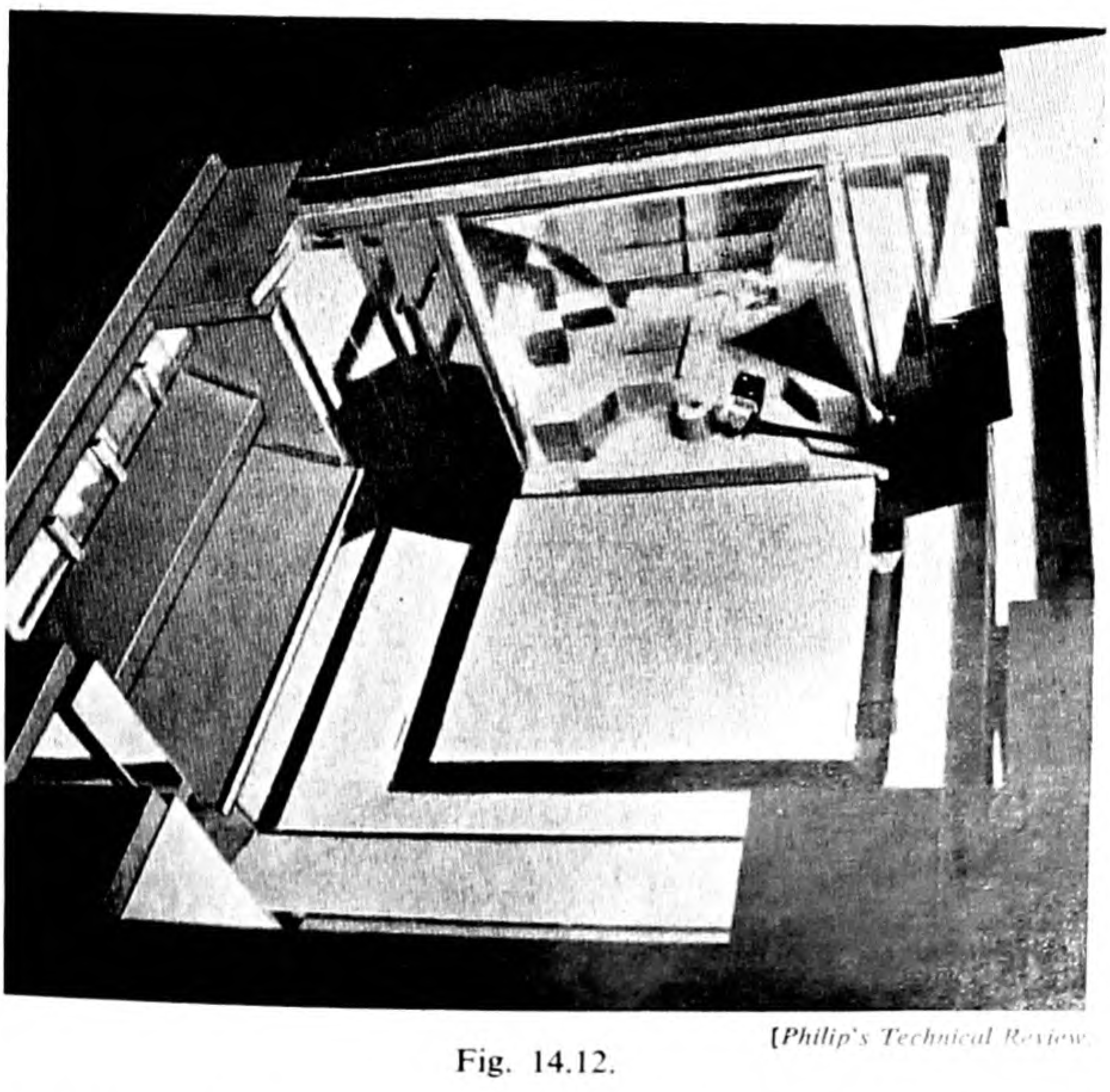
Experimental methods of testing models of auditoria

Before proceeding to erect an auditorium, tests are frequently made on scale models to investigate the behaviour of sound in the proposed structure. Two of the methods—the Schlieren method and the ripple tank—employ sections of the model. In the former the position of the proposed source of sound in the actual auditorium is occupied in the model by a spark gap, so that, when a spark is passed, it gives rise to a sound pulse resembling a sound wave in the actual auditorium. Such a sound pulse can be photographed at any instant in the manner described on p. 155, and in this way its course and that of the sound can be traced.

A ripple tank is a shallow tank containing water. A dipper, which is capable of executing vertical vibrations, touches the surface. When vibrating it sends out ripples whose behaviour can be followed by means

of a stroboscope. A section of the auditorium model is placed horizontally in the tank with its upper surface projecting above the water, the dipper occupying the position of the proposed source of sound. The resulting ripples resemble sound waves *in section*, and the effect of the shape of the auditorium on the waves can be readily observed.

A third method, due to R. Vermeulen and J. de Boer, has the advantage of being three dimensional, for it employs a complete model (Fig. 14.12). This is painted a light matt colour internally, and has a small lamp at the position corresponding to that intended for the source of sound in the actual auditorium. By inserting a



[*Philip's Technical Review*

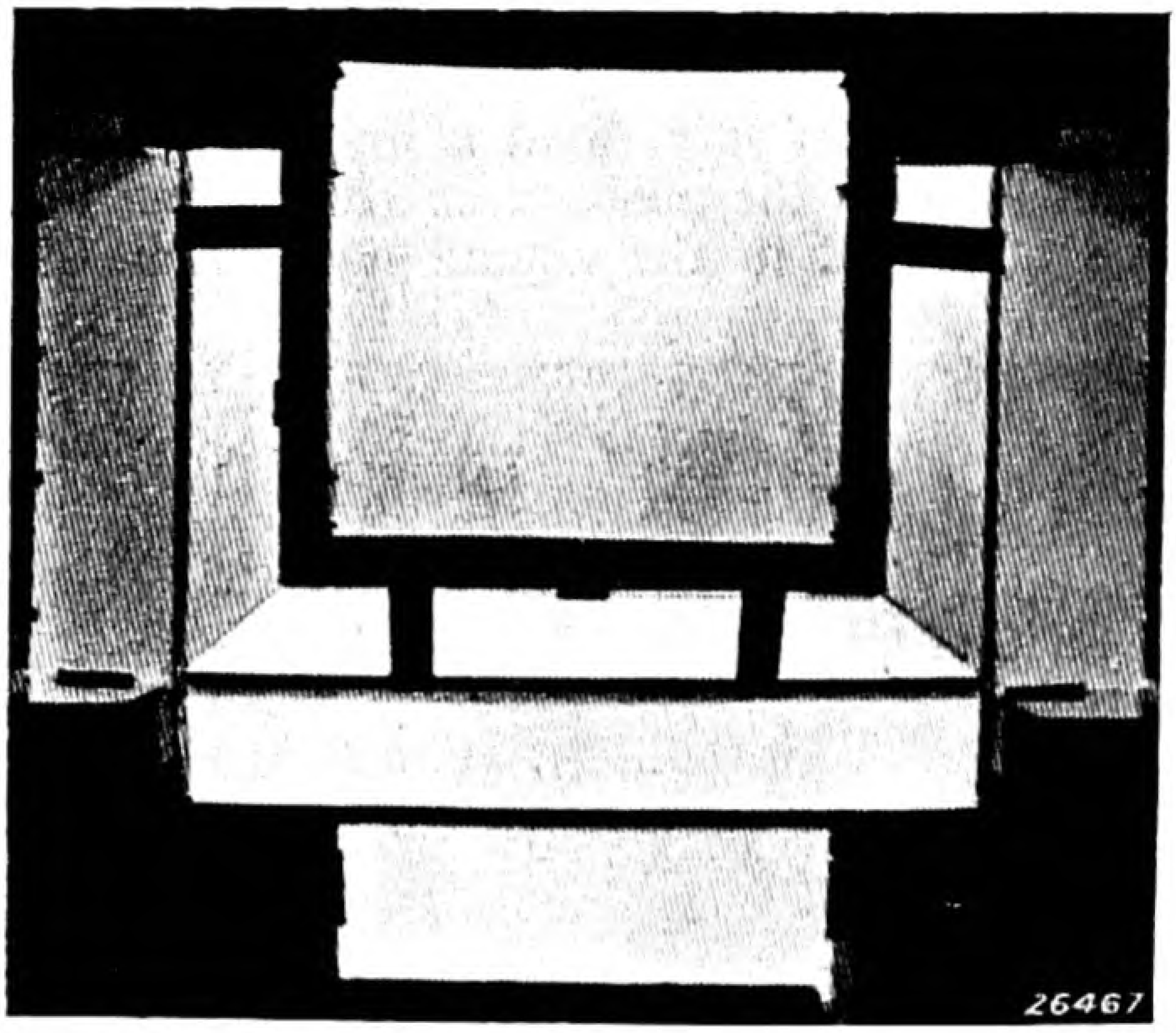
Fig. 14.12.

plate of frosted glass at a particular position the variation of the intensity of illumination reaching that position can be observed, and thus the variation in the intensity of sound in the actual auditorium in the corresponding position can be estimated (Fig. 14.13). Increased sound absorption by certain surfaces can be simulated in the model by painting such surfaces a darker shade. Effects due to individual surfaces can be examined by directing on to them a single beam of light and filling the model with a mist (Fig. 14.14).

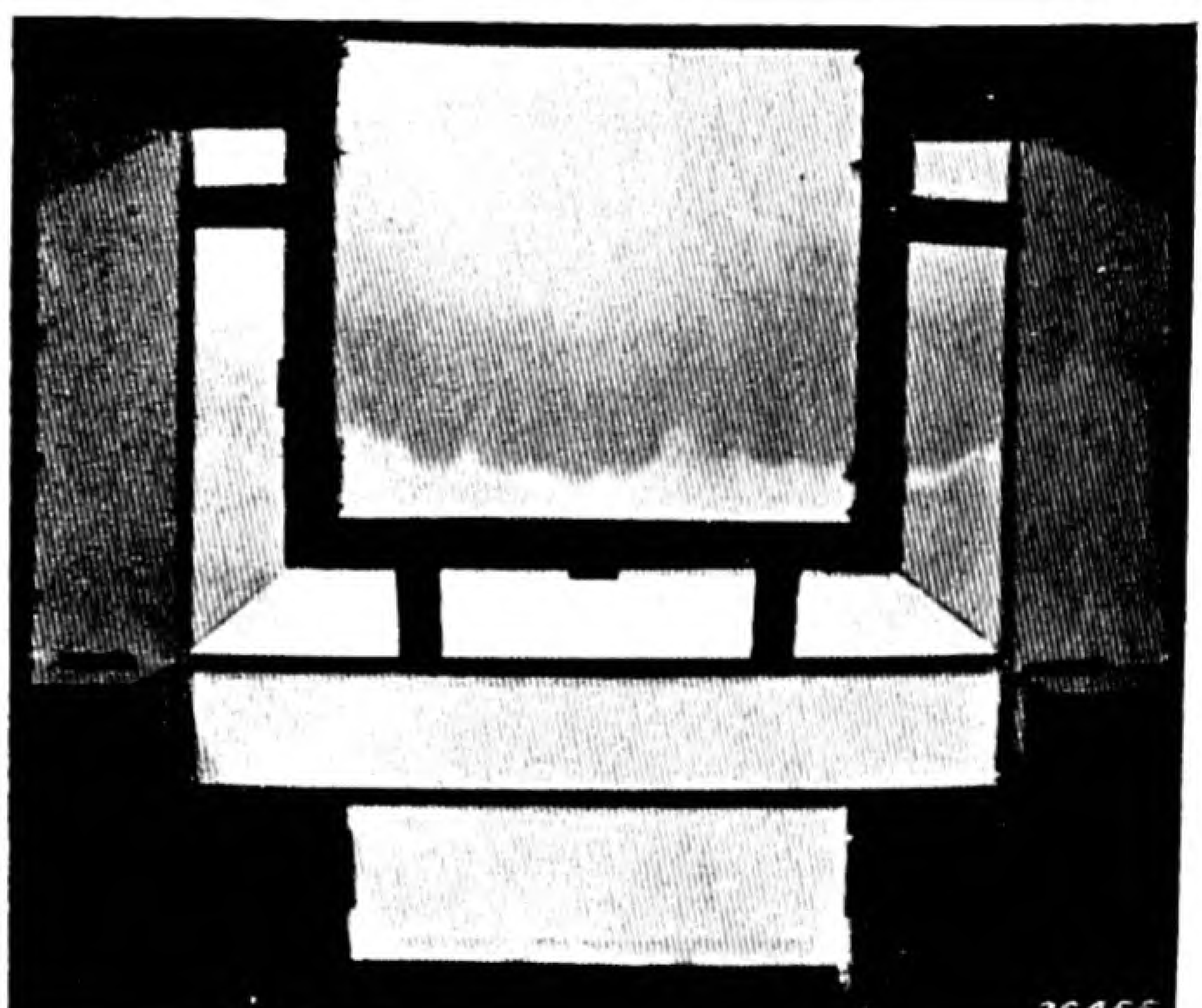
During the last few years it has become apparent that calculations of reverberation time for one frequency—usually 500 c.p.s.—near the middle of the musical scale, do not provide a sufficient criterion

by which auditoria may be classed as satisfactory. In recent practice the reverberation times for more extreme frequencies, such as 100 c.p.s. and 4000 c.p.s., are calculated in addition. As already mentioned, however, the acoustical requirements for music in a particular hall differ from those for speech in the same hall, and yet again the desirable

(*a*)



(*b*)

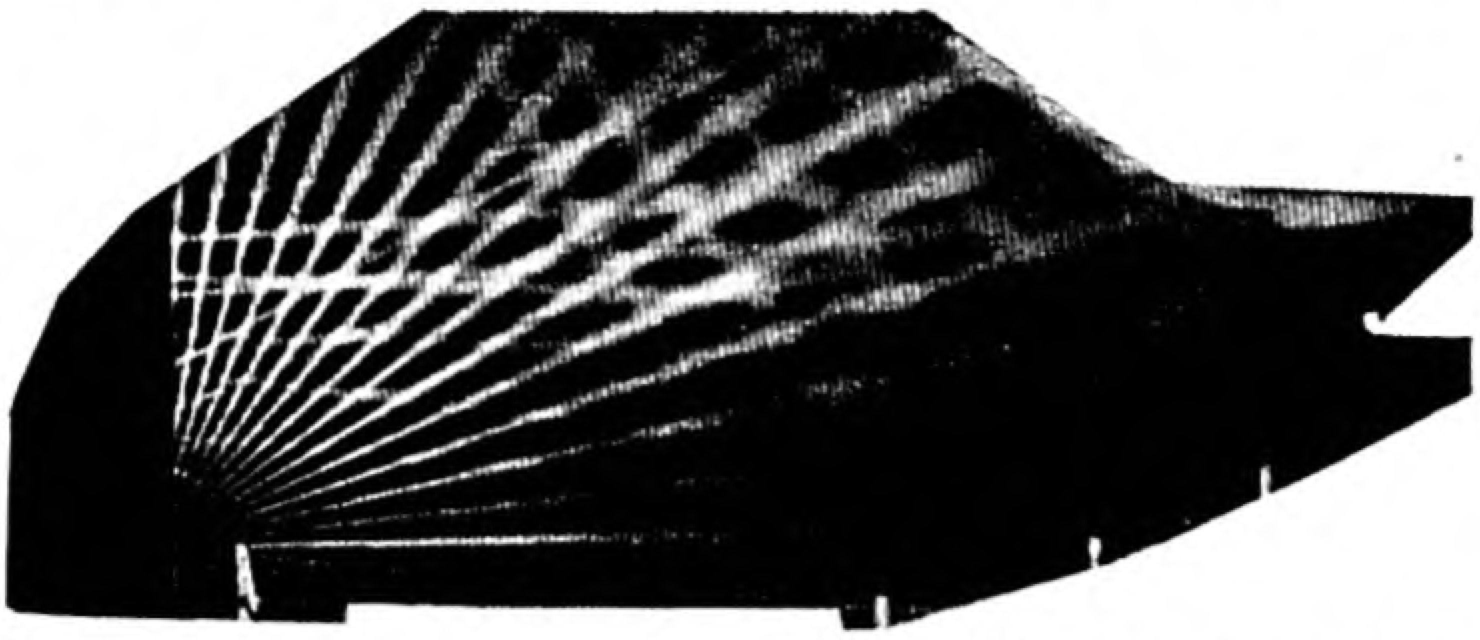


[*Philip's Technical Review*.

Fig. 14.13. (*a*) Distribution of light on the floor and the surrounding seats. (*b*) The absence of a reflector behind the speakers' rostrum is seen to lead to a reduction in intensity along the sides as well as the rear of the hall.

conditions in a small broadcasting studio differ from those in a concert hall. In general, the sounds in the studio should be rapidly damped to give the listener the impression that the performance is taking place in his own room. In the case of an auditorium it is usual to provide the stage with hard surfaces to project the sound into the body of the

hall. Again, in order to bring about a general diffusion of sound, convex reflecting surfaces are often employed. The latest technique in this direction, due to Bonar, which has been employed in broadcasting and recording studios, utilises horizontal semi-cylindrical wooden surfaces of different curvatures with intervening flat spaces.



[*Philip's Technical Review.*

Fig. 14.14.

Although the Sabine formula indicates that the sound intensity within an enclosure has an exponential decay, it is not surprising, considering the assumptions involved in the theory, that no such regular law is obtained in practice; the experimental curve in a typical instance is shown in Fig. 14.15. The perturbations or peaks which occur are largely due to the existence of prominent reflections within the enclosure, and also to the possible presence of standing waves when a particular resonant frequency of the room is emitted by the sound source. Bonar asserts that, by the adoption of the device mentioned above, a closer approximation to an exponential decay is obtained.

A problem of a different type, which has been investigated recently by Mason and Moir, occurs in large cinema auditoria, and refers to the desirable close association of the sound source with the picture screen, whereby the audience receives the impression that the sound

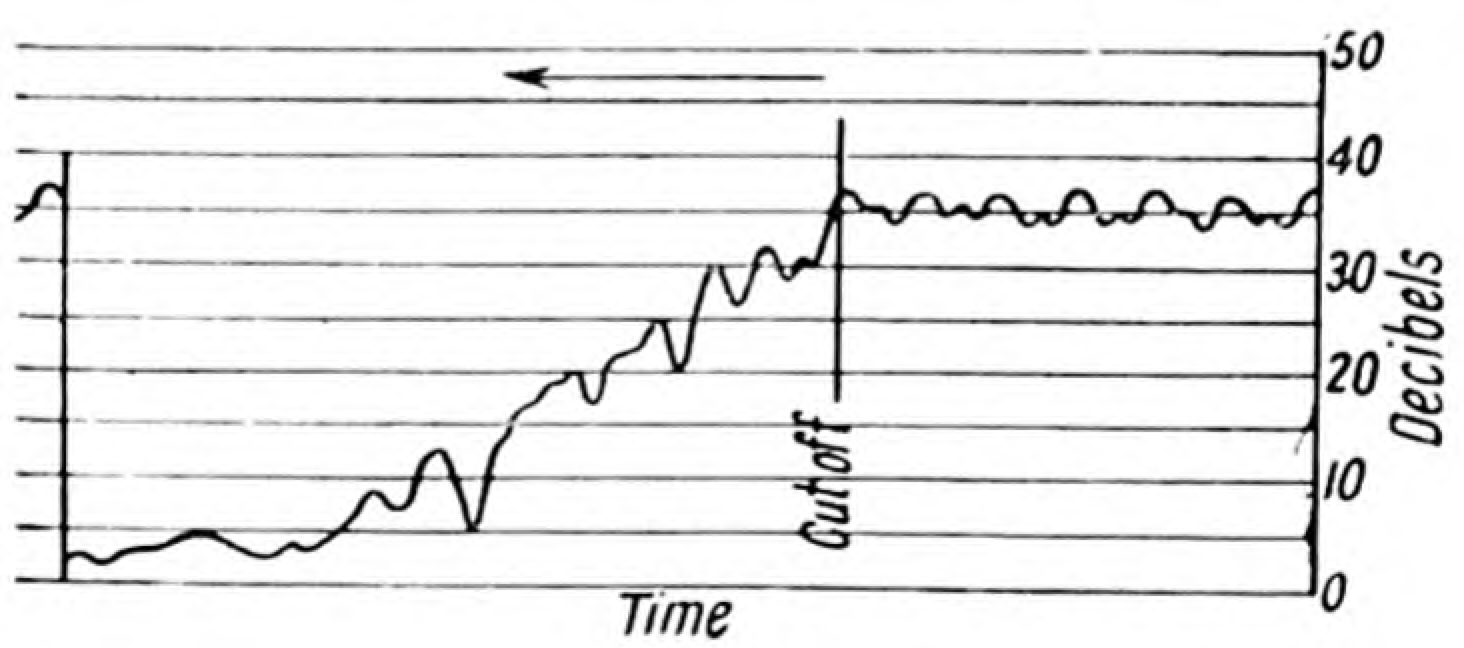


Fig. 14.15.

is actually emanating from the image on the screen. An endeavour is made to achieve this condition of "intimacy," as it is termed, by placing loud-speakers on each side of, or behind, the screen. At the same time, however, care must be taken to prevent the audience receiving any reflected sound of appreciable strength, otherwise the

sense of intimacy is lessened. Mason and Moir showed that it is desirable for the angle between the direct and the prominent reflected rays to be as small as possible. They also point out the necessity for a complete understanding of accurate sound focusing within an auditorium, if stereophonic sound is to be successful in the cinema. The aim in this effect is the synchronisation of the *movement* of the sound focus *across* the screen with the picture of the original source or actor, in such a way as to be followed and appreciated by the audience.

Fig. 14.16 shows the elevation and plan of a theatre of good acoustical design, which possesses a low ceiling, heavily carpeted floors, and seats which are upholstered to be strongly sound-absorbent, but only the

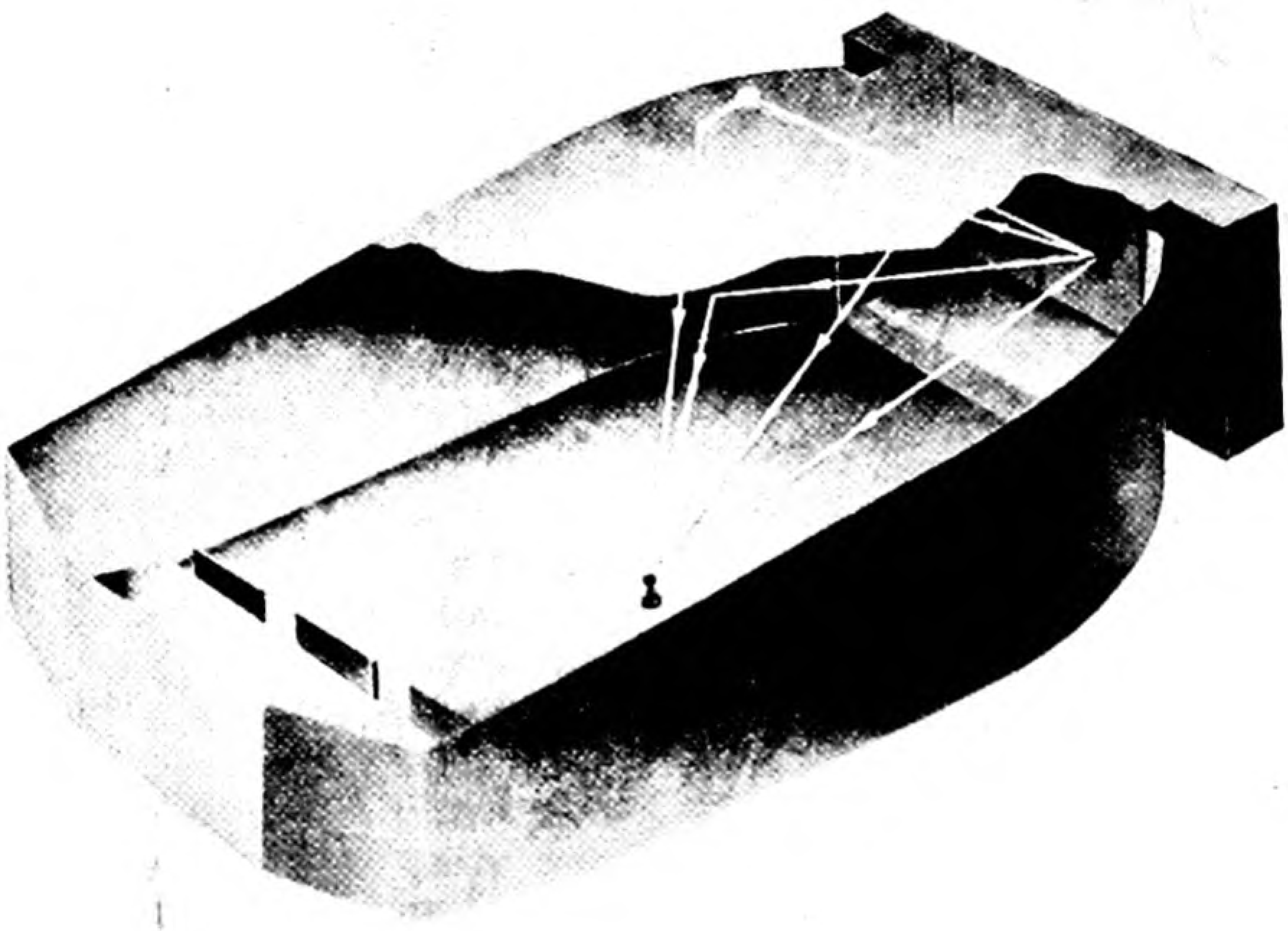


Fig. 14.16. Plan and elevation of theatre of good acoustical design.

back walls and side walls near the screen have received treatment with sound absorbent material. Fig. 14.17 shows an auditorium in which the sound quality is poor. This is borne out by the "impulse photographs" (see below), which show large reflections after both short and long intervals of time. The reverberation time and the frequency characteristics of this hall, however, had previously been presumed to be acoustically satisfactory; this contrary result was shown to occur in other auditoria examined by Mason and Moir. In other words, the impulse measurements are more in accord with aural judgment, and support the contention of these workers that judgment by ear depends chiefly on the form of the reverberation curve during the first few milli-seconds. Now the frequency characteristic of a loud-speaker is normally measured under steady state conditions, and does not represent the conditions which occur in

practice, for the ear receives sound which, in general, is rapidly changing, and which often includes transients. Mason and Moir therefore

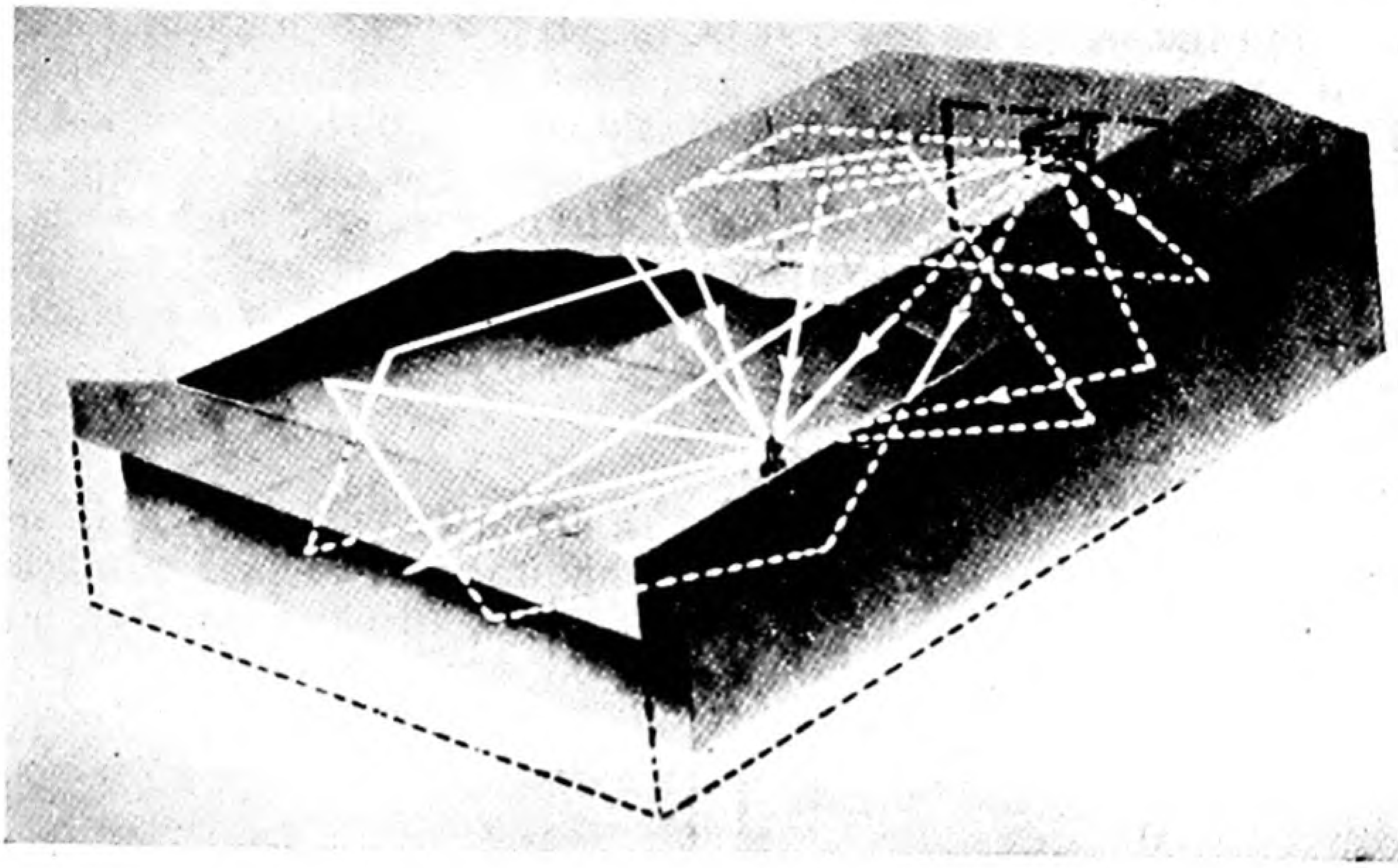
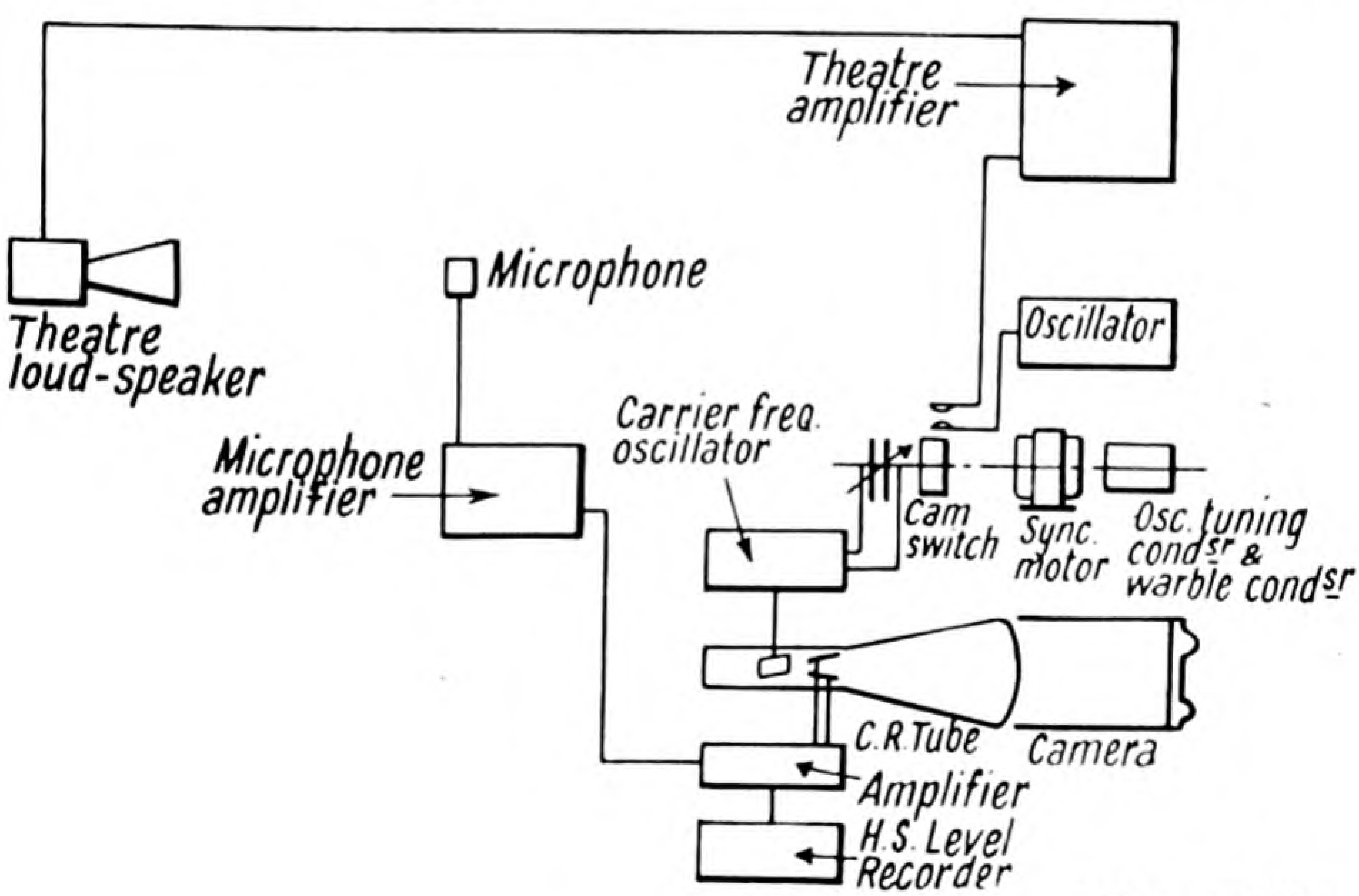


Fig. 14.17. An auditorium of poor acoustical design.

decided to investigate the acoustic properties of an auditorium by means of "impulse" sound, *i.e.* sounds of short duration. Briefly, the method consists in feeding electrical impulses, after amplification,



[Mason and Moir.

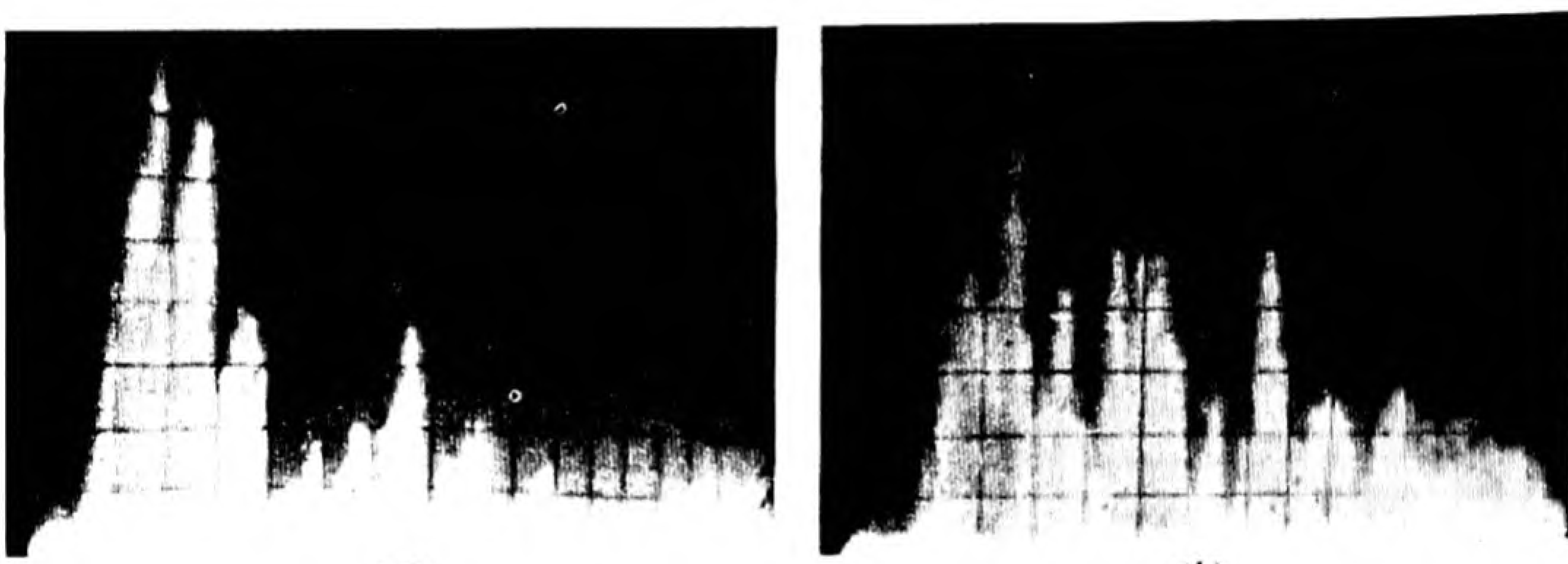
Fig. 14.18.

into a loud-speaker and then picking up the sound impulses by a microphone suitably placed in the auditorium (Fig. 14.18). The

amplified output from the microphone is applied to the vertical deflection plates of a C.R.O. A time-base, synchronised with a cam operating the impulse switch, is applied to the horizontal deflection plates, and the traces on the C.R.O. screen are photographed. Figs. 14.19 *a* and *b* are typical records, and refer to the halls of Figs. 14.16 and 14.17 respectively; the former shows the directly propagated impulse with almost complete absence of reflections, while (*b*) indicates the presence of large reflections after both short and long intervals of time. It should be added that the rate of generation of impulses is arranged to allow the reflected sounds to die away to a negligible amount in the interval between the impulses.

Acoustics of small rooms

A room with a volume of less than about 8000 c. ft. (230 c. m.) in volume cannot be classified as an auditorium, for the problems usually associated with the latter—such as the effect of reverberation on the intelligibility of speech, and good conditions for choral or



(*a*) (*b*)

[*Mason and Moir.*

Fig. 14.19. Pulse photographs.

orchestral music—do not arise. The shorter sound paths lead to more reflections in a given time, but as the direct path to the listener is small, the source need only be of low power. As a consequence, the intensity of the reflected beams will be correspondingly low, and the intelligibility of speech will remain practically unaffected. The character of the music in such a room is limited to chamber music, which requires not more than the optimum number of instruments—trio, quartette or quintette—for rooms of comparatively small volume.

A factor which does arise in small rooms is resonance. Lord Rayleigh showed (see App.14) that the resonant frequencies in rectangular enclosures are given by the expression

$$n=\frac{c}{2}\sqrt{\frac{x^2}{l^2}+\frac{y^2}{b^2}+\frac{z^2}{h^2}},$$

n and c being the frequency and velocity of sound in air respectively; l, b and h the length, breadth and height of the room; x, y and z are integers or zero. When x, y and z have the values 1, 0 and 0

respectively, the expression becomes $n=\frac{c}{(2l)}$, so the wave-length of this frequency is equal to twice the length of the room (*cf.* pipe closed at both ends sounding its fundamental).

The resonant frequencies of a cubical room are readily calculated by the formula. The values for two such rooms, in which $l=b=h$ $= 14$ ft. and 20 ft. respectively, are given in the table, the velocity of sound being taken as 1120 ft. per sec.

x	y	z	1st $[l=14']$	2nd $[l=20']$
1	0	0	40 cycles per sec.	28 cycles per sec.
1	1	0	57 ,, ,,	40 ,, ,,
1	1	1	70 ,, ,,	49 ,, ,,
2	0	0	80 ,, ,,	57 ,, ,,
2	1	0	90 ,, ,,	63 , ,,
2	1	1	98 ,, ,,	69 ,, ,,
2	2	0	115 ,, ,,	80 ,, ,,
2	2	1	120 ,, ,,	84 ,, ,,
2	2	2	140 ,, ,,	98 ,, ,,

Points worthy of notice are:—*

(1) The lowest resonant frequencies are considerably below the lowest reproducible frequency of a radio receiver likely to be used in either room.

(2) Each frequency is associated with a sound pattern. With a sustained musical chord, several patterns are superimposed, hence the relative intensities present in the original are not likely to occur anywhere in the room after reflection, *i.e.* after a few hundredths of a second. The effect is further influenced by the variation of absorption with frequency.

To eliminate the effects, or at least to minimise them, the walls should be draped with highly absorbent material.

The extent to which elimination is desirable for musical purposes depends on the source of the music. Recorded and radio music are influenced by the acoustics of the room in which they are performed, and should be uninfluenced by the room in which they are reproduced, hence the absorption should be high. If, however, the actual performance is in the room itself a smaller degree of absorption is desirable, otherwise the music appears dead, as indicated on p. 292.

In summarising it may be said that in room acoustics the room must be regarded in the nature of a horn for transmitting acoustical energy from the source to the auditor, and the purpose of design of shape and choice of absorption materials is to make this transmission equally efficient at all frequencies to be employed. Care has to be exercised that the absorption is not so large as to lose the advantage of enhancement of sound intensity by repeated wall reflections. The modern trend is not to regard reverberation time alone as a sufficient guide to the acoustic "goodness" of an auditorium, and the idea of "liveness"

* The spacing between eigentones is greater for small rooms and also the lower modes occur within the audible range; the resulting excessive low frequency reverberation is known as *boom*, which is an undesirable feature of small speech studios.

has taken its place. This conception involves such other considerations as the mean square sound pressure being uniform over the space occupied by the audience, that at least a certain percentage of the acoustical energy reaches any particular hearer directly from the source, but that not more than a certain fraction reaches him after reflection from a particular surface, etc. In cases where the log-decay curve is not linear the idea of reverberation time loses its significance and it has been replaced by the conception of *decay rate*, which in effect is the *slope* of the normal decay curve. The natural decay of the stored energy takes place at the eigen frequencies and not at the frequency of the impressed sound, unless these frequencies happen to coincide.

The fundamental requisites for speech and music are similar, but differing personal tastes enter so much into the latter that the optimum musical conditions for a room are very difficult to realise. The accepted reverberation time for music will depend upon its type, but in all cases it will be greater than for speech.

The reasons why reverberation time (T) is not regarded as solely giving a measure of the subjective sound quality of a room are concisely expressed by C. A. Mason, viz. (*a*) T assumes an initially diffuse sound field rarely realised in practice, (*b*) a 60 db. drop in level is defined whereas the ear is only likely to be affected by the first 20 or 30 db. fall, (*c*) T does not take account of the shape of the room and (*d*) although taking account of the amount of absorption in a room, T does not consider the mode of distribution of the absorbing material.

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CHAPTER 15

ELECTRICAL, MECHANICAL AND ACOUSTICAL ANALOGIES

The results obtained from the theoretical analysis of electrical networks are applicable to mechanical and acoustical systems, and vice-versa, only because of the fundamental analogies existing between the two systems. These analogies are based ultimately on the fact that the various motions conform to the same type of differential equation, as will be indicated later in this chapter. By means of the fundamental relations shown in the accompanying table, any acoustical or mechanical system may be transformed into an analogous electrical system, and the problem is then solved by the theoretical analysis of the electrical circuit. In practice the procedure may be carried further, to the stage of designing an acoustical mechanical circuit, which must possess a given frequency-response characteristic. These requirements are met in the first instance by the suitable design of an electrical circuit, and then the components of the latter are translated into their equivalent acoustical or mechanical elements.

It should be mentioned here, however, that the equations of motion of mechanical systems had been developed some years before those of electrical networks, and so in the earlier days it was natural that the process of conversion should be reversed and the propagation of electromagnetic waves regarded as a mechanical phenomenon. A particular case in point is the transmission of energy along an electrical line. This problem was studied originally by reference to its mechanical counterpart, a stretched string. Such an arrangement, similar to that of Melde's experiment, is shown in Fig. 15.1*a*, where the string is fixed to a stationary barrier at one end, and is attached at the other to the prong *T* of an electrically maintained tuning-fork. The argument which follows is that put forward by Pupin.

Now provided the frictional resistance to the motion of the cord is small, then a stationary wave pattern is set up as in Fig. 15.1*b*, since the direct waves generated by the fork and those reflected from *B* will possess nearly equal amplitudes. Suppose, however, a mass is placed on the string at a point *M* (Fig. 15.1*c*), then the outgoing waves from the fork will be reflected there, and less energy will reach *B*, consequently the *attenuation* of the waves becomes appreciable. However, if now the same mass *M* is divided into three or four parts and these are placed at equal distances down the string, it would be found that the attenuation becomes less marked. This process of further subdivision can be extended until a stage is reached where the improved transmission which is obtained differs but little from the transmission of energy down a single unloaded string of the same equivalent mass density per unit length. The general solution to the problem of a string uniformly loaded with beads of appreciable mass was worked out by Lagrange, who showed that all the natural frequencies of the loaded string were less than a certain critical value. In other words, as pointed out later by Routh, if the *period of excitation*

of the string becomes sufficiently short, then no motion in the nature of a wave is transmitted along the string. The importance of this aspect of the problem is that such a system fulfils one of the main functions of a *wave-filter*, which is to transmit certain desired bands of frequencies while highly attenuating neighbouring frequency bands which are not required. In this instance the loaded string would correspond to a *low-pass* filter, since it is the *higher* frequencies which are *suppressed*. The general problem of wave propagation in a medium having a periodic discontinuity of structure, it should be mentioned here, has application in many physical phenomena. In particular the expression for the ratio of reflected to incident amplitude of the wave at the junction of two strings of equal tension but of different loading, originally deduced by Rayleigh, is very similar to the corresponding expression for optical reflection at the boundary of two media of different densities, which is a problem already mentioned

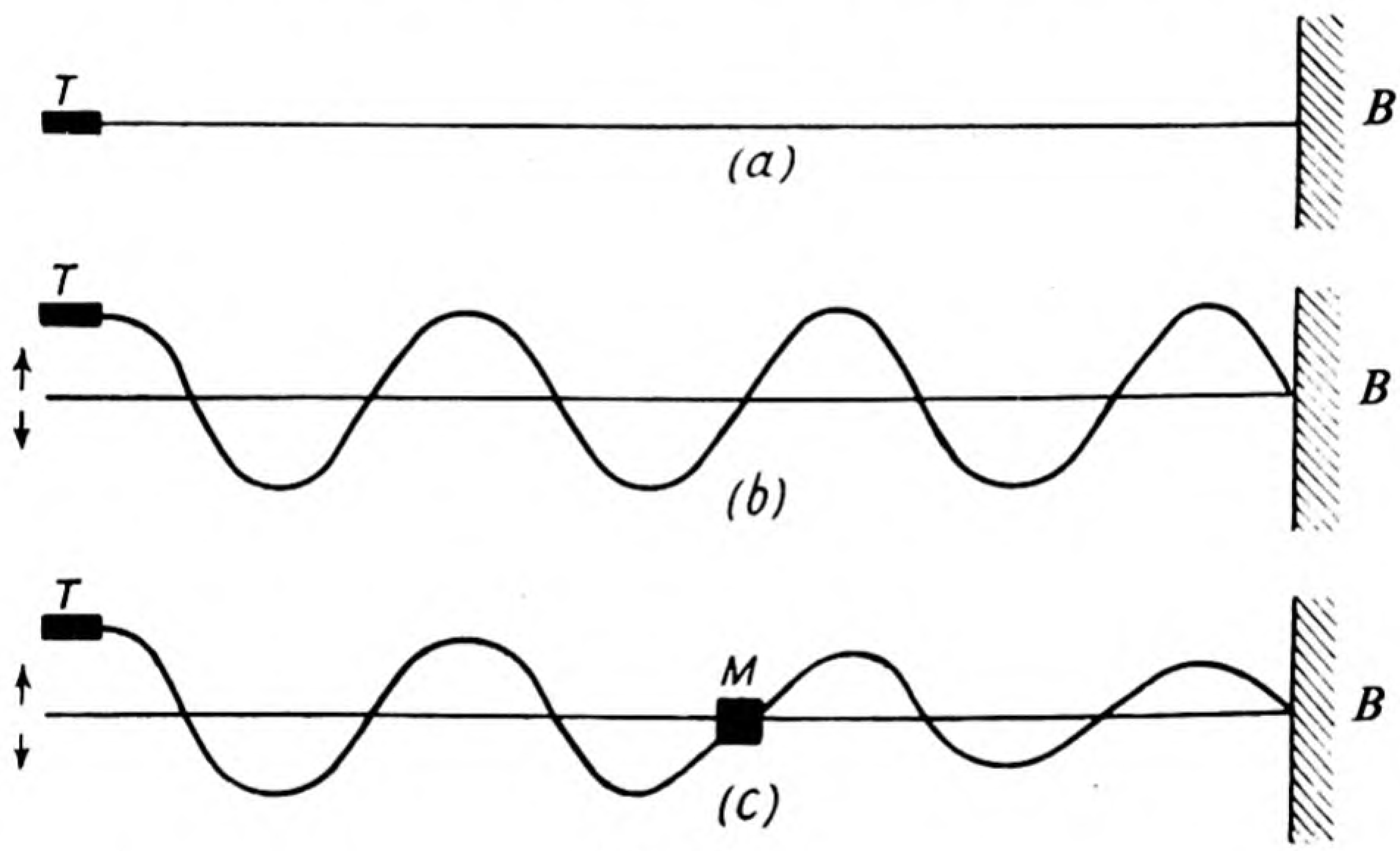


Fig. 15.1.

in the previous chapter. Again, in the propagation of sound waves along a tube which has a series of *equidistant* expansion chambers (see Fig. 15.2), very definite frequency selection is shown, and such an arrangement as schematically shown in the diagram forms a low-pass acoustic filter. A structure of this type, capable of transmitting sound waves in the direction of the tube or channel only, is termed an *acoustic line*.

Fig. 15.2 shows an electrical system which exhibits the property of a low-pass filter when alternating electromotive forces are applied to the line, and the theory of such a system was adapted by Campbell from the corresponding mechanical case by replacing the masses on the string by inductances. The analysis of the electrical system contributed an additional quality to the filter not apparent in the mechanical problem, namely, the dependence of the transmitted *wave-form* on the degree of "loading" of the line. A further development of the theory consequent upon advancement in the knowledge

of electric circuits, led to the idea of a *dissipationless* filter, *i.e.* one in which the *impedance terminations* at the end of the line are *matched to the line*, so that *power* may be put into and *absorbed from* a filter.

It is rather beyond the scope of this book to investigate in detail the complete solution of a particular problem, but the method of obtaining the equations for a simple system will be shown. Firstly, however, it will be necessary to obtain some physical significance of the various acoustic elements concerned, and to define them precisely.

Acoustic impedance

Sound vibrations are of an alternating nature, and hence analogous to the electrical impedance of an alternating current circuit, the acoustic impedance (Z_A) is defined as the *complex quotient* of the pressure applied to the acoustical system divided by the resulting *volume*

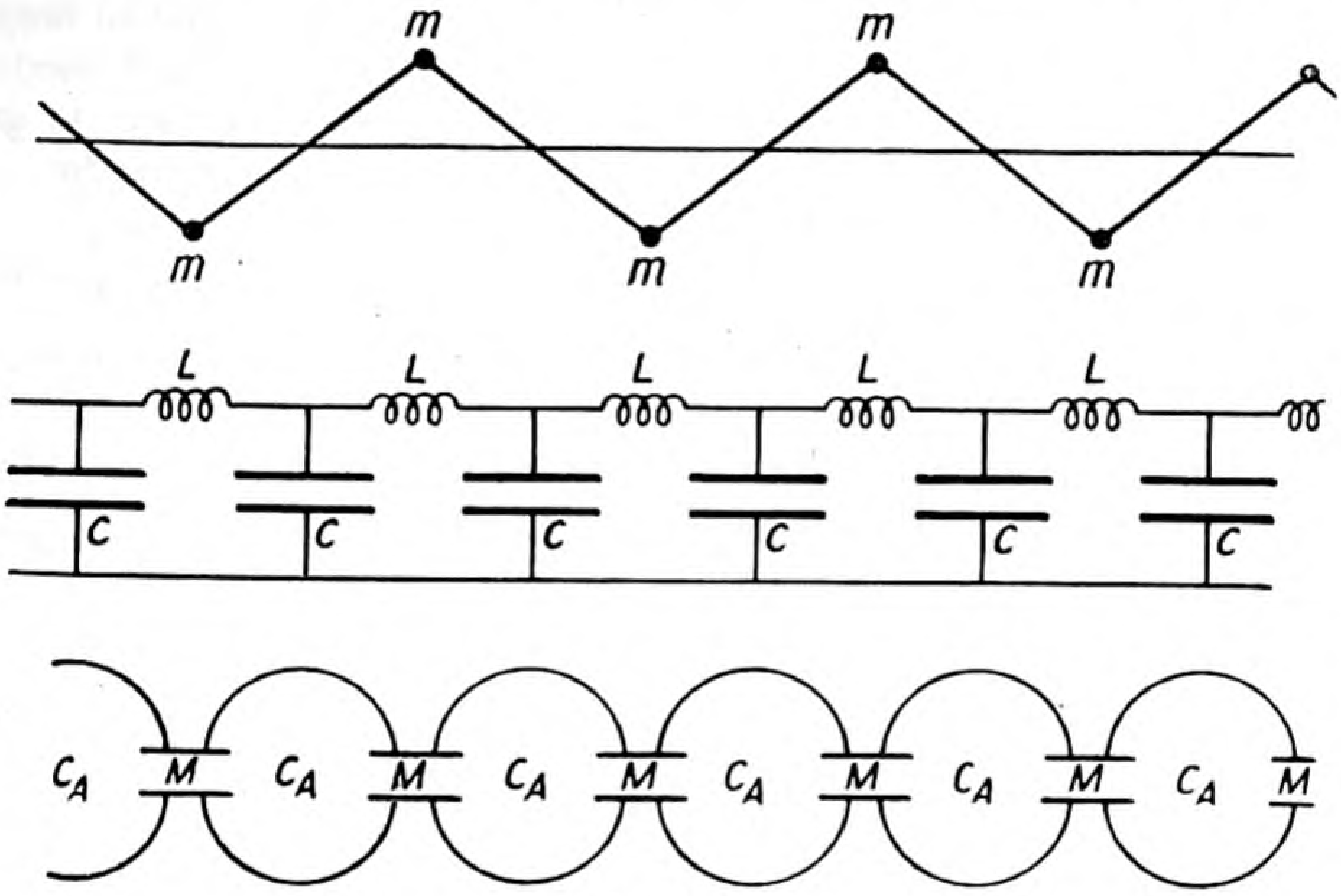


Fig. 15.2.

current ($\dot{X}$). This impedance Z_A may be expressed as the sum of two terms, viz. $Z_A = r_A + jZ$, where $j = \sqrt{-1}$, Z is known as the reactance, and r_A is the acoustic resistance which is a purely resistive component and responsible for the energy dissipated in the element. This resistance r_A is defined by $r_A = \frac{p}{\dot{X}}$, where $\dot{X}$ is the *volume* current in cubic centimetres per second, and p is the *excess* pressure in dynes per square centimetre, and is itself, in general, composed of two terms respectively known as radiation and fluid resistances. The latter component may be likened to the internal resistance of a battery in an electrical circuit, the useful work performed in the circuit external to the battery corresponding to the energy developed in the radiation resistance.

The fluid resistance is the important term in the considerations which follow, and in practice the acoustical resistance experienced

by sound waves in transmission through an aperture or porous body is primarily due to the viscosity of the medium. Just as the passage of an alternating electric current through a resistance generates heat, so also is there a similar dissipation of thermal energy when a fluid medium moves "to and fro" in an acoustic resistance. The impedance offered by a cylindrical conduit has been worked out by Crandall, and shown to comprise a resistive term, involving the viscosity η of the medium and the radius a of the conduit, together with a reactance term dependent upon the density ρ of the medium, and the frequency $\frac{\omega}{2\pi}$ of the sound waves. The purely resistive term predominates for narrow tubes, and its approximate value can be derived by an application of Poiseuille's formula for the orderly flow of a viscous fluid through a capillary tube.

Let p be the pressure *difference* between the ends of a tube of length l and radius a, and η the coefficient of viscosity of the fluid medium. Then the volume Q of fluid flowing through the tube in t sec. is given by $Q=\frac{\pi a^4 pt}{8\eta l}$, and hence the volume current $\dot{X}=\frac{Q}{t}=\frac{\pi a^4 p}{8\eta l}$. The acoustic resistance experienced by the wave is therefore given by $r_A=\frac{p}{\dot{X}}=\frac{8\eta l}{\pi a^4}$. By analogy with the electrical specific resistance, β_E, defined by $r_E=\beta_E\frac{l}{A}$, where l is the length and A is the cross-sectional area of the conductor, it follows that the acoustic specific resistance $\beta_A=\frac{Ar_A}{l}=\frac{\pi a^2}{l}\cdot\frac{8\eta l}{\pi a^4}=\frac{8\eta}{a^2}$.

The *Inertance M* is that property of an acoustical circuit which opposes any change in the volume current, just as the reaction of an inductance does towards any change in the electric current flowing.

Let m be the mass of gas contained within the neck of a vessel, the cross-sectional area of the neck being S. When this mass is subjected to an *excess* pressure p the air particles in the opening of the vessel acquire a velocity $\dot{x}$. Hence the kinetic energy acquired is given by $\frac{1}{2}m\dot{x}^2=\frac{1}{2}m\left(\frac{dx}{dt}\right)^2=\frac{1}{2}\frac{m}{S^2}\left[\frac{d(Sx)}{dt}\right]^2$, *i.e.* K.E.$=\frac{1}{2}\frac{m}{S^2}\left(\frac{dX}{dt}\right)^2$, since the volume displacement $X=Sx$. But by definition $M=\frac{m}{S^2}$, so that for a hollow cylinder of radius a and *effective* length l, the value of M is given by $\frac{\rho\times\pi a^2 l}{(\pi a^2)^2}=\frac{\rho l}{\pi a^2}$, where ρ is the density of the fluid medium.

Actually, the results are generally much more complex, for the resistance and inertance vary with frequency.

One difficulty in using the *equivalent* electrical circuits in acoustics is that all the so-called constants, viz. capacitance, resistance, and self-inductance, generally show a greater variation with frequency than in the electrical circuits.

Acoustic capacitance

The work performed in compressing a medium is stored up in it as potential energy. This energy remains constant only while the dimensions of the body do not change, *i.e.* so long as the external pressure remains unaltered. The property of an acoustical system which tends to oppose any change in the applied *pressure* is known as the acoustical capacitance. Hence, just as the electrical capacitance C_E is defined by $C_E=\frac{Q}{V}$, where Q is the charge on the conductor required to raise its potential by V, so the acoustical capacitance C_A is defined by

$$C_A=\frac{X}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where X is the volume displacement due to an increment of applied pressure p.

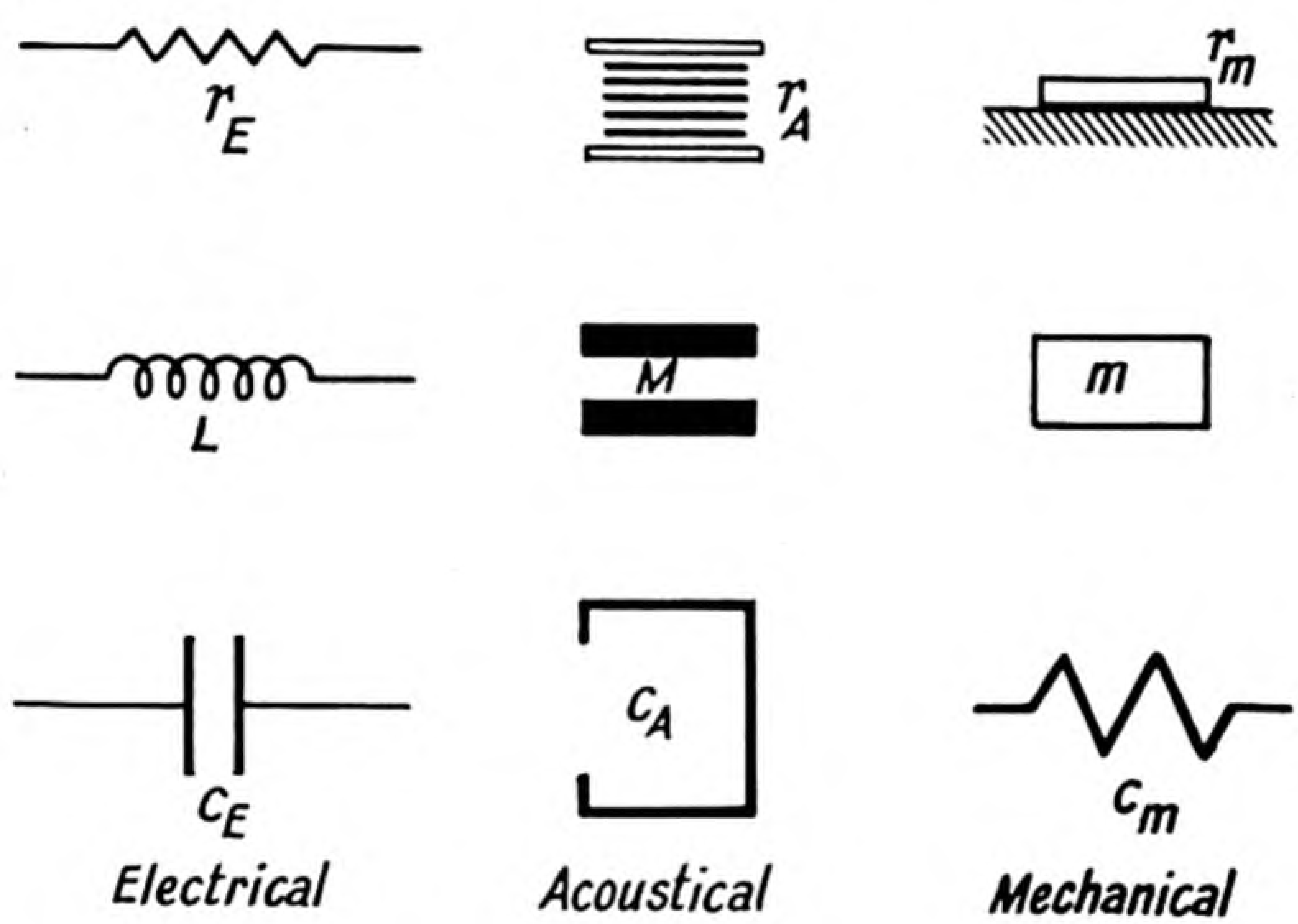


Fig. 15.3.

Now suppose a certain volume v changes to $(v+X)$ as the result of a change of the applied pressure from P to $(P+p)$, then the ratio $\frac{X}{v}$ is defined as the *condensation* s produced in that volume v. But the bulk modulus K of a medium is defined by

$$K=\frac{\text{Stress}}{\text{Volume strain}}=\frac{\text{Applied excess pressure}}{\frac{\text{Change in volume}}{\text{Original volume}}}=\frac{p}{\frac{X}{v}},$$

hence in this case

$$K=\frac{p}{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also the velocity of sound (c) in a fluid medium has been shown (p. 38) to be given by

$$c=\sqrt{\frac{K}{\rho}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

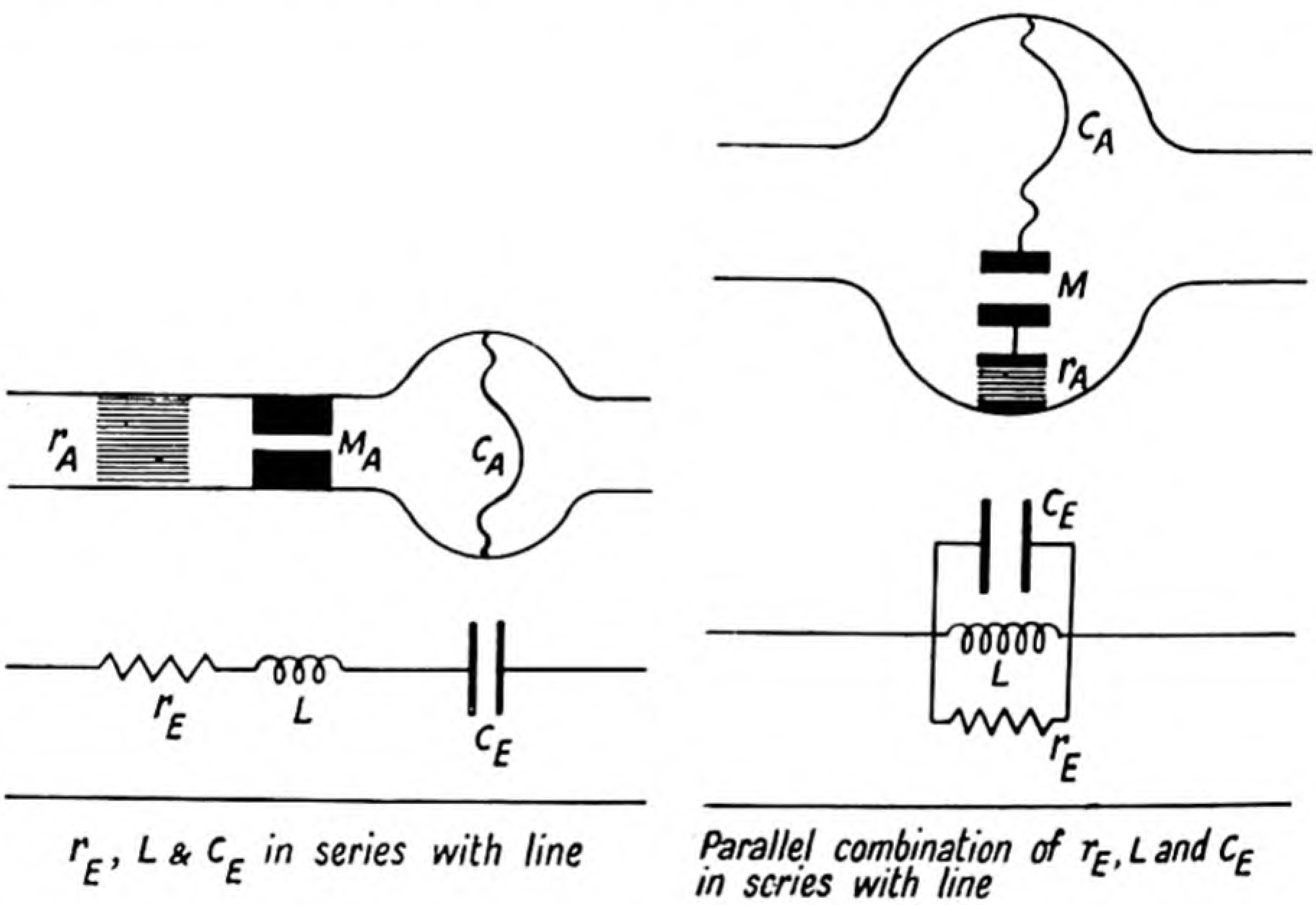
where ρ is the density of the medium. Hence it follows from (2) and (3) that

$$c^2=\frac{K}{\rho}=\frac{p}{\rho s}, \text{ or } p=\rho c^2 s \qquad . \quad . \quad . \quad . \quad . \quad (4)$$

The expression for the acoustical capacitance of a *volume* v of a fluid medium may now be evaluated as

$$C_A=\frac{X}{p}=\frac{sv}{\rho c^2 s}=\frac{v}{\rho c^2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The symbolic representation of the elements of the electrical, acoustical and mechanical systems are shown in Fig. 15.3, which is adapted from the works of Olson and Massa. The acoustical resistance is represented by narrow slits, thus symbolising the energy loss due to the viscous resistance offered by the slits. The capacitance is



(*a*) (*b*)

Fig. 15.4.

shown as a "buffer" volume which acts as a spring or stiffness element, and the mass of fluid in the system stands for the inertance. In the lineal mechanical system, r_m represents energy dissipation due to frictional resistance, m is the mass of the system and C_m is the mechanical capacitance or *compliance*, which is associated with the compression of a spring or similar element. In deriving the equivalent electrical circuit of any mechanical device it is not always immediately evident if a particular compliance should be interpreted as being in parallel or in series with the rest of the circuit. The rule is to represent the stiffness between *two* consecutive *moving* members by a shunt capacitance, and between a moving member and a fixed support by a capacitance in series; in other words, elements which suffer the same displacement are to be placed in series, while those subjected to the same force are connected in parallel. Just as the presence of a series capacitance in an electrical circuit completely impedes the flow

of a *direct* current through the circuit, so by analogy the corresponding acoustical system will not allow a steady flow of gas through it. Following the procedure of Olson and Massa, therefore, the acoustical equivalent of an electrical capacitance in *series* with a line is represented by a "massless" diaphragm, which is considered to be solely controlled by the stiffness of its suspension. The natural resonant frequency of the complete acoustical element is chosen so as to be much higher than the range of frequencies under consideration.

Fig. 15.4*a* shows an acoustic resistance, inertance, and capacitance in series with a line, the elements thus suffering the same displacements, while Fig. 15.4*b* shows the same elements joined together in a parallel combination, which is connected in series with the line. In this latter case the elements are each subjected to the same force. The corresponding electrical elements are shown for each acoustical system.

TABLE OF CORRESPONDING ELECTRICAL, ACOUSTICAL AND MECHANICAL QUANTITIES

Electrical		*Acoustical*		*Mechanical Rectilineal*	
Quantity	*Symbol*	*Quantity*	*Symbol*	*Quantity*	*Symbol*
Self-Inductance	L	Inertance	$*M=\frac{m}{S^2}$	Mass	m
Electrical Charge	q	Volume Displacement	X	Linear Displacement	x
Current	$i=\frac{dq}{dt}$	Volume Current	$\dot{X}=\frac{dX}{dt}$	Linear Velocity	$\dot{x}=\frac{dx}{dt}$
Electromotive Force	$e=L\frac{di}{dt}$	Pressure	$p=M\frac{d\dot{X}}{dt}$	Force	$f=m\frac{d\dot{x}}{dt}$
Electrical Resistance	$r_E=\frac{e}{i}$	Acoustical Resistance	$r_A=\frac{p}{\dot{X}}$	Mechanical Resistance	$r_m=\frac{f}{\dot{x}}$
Electrical Capacitance	$C_E=\frac{q}{e}$	Acoustical Capacitance	$C_A=\frac{X}{p}$	Compliance	$C_m=\frac{x}{f}$

* S is the area over which the mass (m) is distributed.

In an acoustical system the primary interest is the nature of its response to different frequencies, and this characteristic will now be investigated for single elements and simple combinations. Both the acoustical and the corresponding electrical element will be considered, and Fig. 15.5 shows an inductance and an inertance in series with their respective lines. Now the impedance of L is given by $j\omega L$, where $\frac{\omega}{2\pi}$ is the frequency of the alternating current supply, hence the impedance increases with the frequency, so that the higher frequency currents will become considerably attenuated in transmission, as indicated by the graph in Fig. 15.5. Similarly for the acoustic system the constriction in the pipe-line which constitutes the inertance M will offer a greater impedance at higher frequencies, since the impedance of an inertance$=j\omega M$; hence the above curve will also indicate the nature

of the frequency response in the acoustic case. However, if the inductance (or inertance) is connected in parallel with the line (Fig. 15.6), then its impedance, as before, increases with the frequency, but its effect on the energy transmitted down the line will be in an opposite sense to that of the first case considered, for the inductance (or inertance) is now an *alternative* path to the line. It should be

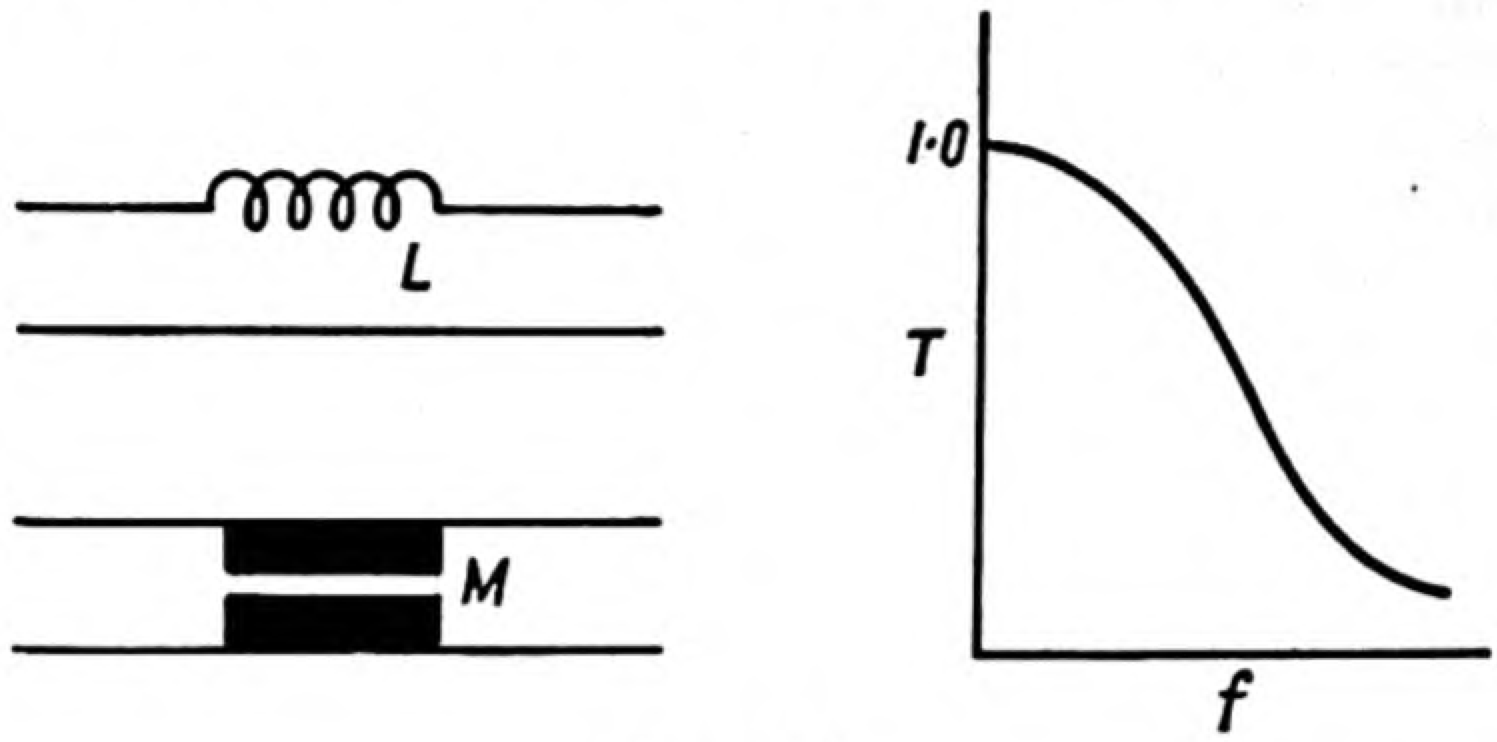


Fig. 15.5.

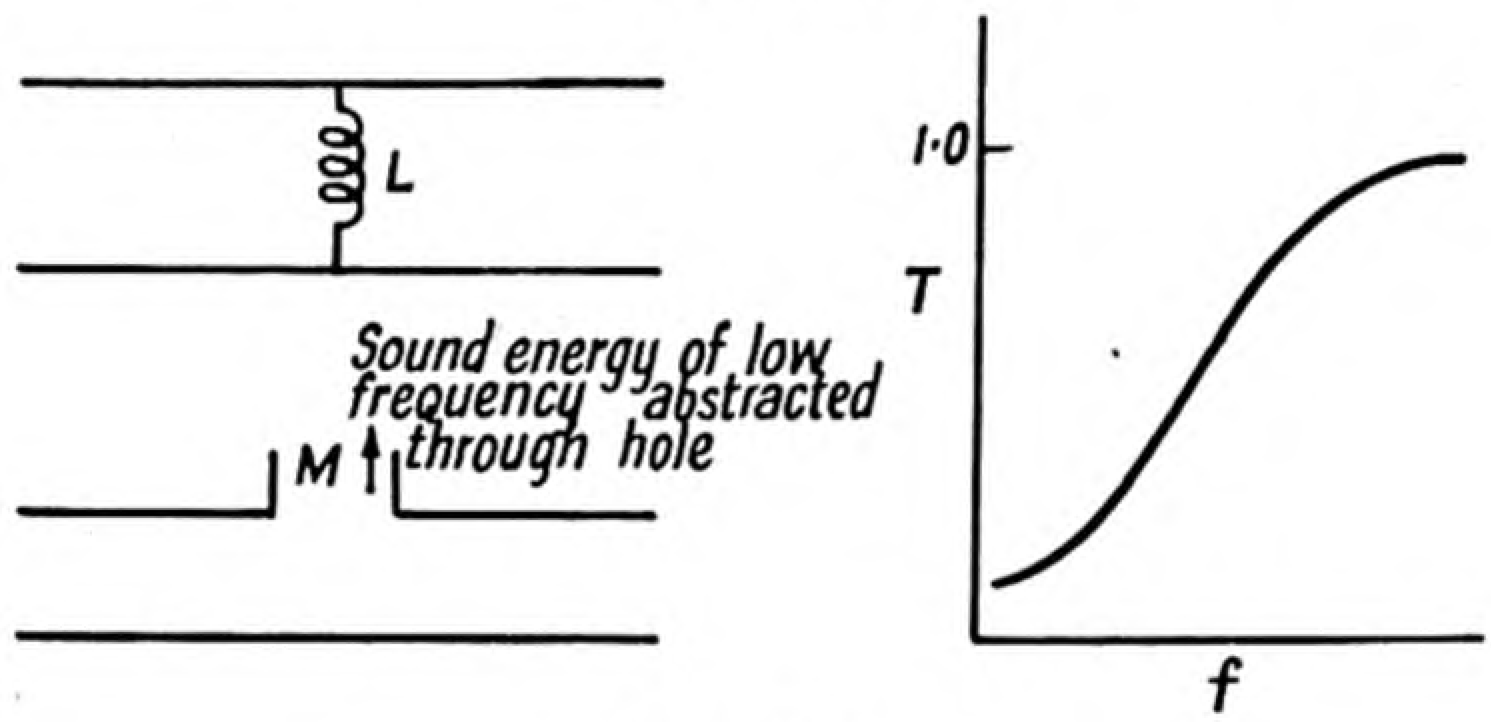


Fig. 15.6.

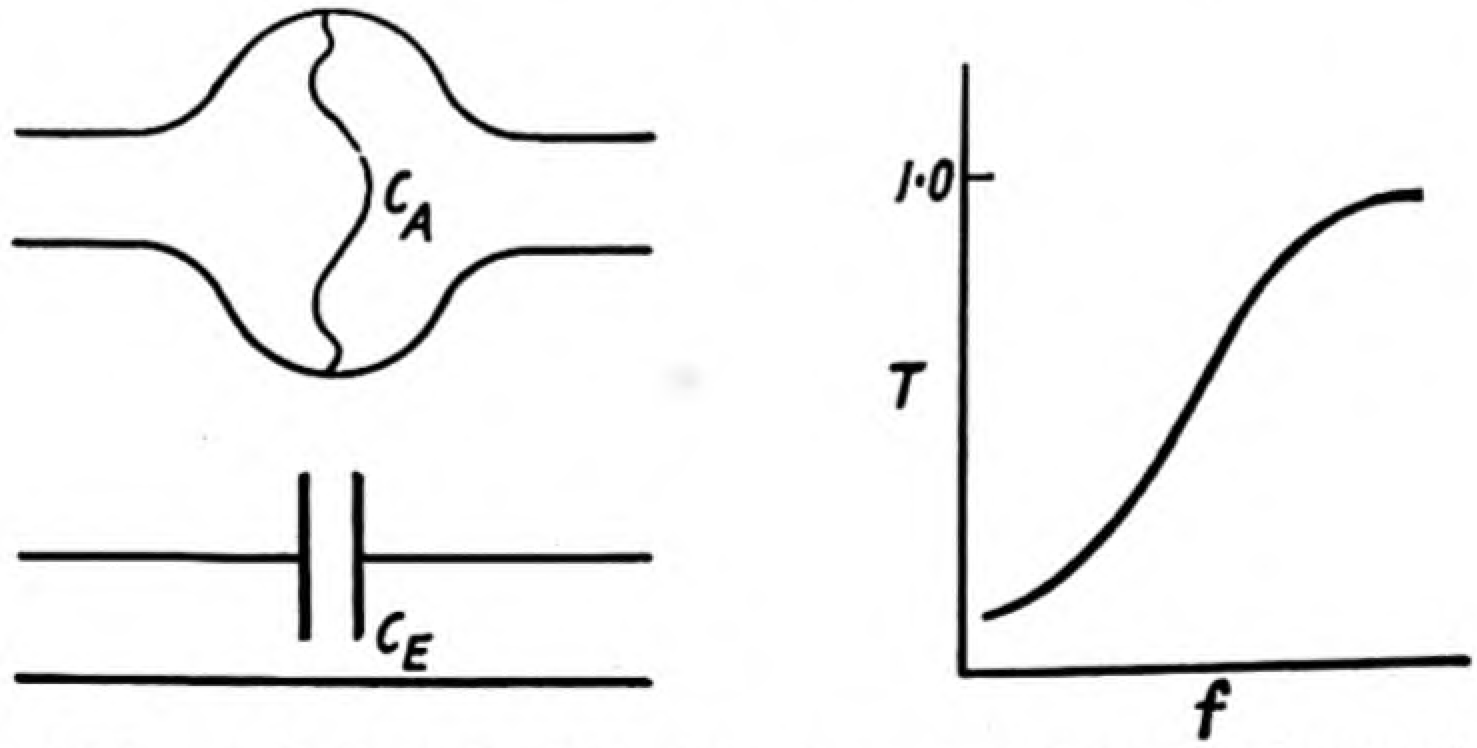


Fig. 15.7. *T* is fraction of incident energy transmitted at various frequencies *f*.

evident, therefore, that the form of the frequency characteristic for both the electric and acoustical systems should be that given in Fig. 15.6. Remembering that the impedance of an electrical condenser, $Z_E=\frac{1}{j\omega C_E}$, and an acoustic compliance, $Z_A=\frac{1}{j\omega C_A}$, both *decrease* with the frequency, and applying the previous reasoning, then the frequency

characteristics of the systems shown in Figs. 15.7 and 15.8 should be self-explanatory.

In order to deduce the reaction of a system of combined elements to electrical (or acoustical) pressures of different frequencies, it is necessary to determine the variation with frequency of their combined

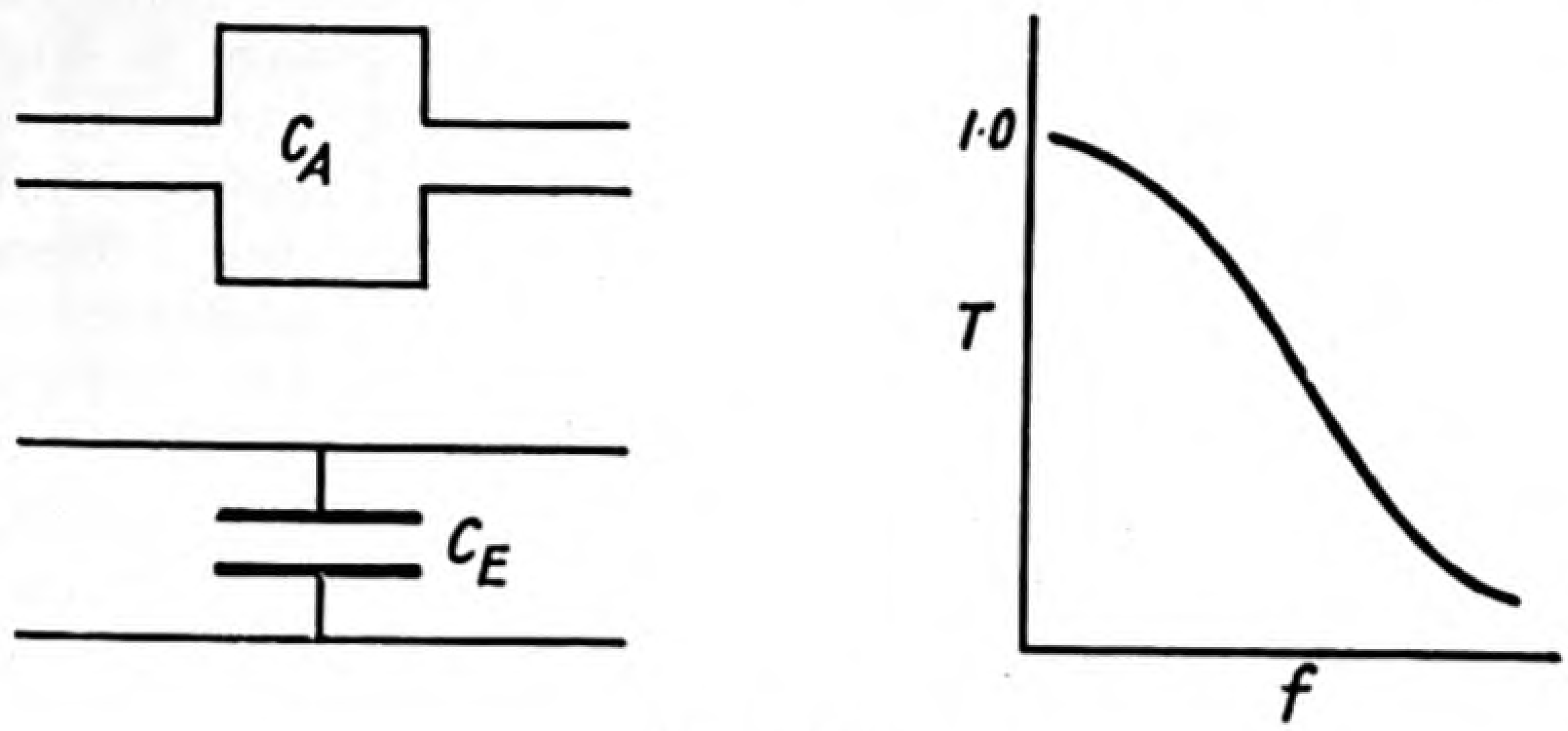


Fig. 15.8.

mpedance. This variation may be conveniently studied by means of a diagram of the type shown in Fig. 15.9, where the reactances of L and C, X_L and X_C respectively, are separately plotted against the frequency. Since, in the simple combination shown, the two reactances are in series, the combined reactance X is obtained by an addition of

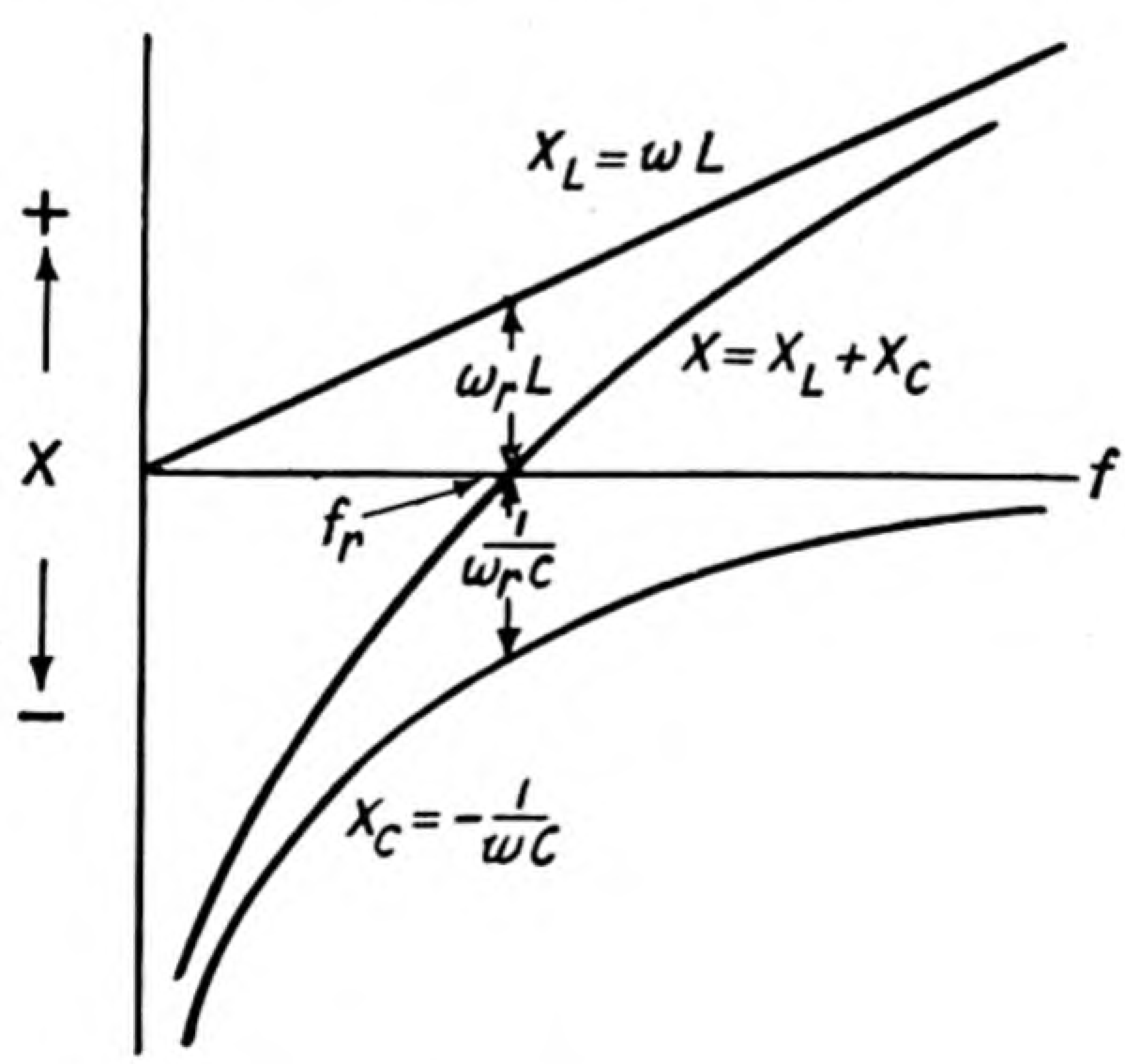


Fig. 15.9.

the ordinates of the separate curves at any frequency. It is seen that at one particular frequency the total reactance is zero, i.e. $\omega_r L-\frac{1}{\omega_r C}=0$, or $\omega_r{}^2=\frac{1}{LC}$; hence this critical frequency f_r is given by $\frac{1}{2\pi\sqrt{LC}}$,

which is the familiar expression for the resonant frequency of a series circuit. In the corresponding acoustical system this particular frequency will be given by $f_r = \frac{1}{2\pi\sqrt{MC_A}}$, and it follows that at this frequency the combined shunt element will act as a short circuit to the line, and therefore, no energy will be transmitted. The frequency response curve will, therefore, take the form shown in Fig. 15.10. Now the use of the above, or other simple combination of elements as a wave-filter, would not provide, in general, the means of obtaining an accurate copy of the desired frequency characteristics. Hence more complicated combined units are designed which are, moreover, repeated a number of times in sequence down the line, in the manner of the loaded string considered previously. The theory of electrical filters has been exhaustively investigated, and this analysis is readily applicable to mechanical and acoustical systems. The ideal frequency characteristics of the four fundamental types of acoustic filter are shown in Fig. 15.11,

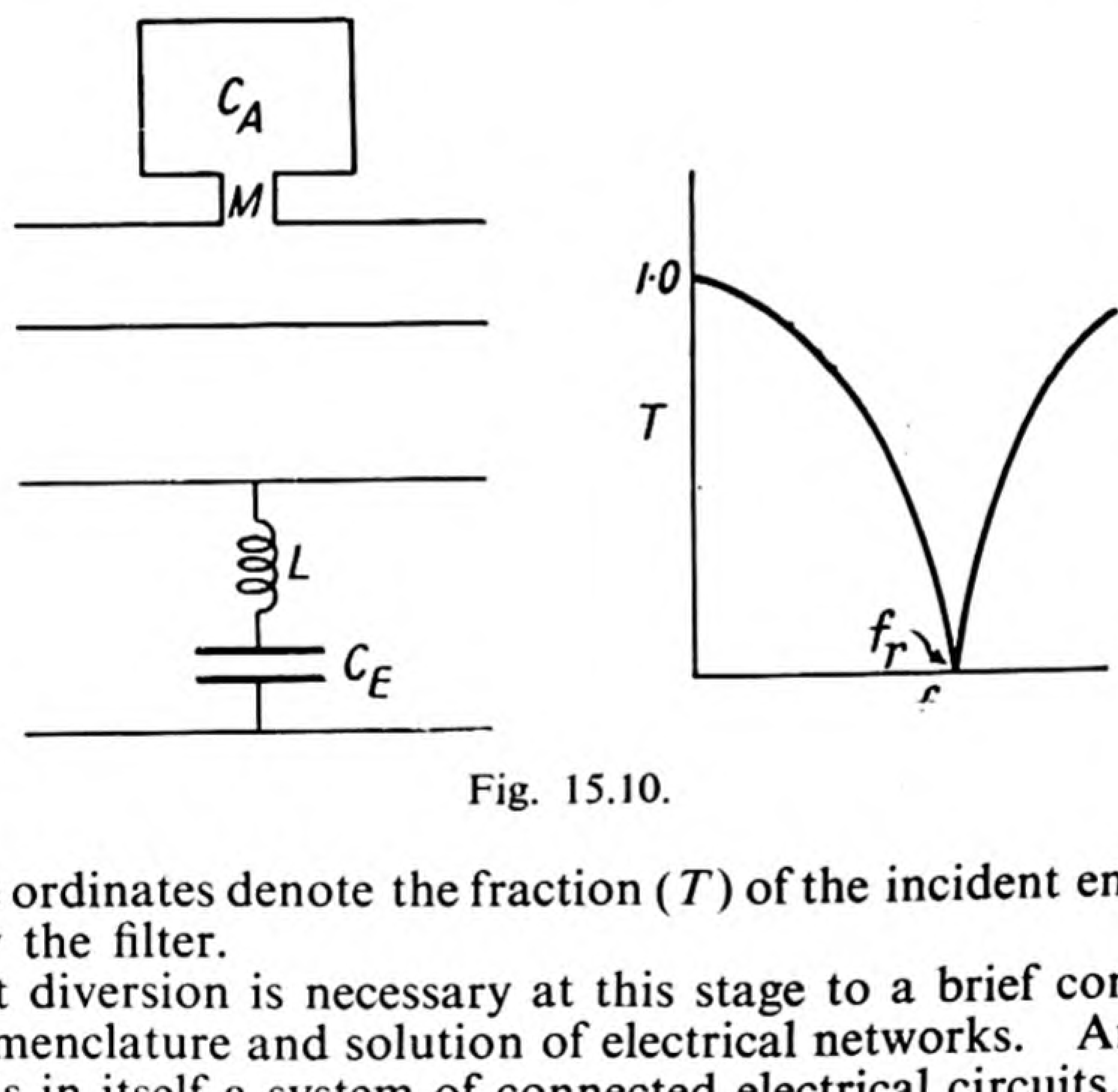


Fig. 15.10.

where the ordinates denote the fraction (T) of the incident energy transmitted by the filter.

A short diversion is necessary at this stage to a brief consideration of the nomenclature and solution of electrical networks. An electrical network is in itself a system of connected electrical circuits, which are known as meshes or branches, and each of these comprises a number of resistive, capacitive or inductive elements. In order to investigate the characteristics of such a network it is, following the method due to Maxwell, considered as a dynamical system in which the electrical currents assume the rôle of *velocities*. Furthermore, on applying an E.M.F. to such a network the effect on the system will be revealed in the form of a number of equations, in which the elements of the circuit will appear as coefficients of the independent variables, *i.e.* of the currents in the various meshes. By way of example, reference may be made to the simple Wheatstone network shown in Fig. 15.12. In this case the independent variables are the three mesh currents x, y and z, and in the familiar Kirchoff's equations derived for the

network, the resistive elements r_1, etc., will appear as the coefficients of these currents. These coefficients are assumed to be constant, as otherwise the mathematical analysis of all but the simplest systems

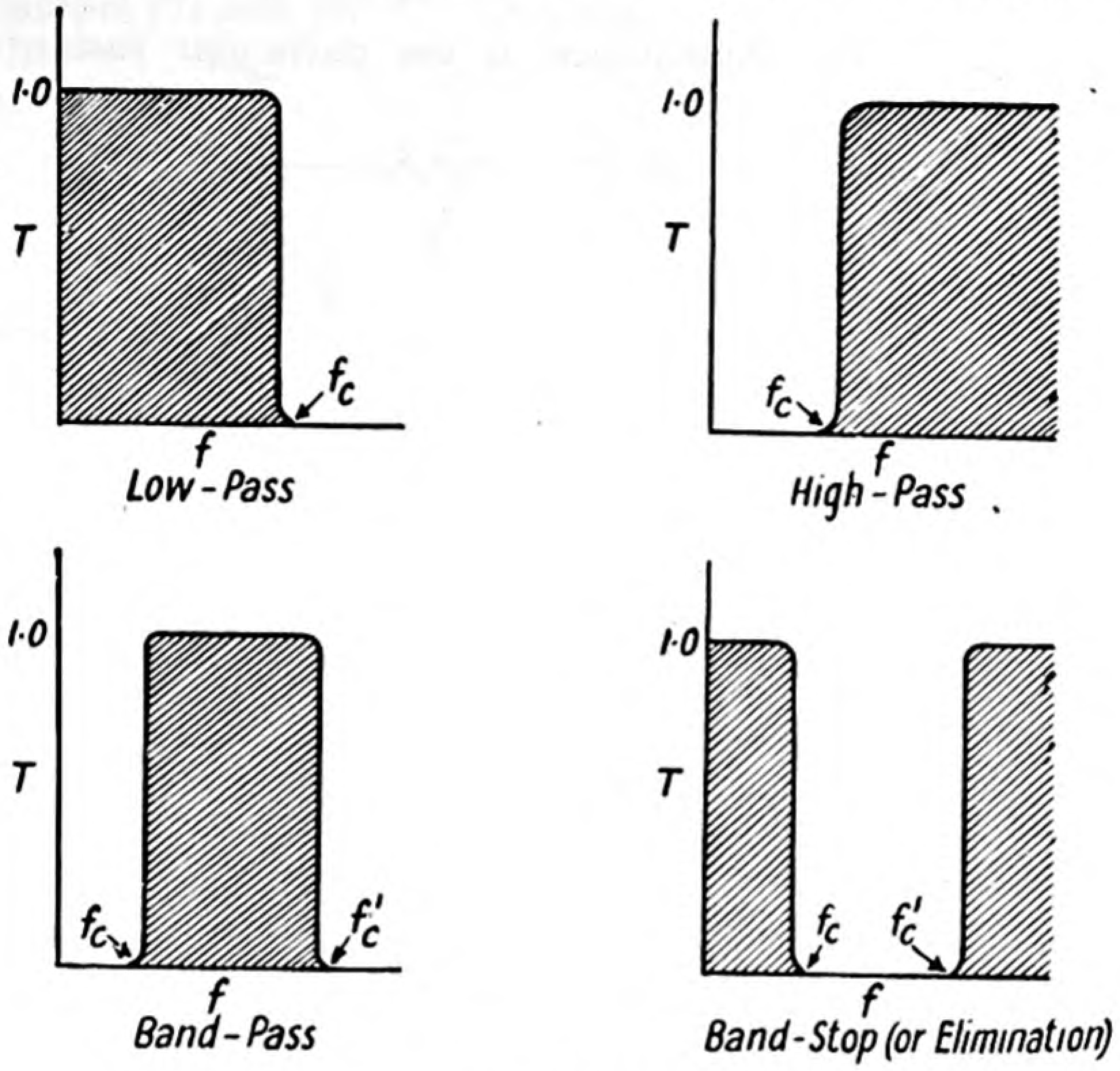


Fig. 15.11.

would be practically impossible. The number of independent variables, *i.e.* electric currents, depends upon the number of degrees of freedom of the system, which in this electrical case is equal to the number of independent meshes, *e.g.* three for the Wheatstone network.

After the above preamble concerning the solution of an electrical network and the various acoustic elements, and how they may be combined to achieve certain desired characteristics, it is appropriate to consider in some detail the theory underlying a particular system. A simple system of one degree of freedom is chosen, and it is represented for all three cases, viz. electrical, acoustical and mechanical, in Fig. 15.13. Attention is firstly directed to the investigation of the electrical circuit, and the results obtained are then interpreted, by analogy, in terms of the elements of the other systems.

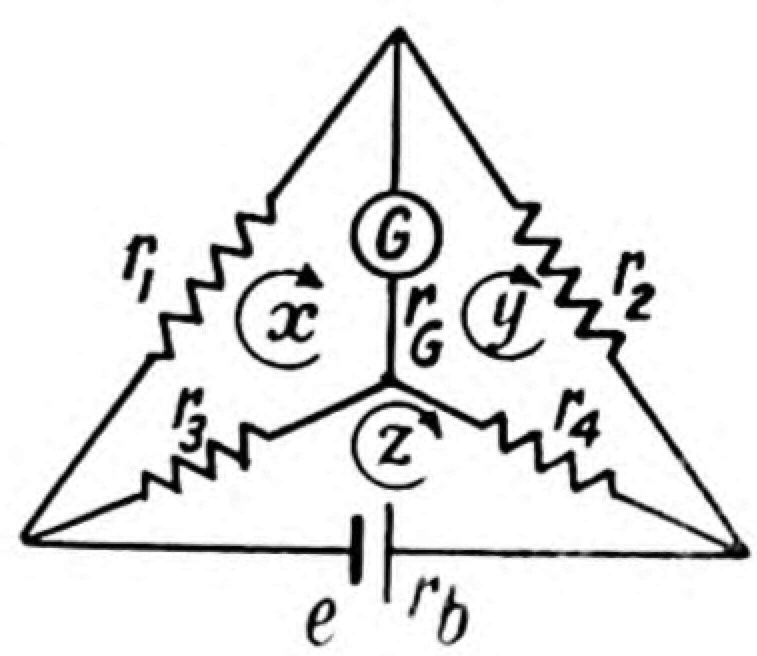


Fig. 15.12.

In the electrical circuit the elements L, r_E and C_E are all in series with the applied periodic E.M.F., $e = e_0 \sin \omega t$, so it follows that the current in *any* element is expressible in terms of *one* variable. Now the *kinetic* energy stored in the magnetic

field of the inductance L at any instant is given by $W_{E.K.}=\frac{1}{2}Li^2$, where i is the instantaneous value of the current, and the *potential* energy stored in the dielectric of the condenser is given by $W_{E.P.}=\frac{\frac{1}{2}q^2}{C_E}$, where q is the charge on the capacitance at the particular instant under consideration.

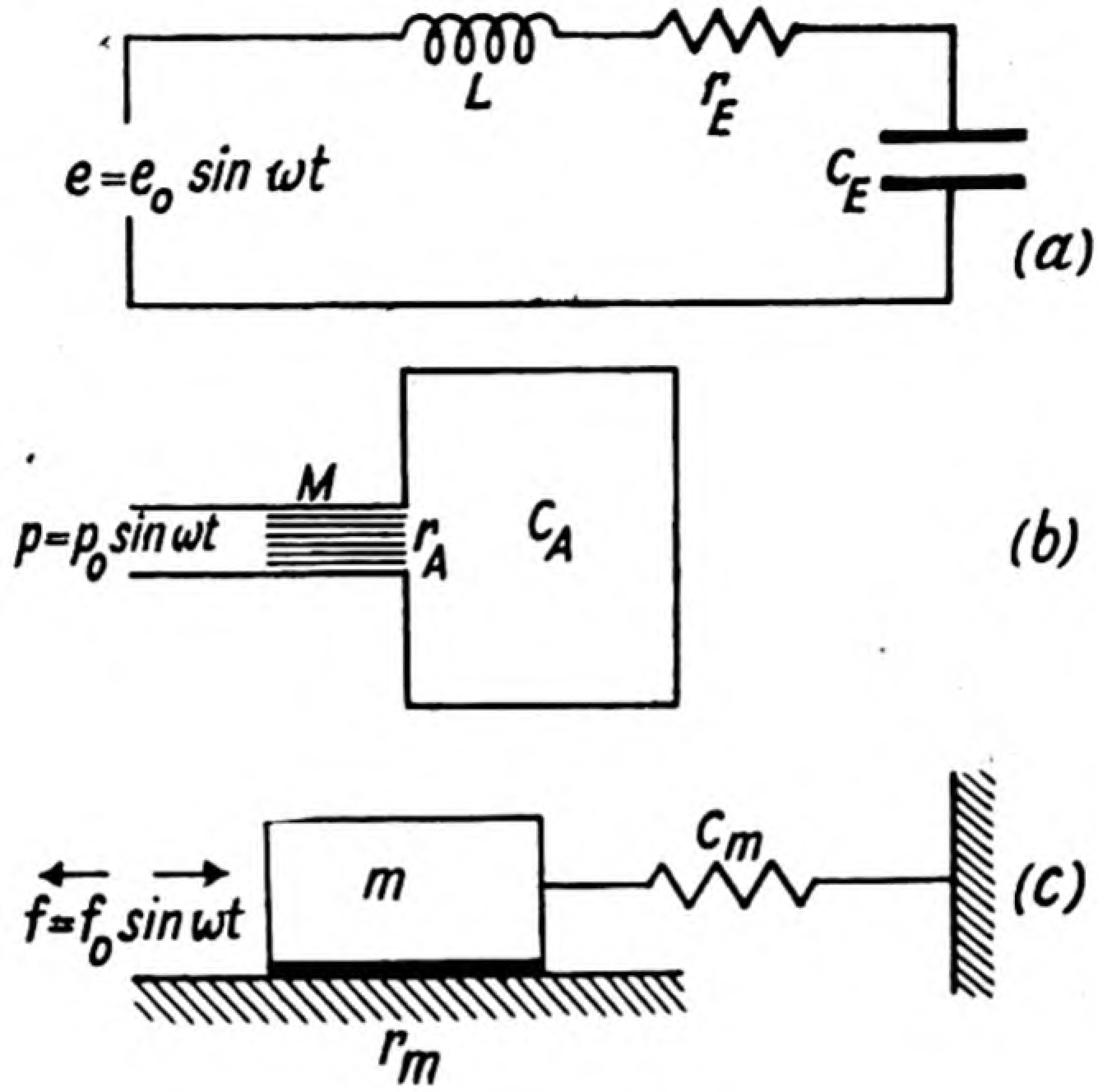


Fig. 15.13.

Hence the total energy stored in the L and C_E elements is

$$W_E=\frac{1}{2}Li^2+\frac{1}{2}\frac{q^2}{C_E}$$

$$=\frac{1}{2}L\left(\frac{dq}{dt}\right)^2+\frac{1}{2}\frac{q^2}{C_E} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and therefore the *rate* at which this energy is stored is given by

$$P_E=\frac{dW_E}{dt}$$

$$=\frac{d}{dt}\left[\frac{1}{2}L\left(\frac{dq}{dt}\right)^2+\frac{1}{2}\frac{q^2}{C_E}\right],$$

i.e.

$$P_E=L\frac{dq}{dt}\cdot\left(\frac{d^2q}{dt^2}\right)+\frac{q}{C_E}\frac{dq}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Again, the *rate* at which energy is being dissipated in the resistance r_E is given by the familiar expression

$$P_{E.R.}=r_Ei^2=r_E\left(\frac{dq}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

But the *power* delivered to the circuit by the applied E.M.F. is $ei=e\left(\frac{dq}{dt}\right)$

$=e_0 \sin \omega t . \left(\frac{dq}{dt}\right)$, and this quantity must be equal to the rate of energy dissipation in the circuit at that instant, i.e. $(P_E+P_{E.R.})$.

Hence from (7) and (8) it follows that

$$L\frac{dq}{dt}\cdot\left(\frac{d^2q}{dt^2}\right)+\frac{q}{C_E}\cdot\frac{dq}{dt}+r_E\cdot\left(\frac{dq}{dt}\right)^2=e_0 \sin \omega t . \left(\frac{dq}{dt}\right),$$

i.e.

$$L\frac{d^2q}{dt^2}+r_E\cdot\frac{dq}{dt}+\frac{q}{C_E}=e_0 \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or written in its alternative notation

$$L\ddot{q}+r_E\dot{q}+\frac{q}{C_E}=e_0 \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The solution of this equation shows (see p. 221) that for a certain frequency the ratio $\frac{i}{e}$, i.e. $\frac{\dot{q}}{e}$, is a maximum, and this resonant frequency in this case is given by

$$f_E=\frac{1}{2\pi\sqrt{LC_E}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

It follows from the analogies already propounded that for the acoustical system the corresponding expression to equation (6) is

$$W_A=\frac{1}{2}M(\dot{X})^2+\frac{1}{2}\frac{X^2}{C_A} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and to equation (8) by

$$P_{AR}=r_A\dot{X}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence the final equation of energy balance is

$$M\ddot{X}+r_A\dot{X}+\frac{X}{C_A}=p_0 \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad (14)$$

By analogy with the electrical problem, the resonant frequency of the acoustical system may be written down immediately as

$$f_A=\frac{1}{2\pi\sqrt{MC_A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Using the previous notation the value of M for the cylindrical neck of the Helmholtz resonator was shown earlier in this chapter to be $M=\frac{\rho l}{\pi a^2}$, and the capacitance C_A for the volume V to be $C_A=\frac{V}{\rho C^2}$.

Hence from the above expression (15) the resonant frequency of such a resonator is

$$f_A=\frac{1}{2\pi\sqrt{\frac{\rho l}{\pi a^2}\cdot\frac{V}{\rho c^2}}}=\frac{c}{2\pi}\sqrt{\frac{\pi a^2}{Vl}}=\frac{c}{2\pi}\sqrt{\frac{S}{Vl}}$$

where S is the cross-sectional area of the neck, and this expression will be seen to be in agreement with the formula developed on p. 164.

In the mechanical system of Fig. 15.13a it is evident that if F_m is the force necessary to produce an extension x of the spring, then by

Hooke's law $F_m = \mu x$, where μ is a constant of the spring (p. 19). Hence $\frac{x}{F_m} = \frac{1}{\mu} = C_m$, which is known as the rectilineal *compliance*, and is the rectilineal mechanical element analogous to the capacitance of the electrical system. The compliance of the spring is a measure of its resistance to compression. It follows from analogy with the electrical system that the resonant frequency of the mechanical system

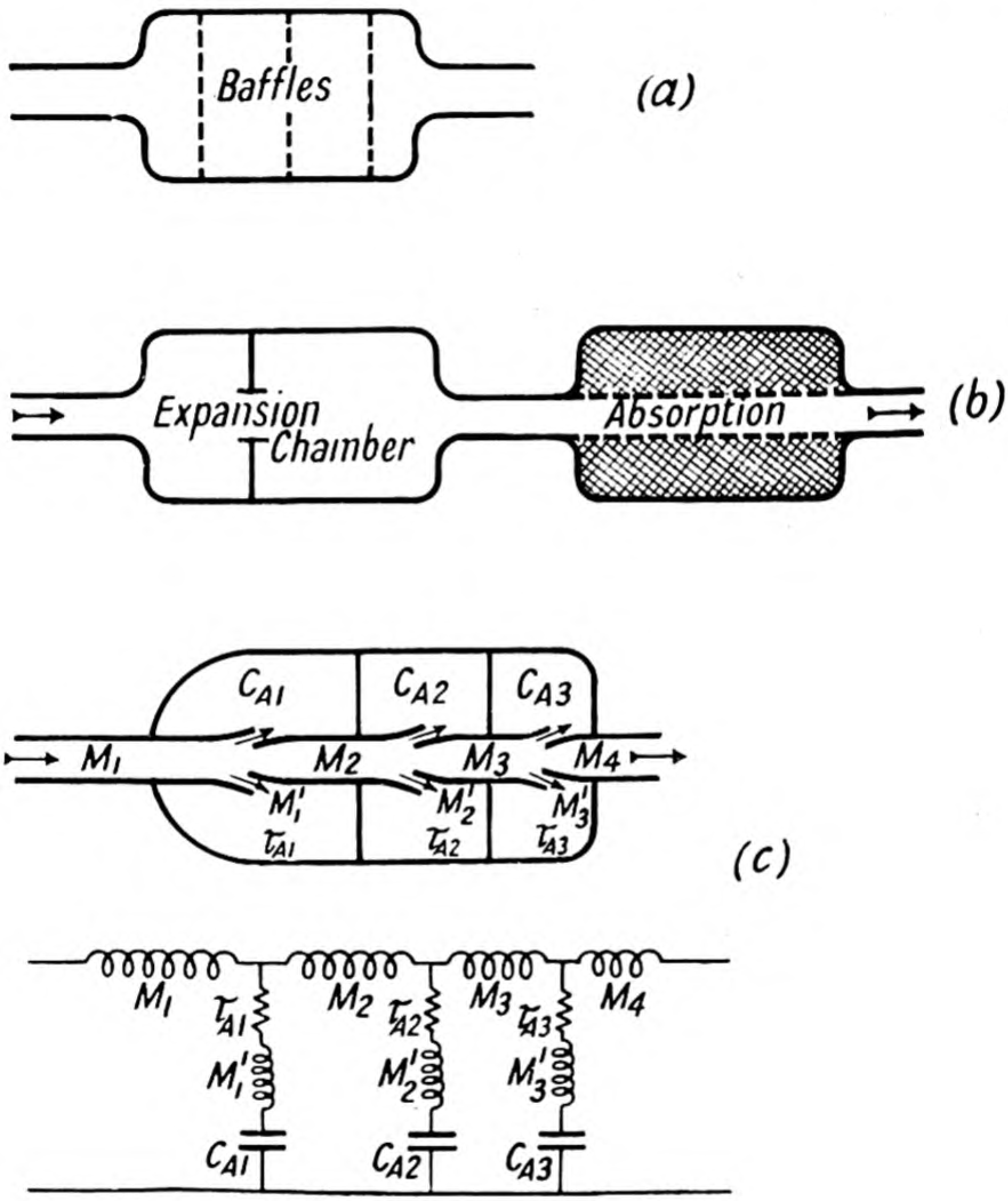


Fig. 15.14.

is $f_m = \frac{1}{2\pi\sqrt{mC_m}}$, where m is the mass of the moving system. If the spring is suspended vertically and the force F_m is supplied by the load, i.e. $F_m = mg$, then $mC_m = \frac{mx}{mg} = \frac{x}{g}$, and the critical frequency $f_m = \frac{1}{2\pi\sqrt{\frac{x}{g}}}$, which is in agreement with the reciprocal of the expression for the *period* obtained on p. 19.

Practical applications of acoustic and mechanical filters

Both acoustical and mechanical filters are becoming of increasing importance in many everyday problems in noise or vibration reduction, such as occur, for example, in air-conditioning systems, etc. An interesting example of the application of *filter theory* arises in connection with the exhaust system of internal combustion engines. In a motor cycle the discharge rate of the exhaust gases into the atmosphere may be anything between 10 and 200 c.p.s., dependent upon the speed of the machine. The low-frequency sounds arising from these discharges are also accompanied by higher-pitched sounds in the frequency range between 2000 and 10,000 c.p.s., the latter being due to the formation of eddies—the exhaust gases in passing through such constrictions as offered by parts of the cylinder. Hence the problem to be overcome in the design of a silencer for such a system involves the suppression, in the main, of two distinct ranges of frequencies. It is essential at the same time that the form of the silencer should not unduly increase the acoustic impedance of the outflow system to low inaudible frequencies, and thus reduce the efficiency of the engine. One of the earliest forms of exhaust silencer, shown in Fig. 15.14*a*, consisted of perforated baffles placed in an enlarged section of the exhaust pipe, and in this case the holes must be sufficiently numerous to avoid appreciable restriction of the gas flow. An attempt to eliminate both high and low frequencies is seen in the composite type of muffler shown in Fig. 15.14*b*. It comprises an expansion chamber fitted with a perforated baffle, followed by an absorption unit consisting of a perforated tube surrounded by glass silk. The dimensions of the expansion chamber, which acts as an acoustic capacitance, are chosen appropriate to the *low*-pitched sound it is desired to suppress, while the function of the glass silk is to *absorb* the *high* frequency components of the sound source. An improved type of silencer designed by the application of the acoustic principles outlined in the chapter, and due to Stewart, is shown schematically in Fig. 15.14*c*. It is essentially a low-pass filter, and the corresponding electrical circuit used in the de sign of such a system is also shown in Fig. 15.14. Historically the apparatus shown in Fig. 15.15, and known as a Quincke's tube, is probably the earliest form of acoustic filter, the dimensions of the side-tube being chosen appropriate to the wave-length of the sound to be eliminated from the source.

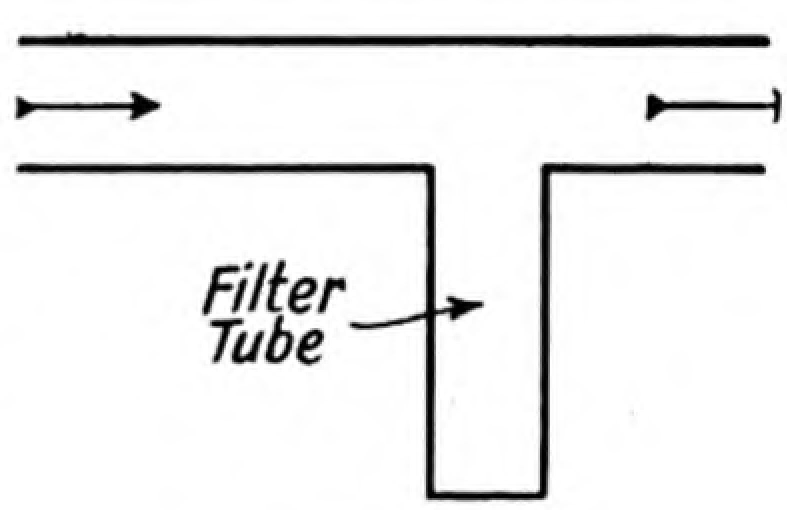


Fig. 15.15.

Isolation of vibrations

The simplest method of isolating any machine or other object m (Fig. 15.16*a*) from vibrations transmitted to it through the building or earth M, is to mount it on an elastic support S (Fig. 15.17*a*). This support is shown schematically in the diagram to possess both elastic and damping properties, defined by C_m and r_m respectively. The latter is represented symbolically by a "dashpot," and comprises the resistance due to air friction and that due to internal friction in the spring.

The external vibrations affecting m are supposed to be periodic of frequency $\frac{\omega}{2\pi}$, and are represented by $F=F_0 \sin \omega t$, where F_0 is the amplitude of the applied force. The electrical circuit corresponding to the non-isolated system is easily seen to be that shown in Fig. 15.16*b*, where L and l are self-inductances corresponding to the masses M and m respectively, and $e=e_0 \sin \omega t$ is the applied E.M.F. The introduction of the elastic element into the mechanical system is simulated by the insertion of C_E and r_E into the analogous electrical

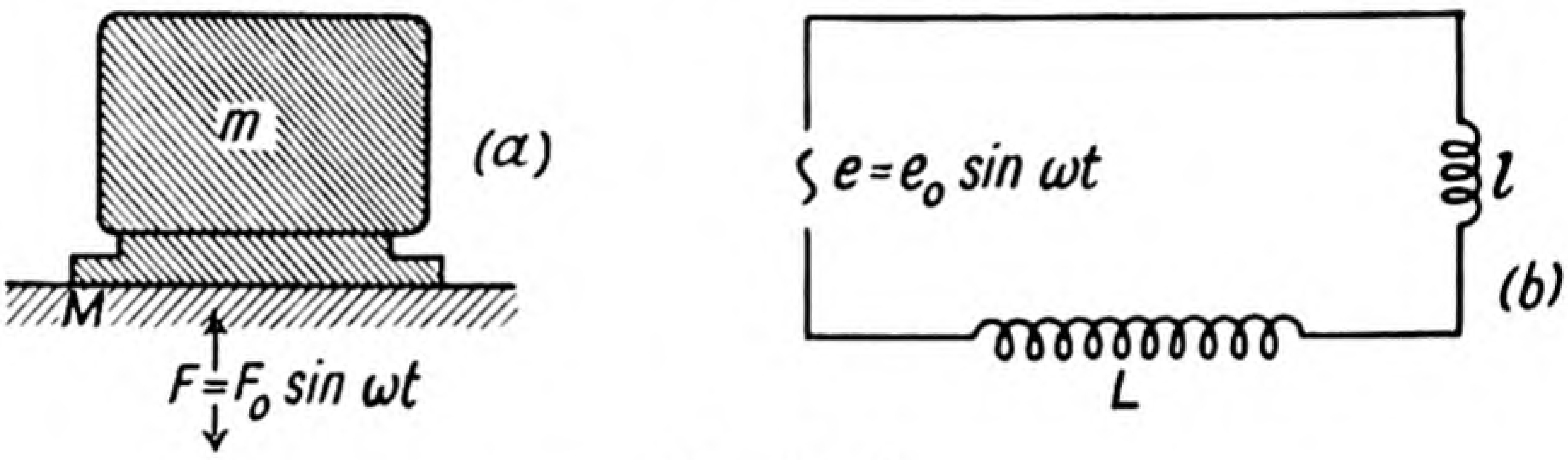


Fig. 15.16.

circuit (Fig. 15.17*b*). The analysis of this circuit is expressed most conveniently in terms of the mesh currents i_1 and i_2, which are the respective currents passing through L and l at any instant.

Let Z_2 be the equivalent series impedance of the circuit elements in the mesh *adcb*, i.e. $Z_2=\sqrt{r_E{}^2+\left(\omega l-\frac{1}{\omega C_E}\right)^2}$; also the series impedance Z_S of C_E and r_E is given by $Z_S=\sqrt{r_E{}^2+\left(\frac{1}{\omega C_E}\right)^2}$. Hence, applying

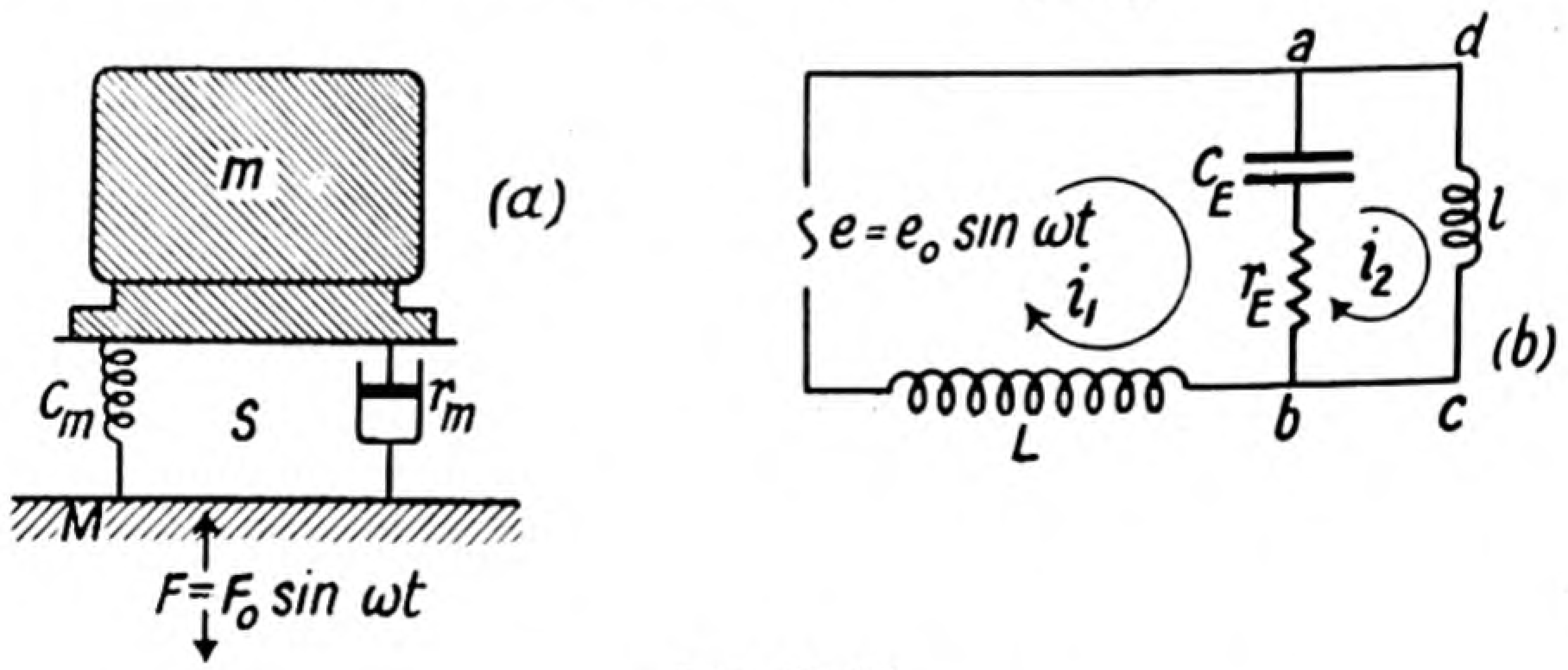


Fig. 15.17.

Kirchhoff's second law to the mesh *adcb* it follows that

$$i_2\sqrt{r_E{}^2+\left(\omega l-\frac{1}{\omega C_E}\right)^2}-i_1\sqrt{r_E{}^2+\left(\frac{1}{\omega C_E}\right)^2}=0,$$

i.e.

$$\frac{i_2}{i_1}=\frac{\sqrt{r_E{}^2+\left(\frac{1}{\omega C_E}\right)^2}}{\sqrt{r_E{}^2+\left(\omega l-\frac{1}{\omega C_E}\right)^2}} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which should be noted is independent of L.

By analogy the ratio of the transmitted amplitude A_2 (*i.e.* the amplitude of m) to the amplitude A_1 of M is given by the ratio $\frac{i_2}{i_1}$, interpreted in terms of the corresponding mechanical elements.

Hence $$\frac{A_2}{A_1}=\frac{\sqrt{r_m{}^2+\left(\frac{1}{\omega C_m}\right)^2}}{\sqrt{r_m{}^2+\left(\omega m-\frac{1}{\omega C_m}\right)^2}} \quad . \quad . \quad . \quad . \quad (17)$$

When the impressed frequency is equal to the resonant frequency $\frac{\omega_0}{2\pi}$ of the insulated system, i.e. $\omega_0 m-\frac{1}{\omega_0 C_m}=0$ or $\omega_0=\frac{1}{\sqrt{mC_m}}$, the

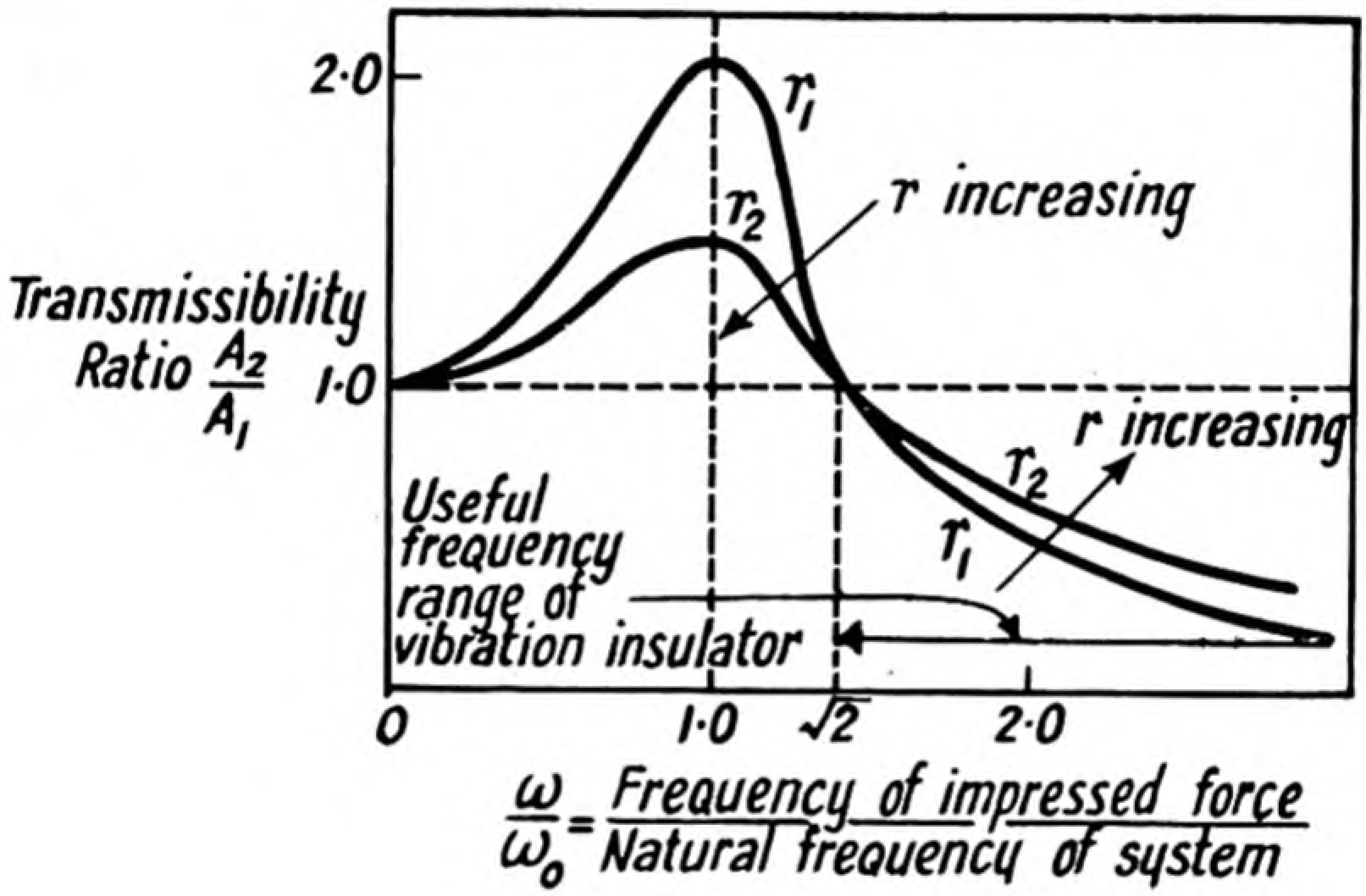


Fig. 15.18.

ratio $\frac{A_2}{A_1}$ is a maximum. The ratio is unity for a particular value of $\omega=\omega_1$ given by the condition

$$\sqrt{r_m{}^2+\left(\frac{1}{\omega_1 C_m}\right)^2}=\sqrt{r_m{}^2+\left(\omega_1 m-\frac{1}{\omega_1 C_m}\right)^2},$$

i.e. when $\omega_1=0$ and for $\omega_1=\sqrt{2}\sqrt{\frac{1}{mC_m}}=\omega_0\sqrt{2}$. (18)

These results are expressed graphically in Fig. 15.18, and show that all impressed vibrations of frequencies greater than $\sqrt{2}\frac{\omega_0}{2\pi}$, *i.e.* $\frac{\omega_0}{\sqrt{2}\pi}$, are attenuated, and hence the above system acts as a low-pass mechanical filter. Since ω_1 should be small for better isolation, it follows from formula (18) that m and C_m should be large, *i.e.* the supported mass should be sufficiently heavy and the elastic support sufficiently compliant, so that the natural frequency of the system is low compared with the frequencies of the vibrations to be isolated.

An inspection of (17) will show that for large values of ω, $\frac{A_2}{A_1}$ approaches the limiting value $\frac{r_m}{\omega m}$ and for greater *efficiency* of isolation, therefore, r_m should not be too large. This fact is evident from the form of the curves in the useful frequency range, shown in Fig. 15.18.

The flexible material to be employed as the vibration insulator will depend, amongst other factors, upon the value of its compliance C_m and its resistance r_m. The compliance may be measured by loading a specimen of the material to the same degree of compression as when used as an insulating support, and then observing the static displacement produced by a steady unit applied force. Expressed symbolically, if x is the depression in centimetres produced by the application of a force of f dynes when the specimen of cross-sectional area A sq. cm. and thickness d cm. is loaded to the working value of P kgm. per sq. cm., then $C_m=\frac{x}{f}\cdot\frac{A}{d}$. This expression may be written as $C_m=\frac{\frac{x}{d}}{\frac{f}{A}}$, which is the *reciprocal* of Young's modulus of elasticity measured under the given conditions. By way of example, if $\frac{x}{f}$ for a specimen of cork, 5 cm. thick and 10 sq. cm. in cross-sectional area, is $0{\cdot}50\times10^{-7}$ cm. per dyne, then the compliance for a cork specimen 1 cm. thick and 1 sq. cm. cross-section is

$$C_m=0{\cdot}50\times10^{-7}\times\frac{A_1}{A_2}\cdot\frac{d_2}{d_1}=0{\cdot}50\times10^{-7}\times\frac{10}{1}\times\frac{1{\cdot}0}{5{\cdot}0},$$

i.e. 10^{-7} cm. per dyne. The internal resistance r_m may be measured, when its value is not large, by suitably loading a specimen of the material and observing the *decay* of the amplitude of the free oscillations of the system after it has been slightly displaced from its equilibrium position. The logarithmic decrement λ of such a motion is shown to be given (p. 222) by $\lambda=\frac{r_m T}{4m}$, where T is the observable time-period of the motion for small damping, so that r_m may be readily calculated. If the resistance of a cork sheet of 10 sq. cm. area of cross-section and thickness 5 cm. is $0{\cdot}22\times10^6$ c.g.s., then r_m for a specimen 1 cm. thick and 1 sq. cm. cross-section is $0{\cdot}22\times10^6\times\frac{A_2}{A_1}\times\frac{d_1}{d_2}=0{\cdot}22\times10^6\times\frac{1}{10}\times\frac{5}{1}$, *i.e.* $0{\cdot}11\times10^6$ c.g.s. An important point in the choice of material which must not be overlooked is a knowledge of the maximum load it will be required to sustain, thus enabling the dimensions of the specimen to be suitably chosen so that it is used within its elastic limit.

Fig. 15.19 shows the methods used in practice for insulating machines from their foundations, the advantage of using the spring as the elastic support instead of a material like cork or rubber, is that its compliance is independent of the load.

However, it is necessary to insert a layer of cork in this case also, as shown, to "damp out" any compressional waves transmitted through the material of the spring.

The transformer

The transformer is a device for the transference of energy between two impedances of different values, without appreciable loss. The most familiar type is the electrical transformer and the theory of this instrument will now be developed. The laws of electromagnetic induction show that if $\frac{d\phi}{dt}$ is the rate of change of flux in the core of the transformer (Fig. 15.20*a*), due to an applied E.M.F. e_P, then the

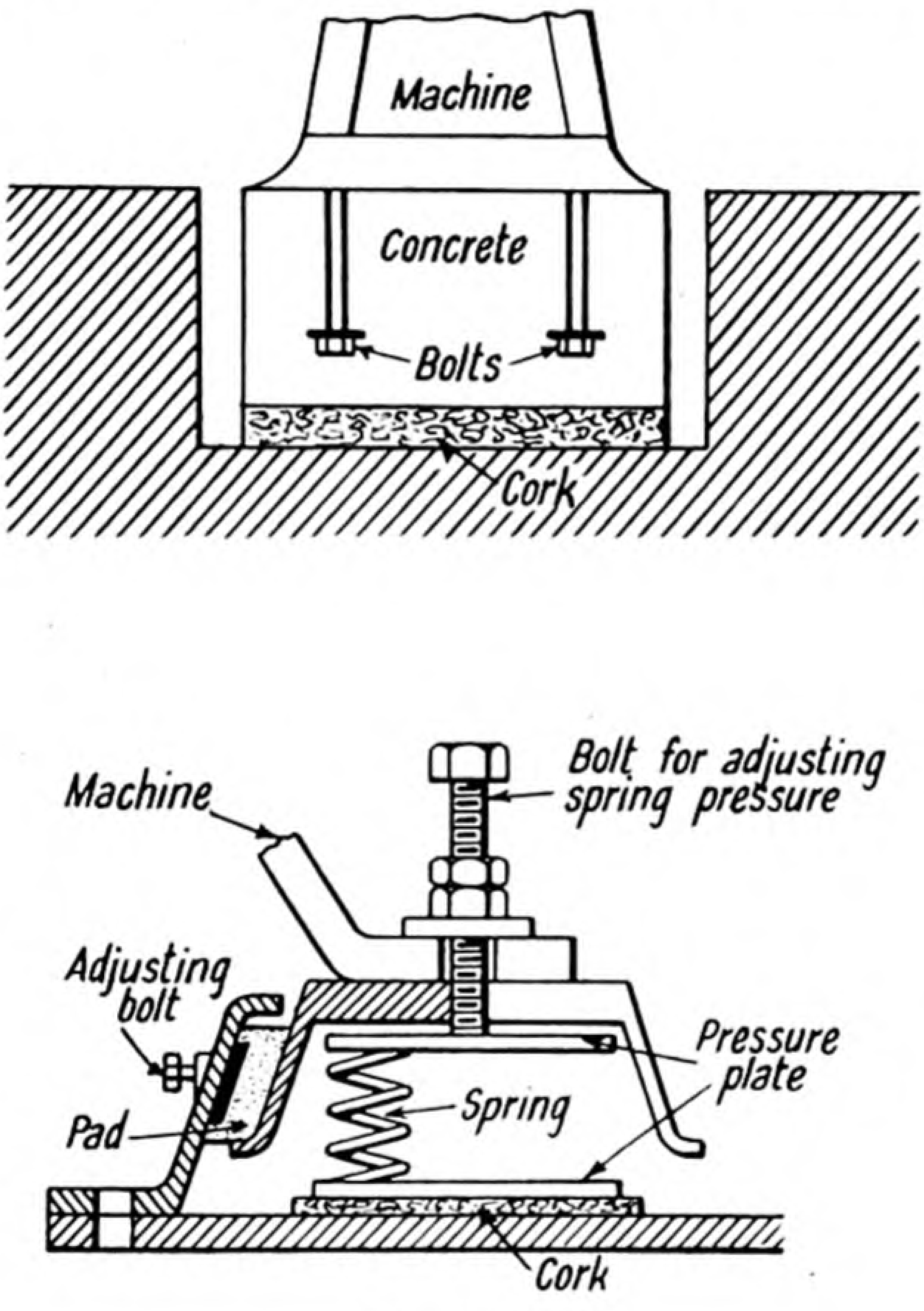


Fig. 15.19.

back E.M.F. induced in the primary winding of N_P turns, and which is balanced by the applied E.M.F., will be given by

$$e_P = -N_P . \frac{d\phi}{dt}.$$

As this flux is identical to both primary and secondary windings, the induced E.M.F. e_S in the latter (of N_S turns) is given by

$$e_S = -N_S . \frac{d\phi}{dt}.$$

It follows that

$$e_S = \frac{N_S}{N_P} . e_P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

In the ideal case there is no loss of energy due to "transformation," and therefore

$$i_P e_P = i_S e_S$$

or

$$i_S = \frac{e_P}{e_S} . i_P = \frac{N_P}{N_S} . i_P \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

Now the impedance of the primary circuit is

$$Z_P = \frac{e_P}{i_P}$$

and that of the secondary

$$Z_S = \frac{e_S}{i_S}$$

hence

$$Z_S = \frac{\frac{N_S}{N_P} . e_P}{\frac{N_P}{N_S} . i_P} = \left(\frac{N_S}{N_P}\right)^2 . \frac{e_P}{i_P} = \left(\frac{N_S}{N_P}\right)^2 Z_P \quad . \quad . \quad . \qquad (21)$$

Now the ideal acoustical transformer has been represented by Olsen as two rigid diaphragms, d_1Od_2 and D_1OD_2 (Fig. 15.20*b*), both of negligible mass and suspension stiffness. Since the combined system is in equilibrium at any instant, the *total* force exerted by the first diaphragm must be equal to the *total* reaction of the second,

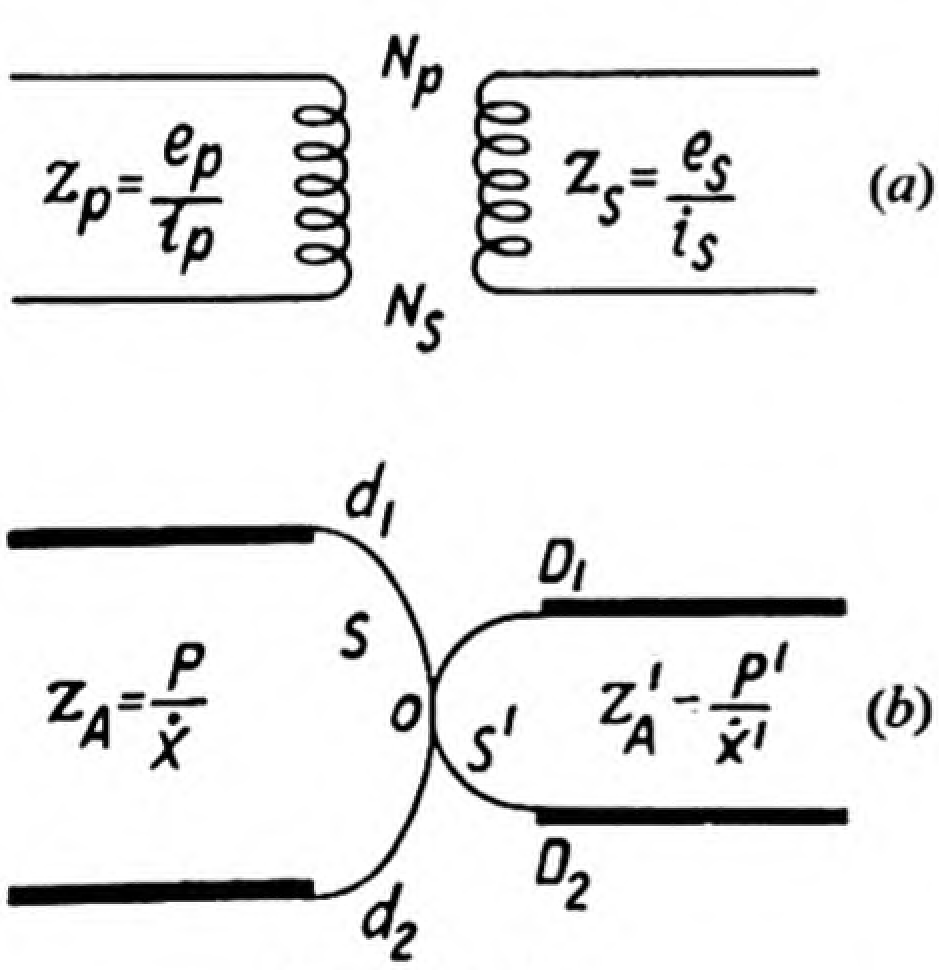


Fig. 15.20.

i.e. $\quad p'S' = pS \quad$ or $\quad p' = \frac{S'}{S} \cdot p$

$$. \quad . \quad . \quad . \qquad (22)$$

where S and S', p and p' are the corresponding areas of, and the pressures exerted on, the two diaphragms. Comparing the relevant formulae (19) and (22) it should be noted that $\frac{1}{S}$ corresponds to N_P.

Again, the air volume current in cubic centimetres per second is S times the velocity of the air particles in the conduit, i.e. $S\dot{x} = \dot{X}$, for the system closed by the first diaphragm. The corresponding quantity for the second system is denoted by $\dot{X}'$, and since the two systems are connected in series, the particle-velocities $\dot{x}$ and $\dot{x}'$ are equal. But $\dot{x} = \frac{\dot{X}}{S}$ and $\dot{x}' = \frac{\dot{X}'}{S'}$, therefore it follows that $\frac{\dot{X}}{S} = \frac{\dot{X}'}{S'}$ or

$$\dot{X}' = \frac{S'}{S} . \dot{X} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

Now the acoustical impedance is defined by the ratio of the excess pressure to the volume current, i.e. $Z=\frac{p}{\dot{X}}$ and $Z'=\frac{p'}{\dot{X}'}$ represent the impedances for the two systems. Hence it follows from (22) and (23) that in the ideal case $\frac{Z}{Z'}=\frac{p}{p'}\cdot\frac{\dot{X}'}{\dot{X}}=\left(\frac{S'}{S}\right)^2$,

or

$$Z'=\left(\frac{S}{S'}\right)^2 Z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

In the purely acoustical transformer action no mechanical element is involved. Thus a horn, by reason of its shape, performs the function of coupling together the relatively high impedance of the driving system to the external gaseous medium of small density in which the sound is to be propagated.

The mechanical analogue of the transformer is the ordinary lever, which transmits and steps up or down the force exerted in one system to that exerted in another. An example of its application in an acoustic system is the hammer and stirrup mechanism of the middle ear (p. 169).

For further reading

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CHAPTER 16

ULTRASONICS

Following present-day usage, ultrasonic radiation is considered to refer to sound waves whose frequencies are higher than the upper limit of human audibility, *i.e.* greater than about 20,000 c.p.s. The term supersonic is now restricted to describe *velocities* greater than that of sound. Recently the term hypersonic has crept into the literature and it refers to phenomena at very high frequencies, *i.e.* 1000 Mc.p.s. or more. Taking the upper limit of frequency for ultrasonic generators to be of the order of 500 Mc.p.s. it means that in air the wave-lengths will lie approximately between 1·5 cm. and $0{\cdot}5 \times 10^{-4}$ cm., while in solids and liquids the corresponding wave-lengths will be respectively about twelve and four times as great. In order to generate this high frequency radiation it is necessary to abandon the possibility of adapting ordinary mechanical vibrators with their inherently lower natural frequencies, although it should be mentioned that very small tuning-forks have been constructed for frequencies up to 90 kc.p.s. A special type of whistle, due to Galton, may also be employed up to this frequency. It consists essentially of a very small resonant chamber situated immediately opposite a jet through which air is forced, the wave-length associated with the air pulses from the chamber being given by $\lambda = 4l + k$, where l is the length of the resonant chamber and k a correction factor dependent on the pressure. An improvement on the Galton whistle, from the point of view of energy output, is the Hartmann generator, shown diagrammatically in Fig. 16.1. Air under a pressure excess of nearly an atmosphere is directed towards a small resonant cavity (diameter d and length l), the opening of which forms a circular knife-edge. Unstable regions of rising pressure are created in the space between the jet and the resonant chamber, which is adjusted to one of these positions by a micrometer screw. When the speed of the jet becomes supersonic, oblique shock waves are produced at the edge of the jet. They are inclined at the Mach angle, being reflected to and fro at the confines of the jet where the moving air and the static air meet, and give rise to the criss-cross appearance. Fig. 16.2 shows this effect, the high density obtaining in the wave-fronts of the shock wave permitting shadow photography to be utilised, as in the case of high velocity bullets (p. 153). This type of ultrasonic generator has a drawback due to the many overtones which accompany the fundamental of the pipe. A recent development in Germany has been the use of a liquid instead of a gas as the jet fluid and one form of apparatus

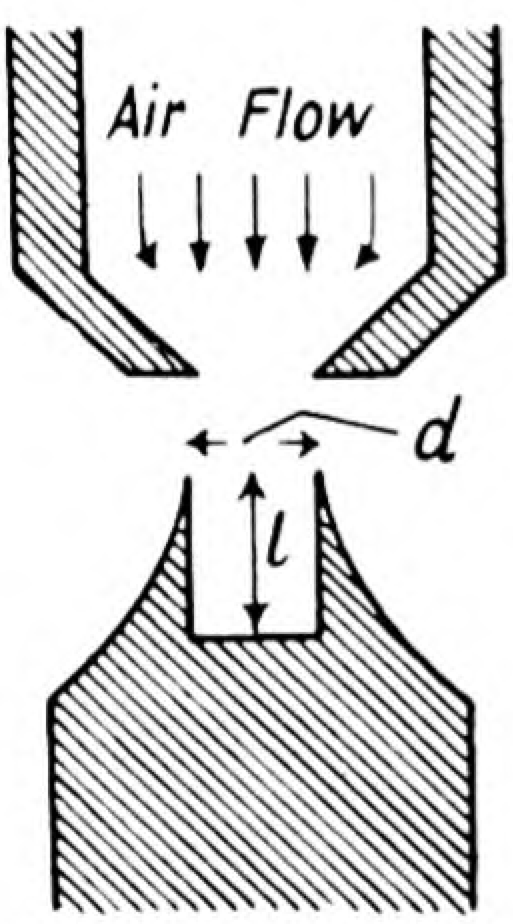
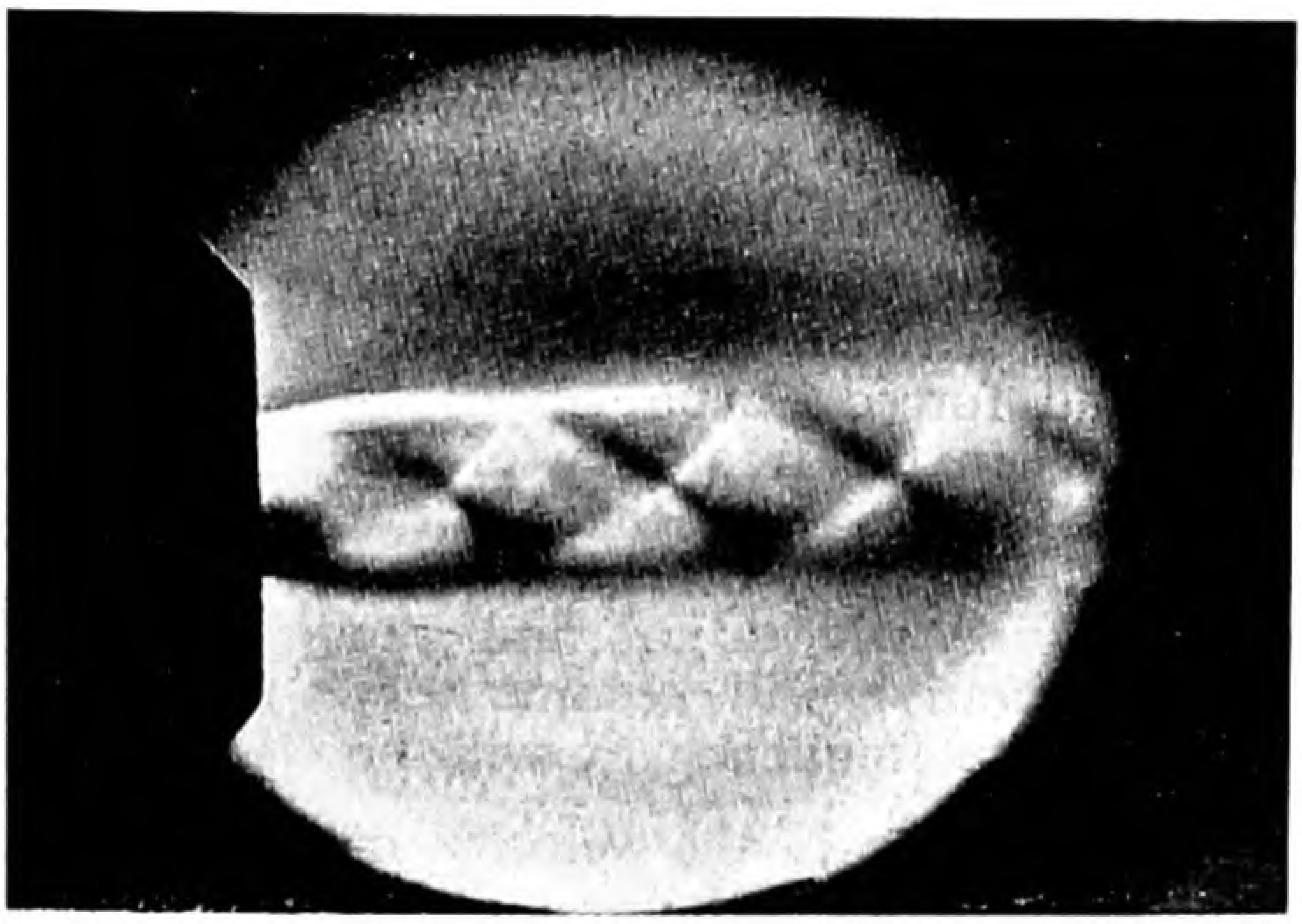


Fig. 16.1.

follows that of the air-jet generator. The jet and resonant cavity are immersed near the bottom of the vessel containing the liquid in which, for example, emulsification is to take place. For a prolonged time of oscillation the liquid may be continually circulated through the whistle by a compression pump. It is usual to make l equal to the diameter d, and for a frequency of approximately 60 kc.p.s. each would be made equal to 1 mm. approximately. An energy output of sound up to 50 W. is reputed to be possible with this generator, and by the substitution of hydrogen for air in the jet a frequency of the order of 0·5 Mc.p.s. may be attained (velocity of sound in hydrogen = 4 times that in air).

The singing arc and electric spark were other methods used for producing high frequency aerial vibrations. The Duddell singing arc, for example, depends for its action upon the fact that in an electric arc the current *decreases* as the potential difference across the arc



[*Hartmann and Lazarus*.

Fig. 16.2. Photograph of an air-jet with a velocity exceeding that of sound.

increases. Hence as regards *changes* of voltage the arc behaves like a negative resistance, so that if it is connected in series with an oscillatory circuit (Fig. 16.3) containing an inductance L and capacity C, the positive reactance of the latter may be neutralised and the circuit become oscillatory. The frequency (f) of the oscillations will be given by

$$f = \frac{1}{2\pi\sqrt{LC}}.$$

The original experiments of Duddell were restricted to audio-frequencies, but the method was later adapted by Poulsen as an ultra-high frequency source, although his primary purpose was to use it as a transmitter of electromagnetic waves. It was not until the first World War, however, that sources were developed by Langevin in which a dependable control of intensity and frequency could be affected. This class of controllable generator is, in general, a suitably shaped solid body which is so excited that it oscillates in its natural

elastic-periods. The excitation may be produced by electrostatic fields or electromagnetically. In the former case it is dependent on the phenomenon of piezo-electricity, which is a property associated with certain crystals and other orientated materials. When a mechanical force is applied to these crystals in certain directions, electrostatic stresses are, in general, produced in other directions, and are made evident by electric charges developed on the end faces *perpendicular* to these directions. The amount of the electric charge produced is proportional to and *changes sign* with the applied mechanical stress; in contrast to this "direct" piezo-electric effect there is a "converse" effect whereby the crystal may be mechanically strained if subjected to a suitable electric field. The mechanical strain also changes sign with that of the applied E.M.F., so that if the latter is alternating, the plate will alternately expand and contract with the same frequency as the supply, and forced elastic vibrations will be set up. When the frequency of the alternating P.D. coincides with that of the natural mechanical frequency of vibration of the crystal, resonance will occur, and the amplitude of the vibration will become very large. This resonance frequency f_r, will be given by $f_r = \frac{v}{2d}$, where v is the velocity of propagation of longitudinal waves in the direction of the thickness (d) of the quartz, and will depend on the orientation of this direction with respect to the crystallographic axes.

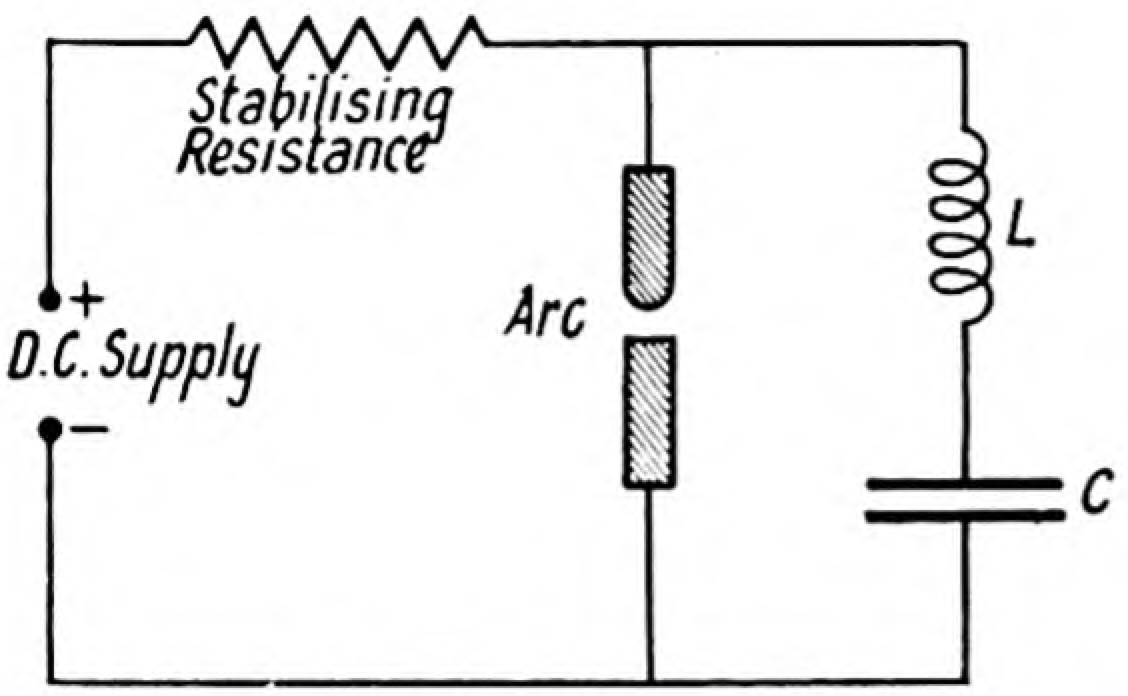


Fig. 16.3.

The crystals chiefly used are quartz and Rochelle salt; the latter shows a much greater piezo-electric effect, and is utilised in microphones, etc., but the quartz has the advantages of greater stability and smaller response to changes of temperature. Quartz belongs to the trigonal system of crystals, and usually occurs in the form of hexagonal pyramids, and the line ZZ (Fig. 16.4) is known as the principal axis and any parallel direction defines an *optic* axis; no electric polarisation is produced by mechanical stresses along this direction. A hexagonal cross-section of a quartz crystal obtained by taking a plane perpendicular to the optic axis is shown in Fig. 16.5, the axes X_1OX_1, X_2OX_2 and

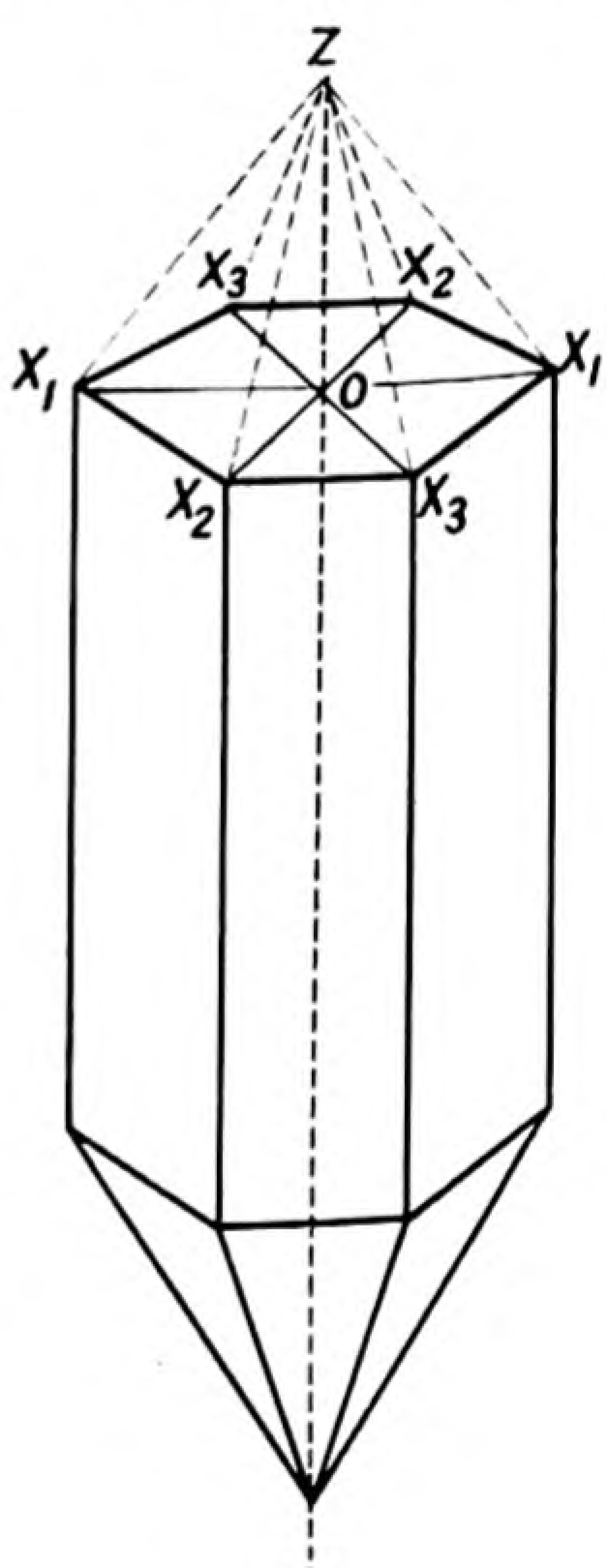


Fig. 16.4.

X_3OX_3 passing through the corners of the hexagon being known as the electric axes, whereas the axes Y_1OY_1, etc., which are normal to the faces of the crystal are referred to as the *mechanical* or *third* axes. If a mechanical force is applied to the quartz specimen in the direction $X_3'OX_3$ (Fig. 16.6), equal and opposite charges are developed on faces *AB* and *DC*, and are expressed by the law $P=0{\cdot}16\eta$, where P is the total electric charge developed in coulombs per square metre, and η is the applied mechanical strain. A tensile force applied along the third axis $Y_3'OY_3$ produces charges of the same sign on corresponding faces as a compressive force along the electric axis $X_3'OX_3$, and the magnitude of the charges will be equal if the mechanical stresses are the same in the two cases. The law for the "converse" piezo-electric effect is expressed by $e=2{\cdot}15\times10^{-12}X$, where e is the strain in $X_3'OX_3$ direction, and X is the applied electric field (in $X_3'OX_3$ direction) given in volts per metre.

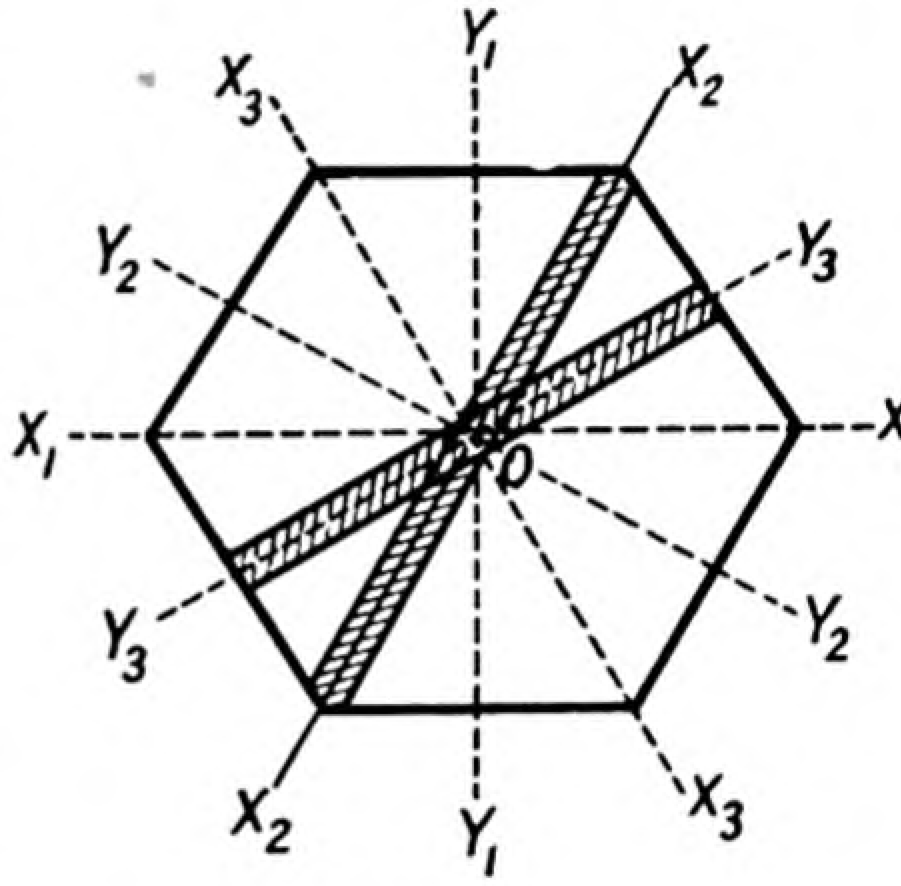


Fig. 16.5.

The mechanism of the operation of a crystal in an oscillatory circuit is very similar to that of a tuned electrical circuit, such as is shown in Fig. 16.7, where K represents the capacitance between the electrodes of the crystal when *quiescent*, and the inductance L_1, capacitance C_1 and resistance R_1 represent respectively the *electrical equivalents* of the effective mass, resilience and frictional loss of the crystal when in *vibration*. If the appropriate frequency of supply is applied between a and b the reactances of L_1 and C_1 will together be equal to that of K, and thus form a parallel resonant circuit. L_1 and C_1 can also form a series resonant circuit, the frequency being slightly lower than that of the parallel resonance which is quoted as the fundamental frequency of the crystal. The "Q" (see p. 234) of a vibrating quartz crystal* may be between 10,000 and 15,000, which is much larger than can be obtained with any ordinary tuned circuit. This high value of "Q"$=\frac{2\pi fL_1}{R_1}$, where f is the resonant frequency, arises from the fact that its effective mass (L_1) is very large compared with the frictional loss (R_1). The method of

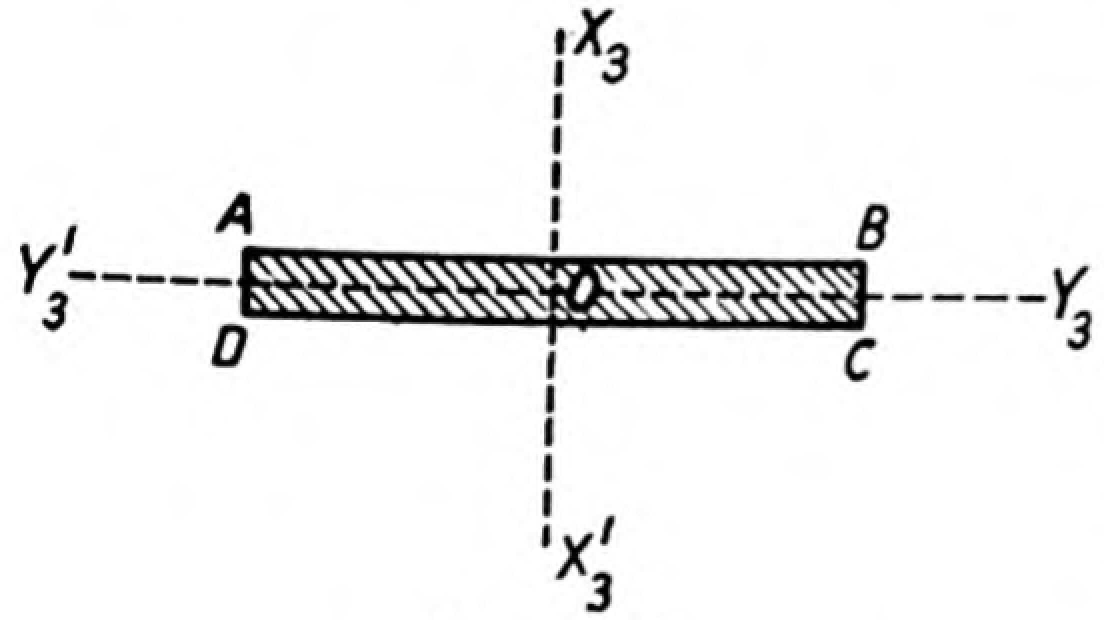


Fig. 16.6.

* The "Q" of a quartz crystal used in an electrical wave-filter may be 10^5 or greater, and Van Dyke by carefully etching the surface of crystal to remove surface cracks and by suspending it in a vacuum obtained a "Q" of 6×10^6.

incorporating such a crystal in a valve circuit for the purpose of generating ultrasonic vibrations is shown by the simple Pierce circuit in Fig. 16.8. It is "tuned-grid–tuned-anode" arrangement, using a triode in which the former part is supplied by the crystal itself, the two portions of the circuit being coupled electrostatically by the plate-grid electrode capacitance which, if necessary, may be increased by the insertion of a very small capacitance C' between grid and plate.

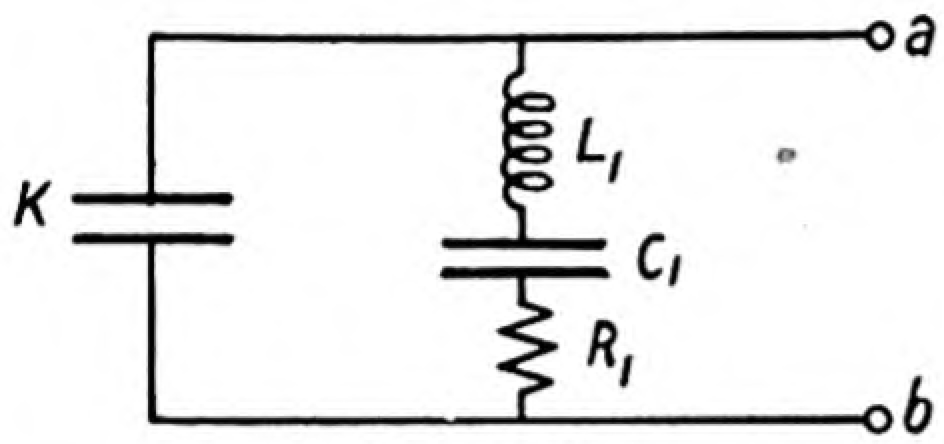


Fig. 16.7.

Suppose there is a small change in the grid voltage, then since this provides the electrostatic field across the crystal it will give rise, as a result of the "converse" piezo-electric effect, to a small change in the dimensions of the crystal. The "direct" effect will now operate and affect the potential of the grid, which in turn will produce a change in the anode current, and hence in the anode potential, if the impedance of the LC circuit is high at the resonant frequency of the quartz. This change in anode potential is communicated to the grid by virtue of the inter-electrode capacitance and provided the phases of the various changes are favourable the "converse" piezo-electric effect will operate in the sense of maintaining the elastic vibrations of the crystal. The response, as measured by the "Q" of the crystal at resonance is extremely high, so that the frequency of the oscillations in the circuit will correspond to that

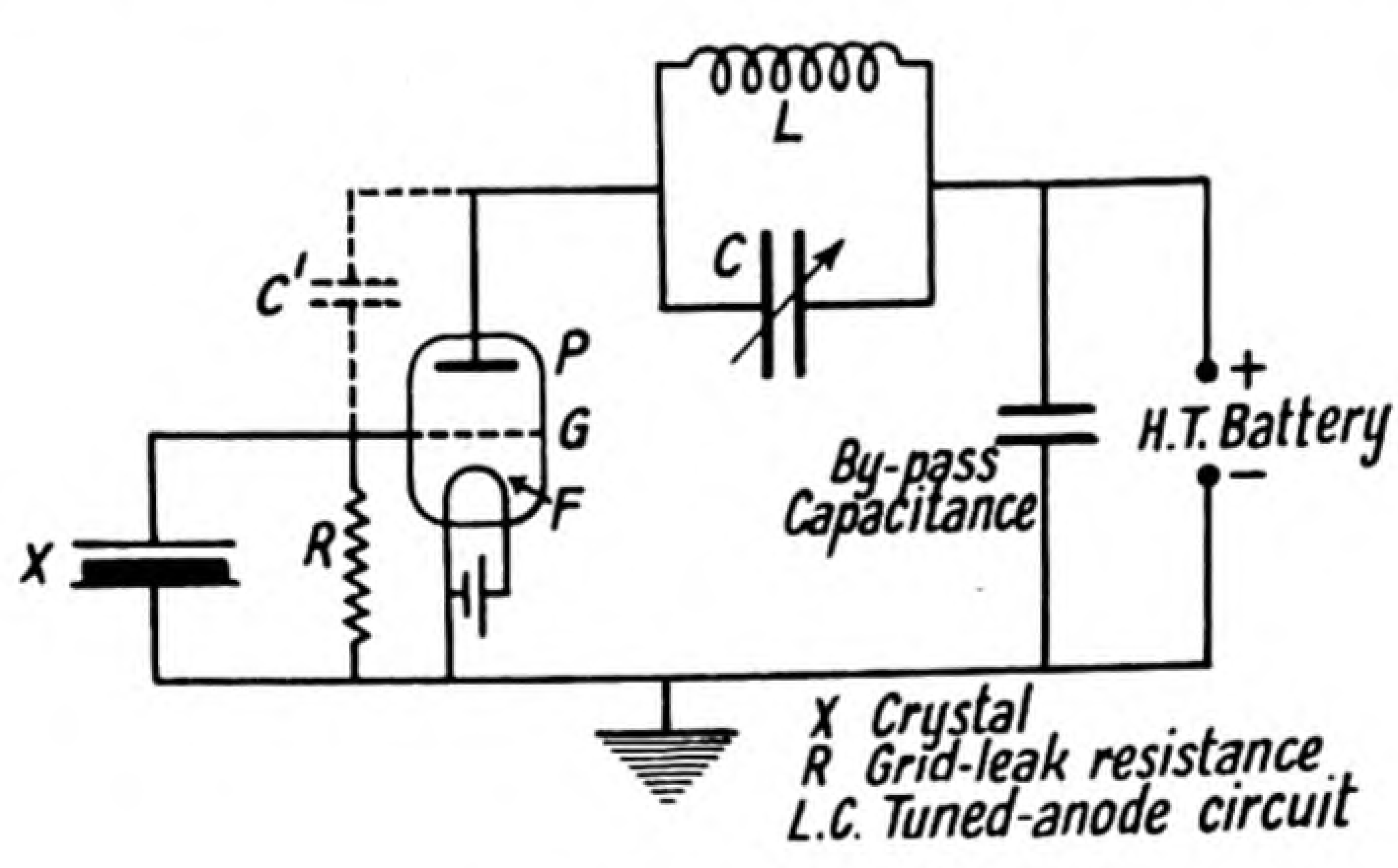


Fig. 16.8.

of the natural frequency of the crystal, even if the tuned plate circuit is slightly off-tune; the frequency of a crystal oscillator is, in consequence, extremely stable.

The modes of motion of a quartz crystal may be quite complicated, for even if only the three basic types of motion are considered, viz. extensional, flexural, and shear, there is the possibility of coupling between these various types. In practice the resultant motion of a bar or plate is almost completely determined by its dimensions and the specific type of wave generated, and is little influenced by the driving

system, if the latter is only loosely coupled to the crystal. An example of how this intra-type coupling can occur is given by Fig. 16.9 which shows the motions of a thick vibrating bar and the similarities between even order shears to odd order flexures and of even order flexures to odd order shears.

When it is required to radiate sound waves from as large a surface as possible, as, for example, in submarine detection work, use may be made of the fact that the velocity of sound in steel is approximately equal to that in quartz. By sandwiching a mosaic of quartz crystal plates between two thick steel plates, to which the crystals are cemented, the equivalent of a single oscillator is obtained whose frequency will be given approximately by $c/2t$, where t is the total thickness of the sandwich and c is the velocity of sound in the steel (or quartz). This type of transmitter was first described by Langevin.

A quartz crystal oscillator provides a most valuable means for calibrating test oscillators by using the harmonics of the crystal, which will be excited at the same time as, but to a much weaker intensity than, the fundamental. The output of the test oscillator, *e.g.* signal generator, is fed into a small coil as loosely coupled as possible to the inductance L of the tuned-anode circuit of the crystal oscillator (Fig. 16.8) and the frequency adjusted until there is zero beat note with a particular harmonic; this condition being detected by a third loosely coupled circuit containing headphones. If a large number of higher harmonics are required care has to be taken when increasing their intensity that the safe current through the crystal is not exceeded, otherwise it will be fractured.

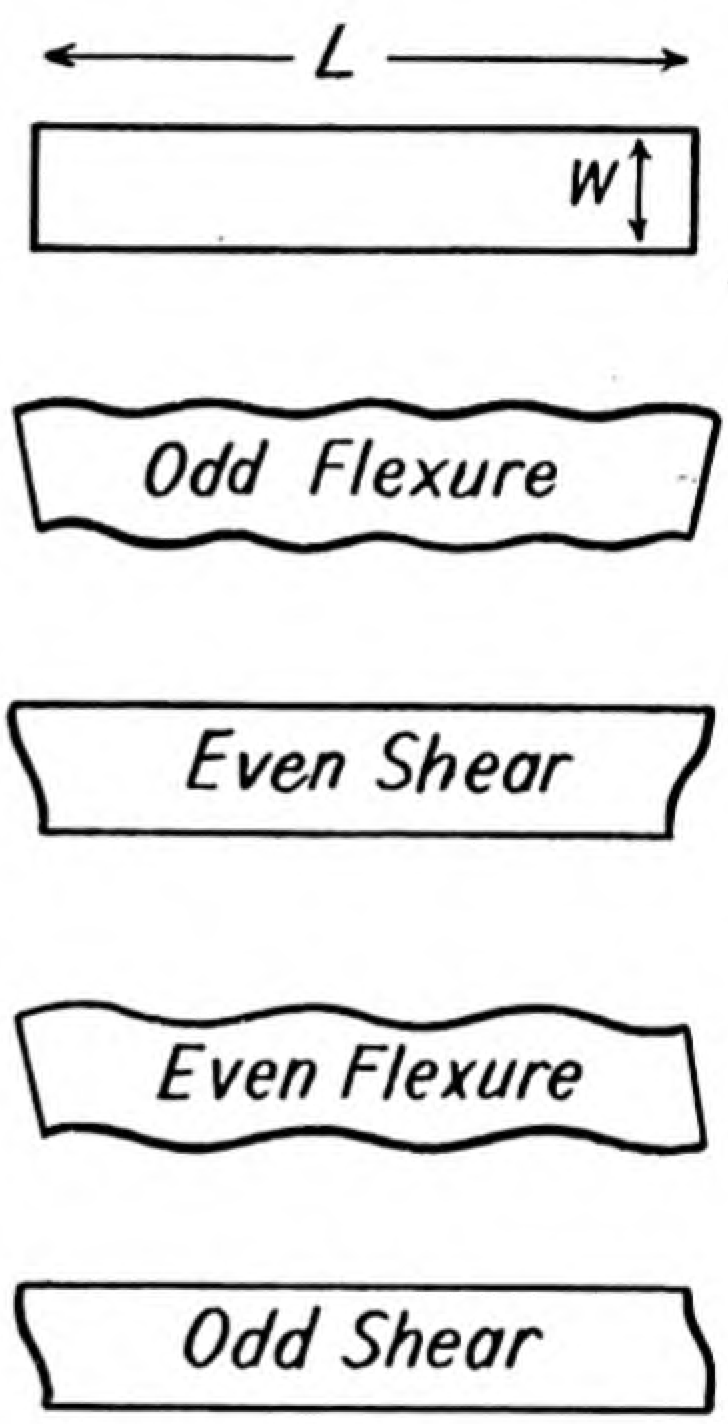


Fig. 16.9. End motions of a plate in flexural and shear vibration.

Magnetostrictive generators

This class of supersonic source is dependent upon the change in length of a rod of ferromagnetic material when placed in a magnetic field acting parallel to its length. This phenomenon was first recorded by Joule in 1847, and is a small effect, as will be seen from the graph (Fig. 16.10); in the case of nickel for small flux densities, the change in length is proportional to the square of the flux density. Since the change of length is independent of the direction of the magnetic field, it is necessary to polarise the specimen so that if subjected to an alternating magnetic field the rod will alternately lengthen and contract, *i.e.* will give rise to waves of compression and rarefaction in the surrounding medium. If the frequency of the alternating magnetic field is the same as the natural frequency of the rod, then resonance

will occur, and the amplitude of vibration of the specimen will be much in excess of that due to an equal static field. The material usually employed is nickel, which shows a contraction with increasing field strength, or permalloy which exhibits a corresponding elongation. Invar, nichrome, and monel metal are also employed. In practice, rods are not employed because of the energy loss due to eddy currents

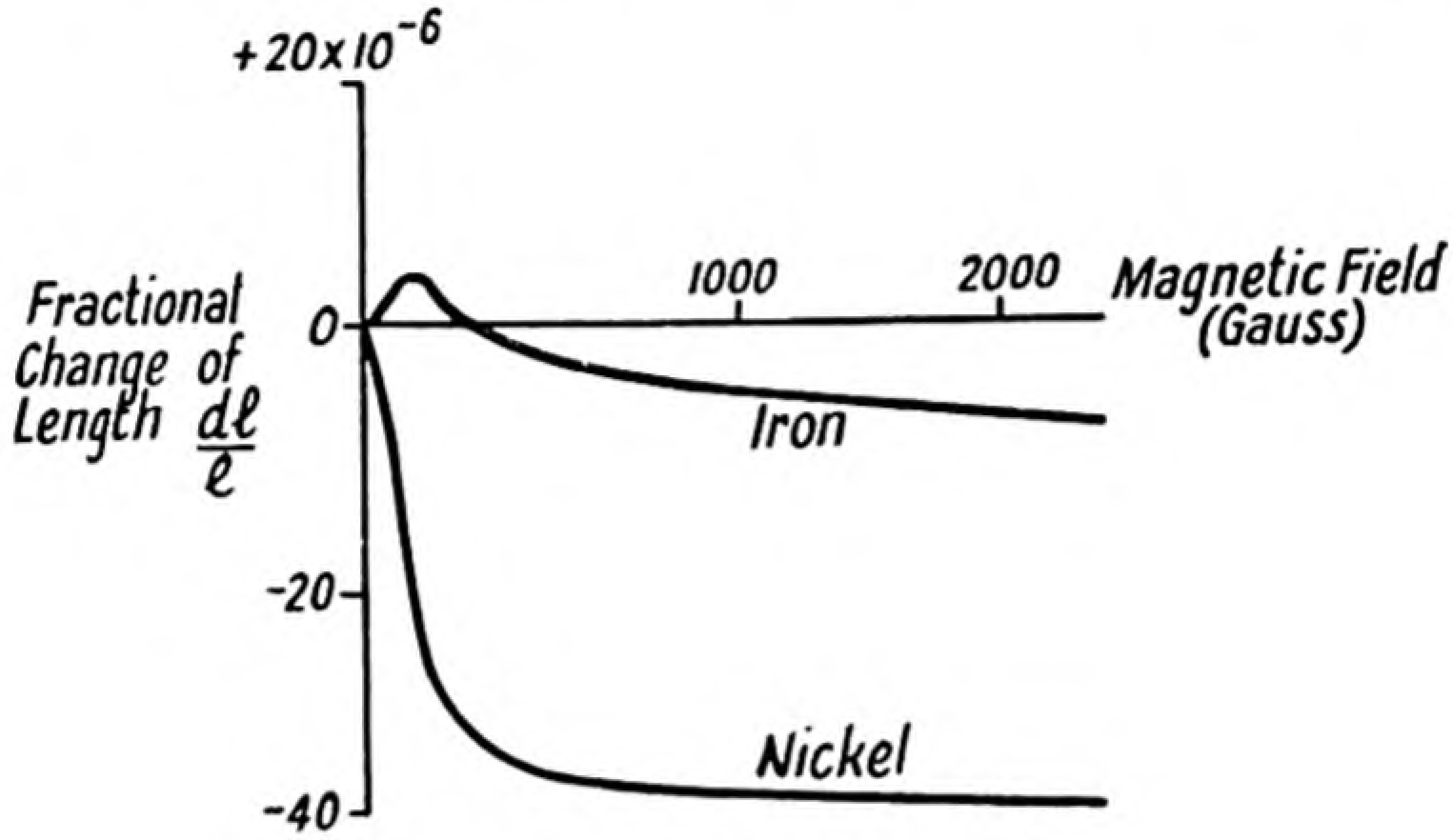


Fig. 16.10.

in the solid material, and the specimen is usually in the form of a thin tube, a bundle of finely laminated wires glued together, or a composite cylinder made up of layers of thin metal foil separated from each other by thin layers of paraffin-wax.

The use of a magnetostrictive element in an oscillatory circuit is due to Pierce (1925), and a circuit typical of his design is shown in Fig. 16.11. The specimen nickel tube *SS* is clamped at the centre and around each half of the cylinder are wound two coils *G* and *P*, which are respectively included in the grid and plate circuits of the valve *V*. The specimen is polarised by a suitable steady current passed through the winding *MM*, *C* is a small variable condenser to introduce extra coupling between the grid and plate circuits if necessary, and *A* is a direct current meter to indicate the onset of oscillations in the circuit. The mode of action of the circuit is dependent on the successive operation of the "direct" and of the "inverse" magnetostrictive effects. The "inverse" effect refers to the *change of* magnetic *flux* through the specimen when its length is changed, and in consequence of which an E.M.F. will be induced in any coil surrounding it. Hence, if any fortuitous change of plate current occurs, the magnetic field within the coil *P* will be altered correspondingly, and will lead to a change of length of the specimen on the right-hand side of the clamping point, and this change will be

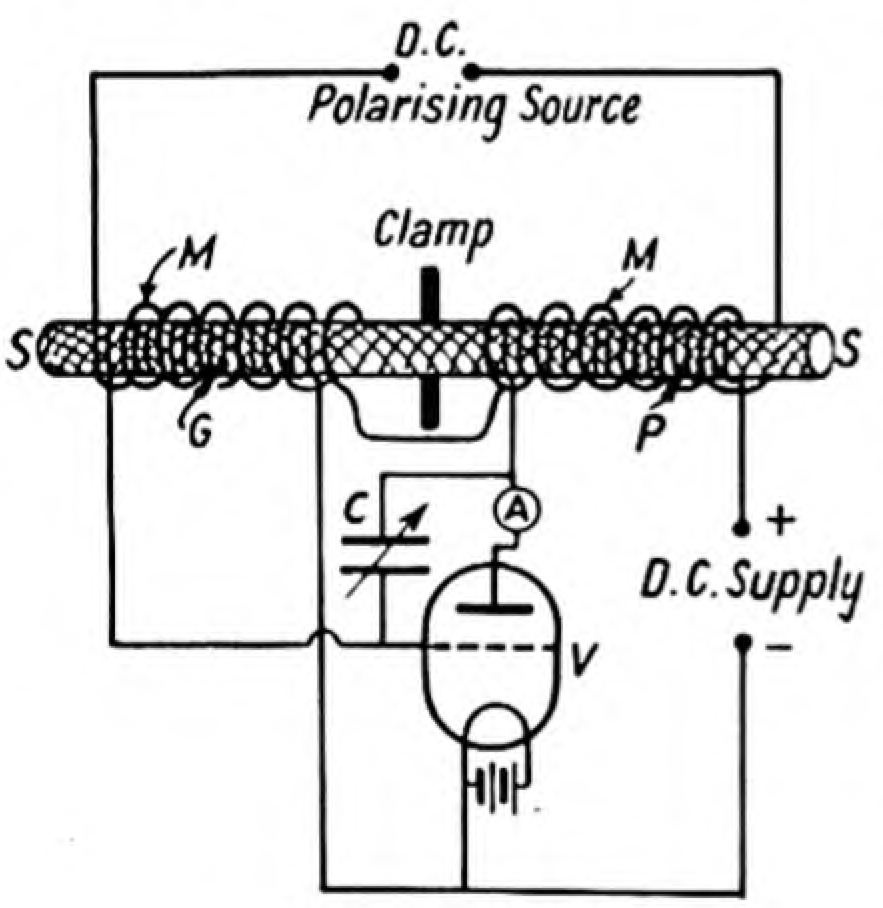


Fig. 16.11.

transmitted to the left-hand portion of the specimen. The "inverse" effect now becomes operative, and the E.M.F. induced in coil G will modify the grid potential of the valve and so react upon the anode current. If the phases of the changes are correct, then the amplitude of the oscillating current will build up and the vibration of the specimen will be maintained.

The natural fundamental frequency of the longitudinal vibrations of a rod of length l is given by

$$f=\frac{1}{2l}\sqrt{\frac{E}{\rho}}$$

where E is Young's modulus of elasticity and ρ is the density of the rod which gives a frequency of 25 kc.p.s., approximately, for a nickel rod of 10 cm. length. Higher frequencies up to 60 kc.p.s. can be obtained with shorter specimens, but with these it becomes increasingly difficult to excite their fundamentals, and although recourse may be made to excite the harmonics of rods this involves a big loss in intensity. Magnetostrictive oscillators are particularly advantageous at the low ultrasonic frequencies when a considerable output can be obtained without the danger of fracture of the oscillating element as exists with the quartz crystal. One particular form of magnetostrictive oscillator is shown, in section, in Fig. 16.12, where the "element" is a tube formed of layers of nickel foil, which are electrically insulated from one another by thin layers of paraffin-wax to reduce eddy-current losses. The winding upon the tube in this instance has been made in toroidal fashion, and the end plate transmitting the magnetostrictive oscillations into the surrounding medium is suitably cemented to the tube.

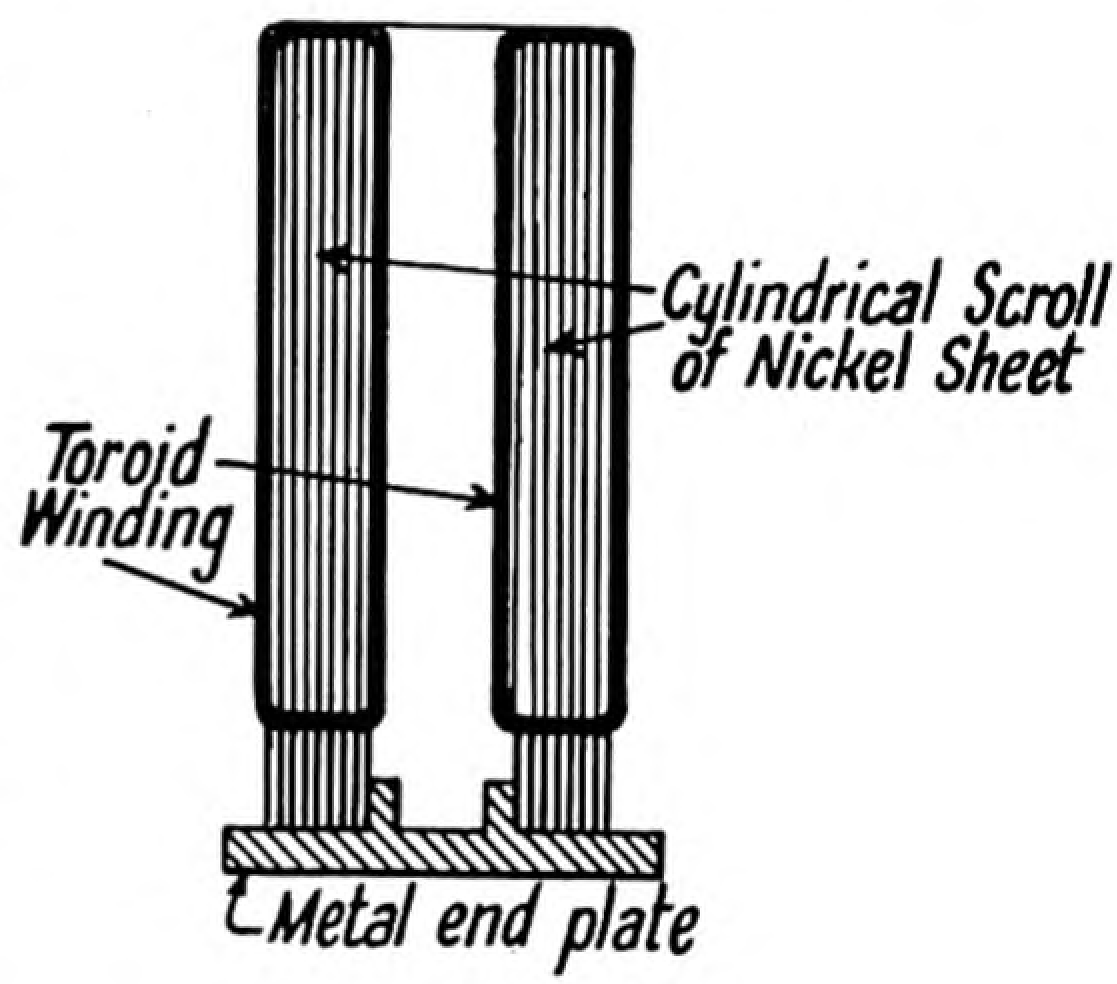


Fig. 16.12.

By taking care in only lightly clamping a nickel rod or tube at its centre, Knight has obtained values of "Q" in excess of 30,000 for this form of magnetostrictive oscillator. Such vibrators may be used in the manner of a quartz crystal in an acoustic interferometer (see later) and they have also been employed as filters. The temperature stability of a magnetostrictive oscillator approaches that of a quartz crystal if two metals having opposite temperature coefficients are combined, one as a rod fitting closely into a tube of the other.

An electromagnetic sound generator

Another type of high frequency sound generator which should be mentioned is a modification of a dynamic loud-speaker and it has been

developed for the purpose of flocculating suspended matter in smoke and fogs. The vibrator (Fig. 16.13) consists of a solid cylinder of a metal, *e.g.* duralumin, which should have both a "high electrical conductivity" and a low internal damping. The manner of supporting the bar at its centre has a great influence on its efficiency as a sound generator, for lateral contractions must necessarily accompany the longitudinal vibrations of the rod and any tight clamping at the centre will produce considerable damping and hence mechanical losses. In order to overcome this difficulty the supporting thin metal "web" extending radially from the cylinder at its mid-section was separated by rubber shims from the metal housing in which it was clamped. The presence of radial motion reduces the resonant frequency of a cylinder from

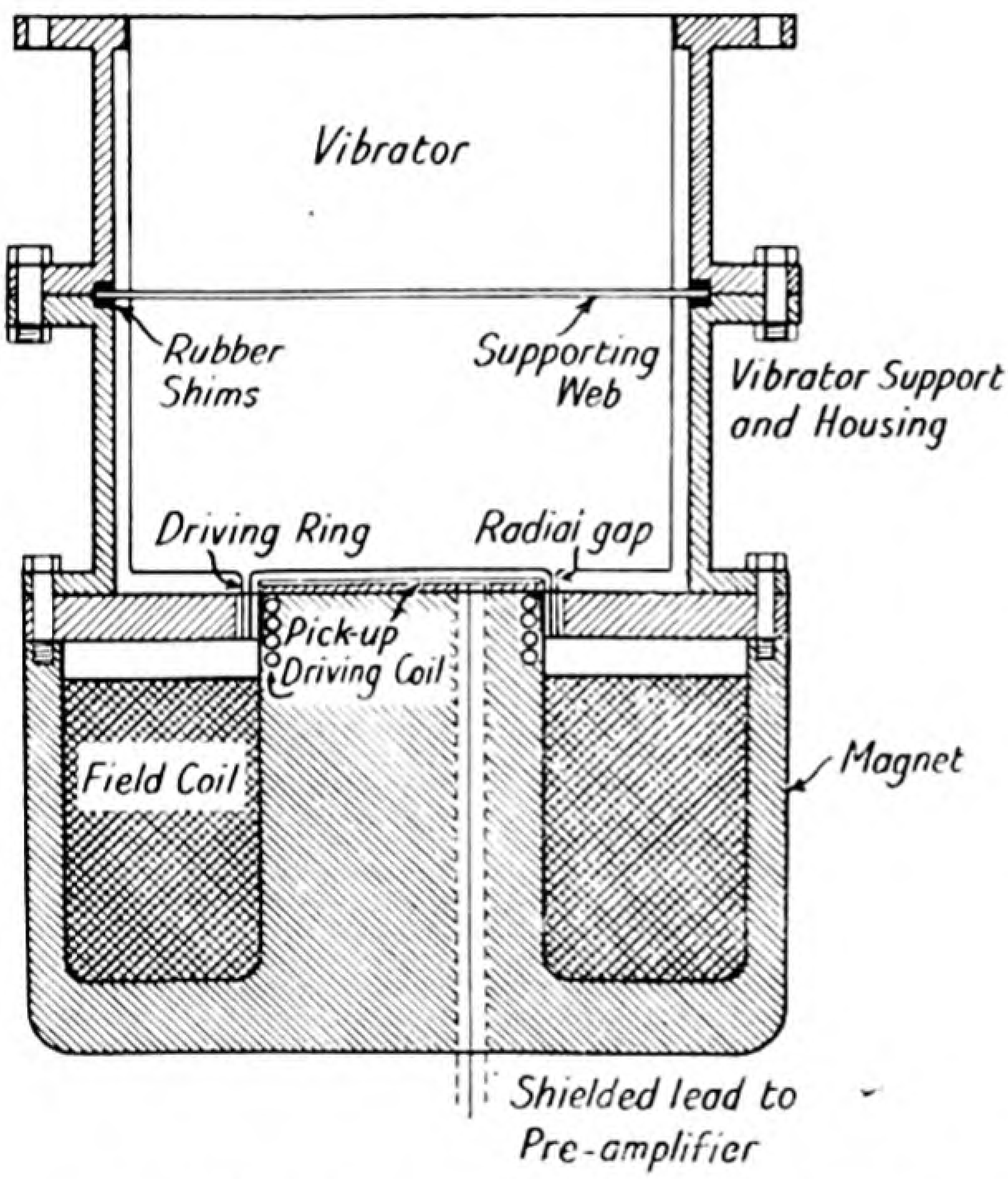


Fig. 16.13. Sound generator, showing construction.

$$f_o=\frac{1}{2l}\sqrt{\frac{E}{\rho}}$$

to $f_o'=\left(\frac{1-\sigma^2\pi^2a^2}{4l^2}\right)f_o$,

the latter expression being due to Rayleigh ("Theory of Sound," p. 252, Vol. 1). E, σ, and ρ refer respectively to Young's modulus, Poisson's ratio, and the density of the material, and l and a respectively to the length and radius of the cylinder, l being assumed greater than $2a$.

Chree has shown that the boundary conditions which are necessary to allow *rigid* clamping at any point on a *free-free* bar are only possible if $\frac{2a}{l}=\frac{2\beta}{\pi}$, where β is a root of the equation $J_1(x)=0$. The first two roots of this equation give $\frac{a}{l}=0{\cdot}589$ and $1{\cdot}696$ respectively, and for such cases the rod may be supported rigidly at its centre of gravity. Hillary W. St. Clair, in America, has constructed an electromagnetic sound generator conforming with the above requirements and he remarks on its low damping. Fig. 16.13 shows the original form of generator made by St. Clair in which the electrical oscillations are initiated by an electrostatic pick-up formed by the end of the vibrator, acting as the moving electrode, and a fixed electrode suitably insulated from the central pole of the magnet. This fixed electrode is connected to a high-voltage supply, *i.e.* of the order of 300 volts, by a two megohm

resistance, the alternating voltage across which is amplified and phase-shifted by a preamplifier which, in turn, provides the input to an amplifier feeding the driving coil of the generator. This self-exciting system, whereby the vibrator itself controls the electrical oscillations which drive it, was found to be necessary owing to the very sharp resonance of the metal cylinders and the loss of amplitude that would result from a very small drift from the resonant frequency, as is likely to occur with a separately controlled oscillator.

Ultrasonic wave-length measurements

These measurements are noteworthy for the application of optical experimental technique to acoustical measurements, which is made possible because of the shorter wave-lengths in the ultrasonic region.

The acoustic interferometer

This instrument was originally devised by Pierce for experiments on gases, and has been widely used in various modifications for both velocity and absorption measurements. It consists essentially of a quartz crystal *P* (Fig. 16.14) maintained in vibration by a suitable oscillatory circuit, the linear dimensions of the vibrating crystal being sufficiently large compared with the ultrasonic wave-length to ensure

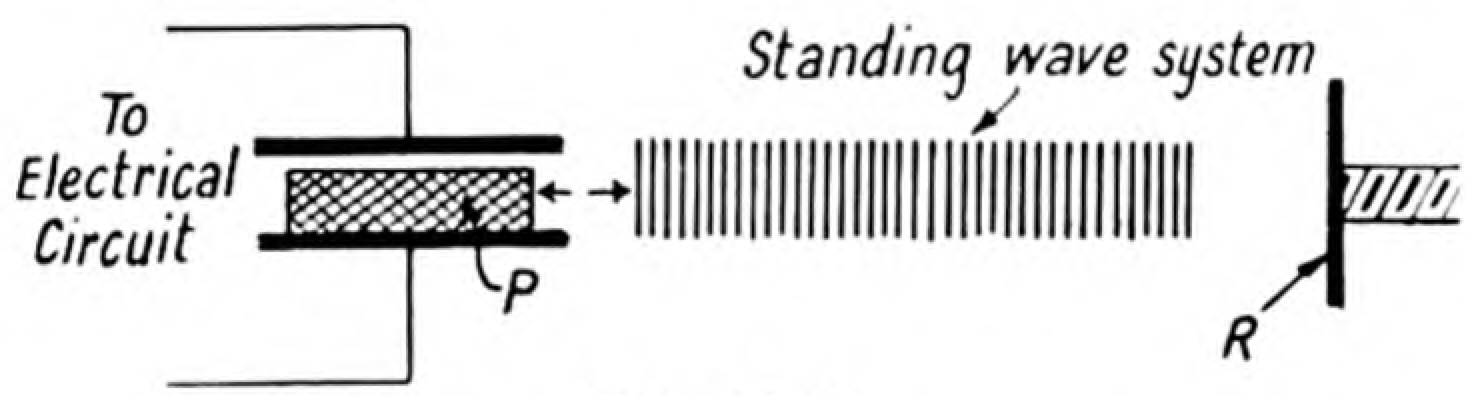


Fig. 16.14.

that *plane* waves are generated. Parallel to the vibrating face of the quartz is a movable reflecting plate *R*, which is attached to a micrometer screw, so, in fact, the set-up is similar to that of the Kundt's tube with the added refinements necessary for measuring shorter wave-lengths. In this case the existence of a standing wave-system is detected by the quartz crystal itself, use being made of the changing phase of the reflected waves reaching *P* as the reflector is moved backwards and forwards. The reaction of the reflected waves on the source will be a maximum when the out-going and returning waves at the crystal face are 180 degrees out of phase. As the reflector is moved, this reaction will vary, and the effect may be likened to a changing load on an electric motor; furthermore, it may also be indicated in a similar manner by a current-measuring instrument, the meter in this case being included in the plate circuit of the transmitting valve. Fig. 16.15 shows the type of record which is obtained by plotting the readings of plate current against the position of the reflector; in practice, the value of the maximum current readings will decrease with increasing distance of *R* from the source, due to sound absorption in the intervening medium. Although the wave-length is small, by observing a large number of internodal distances, a high degree of accuracy may be

attained. In order to calculate the velocity of propagation of the ultrasonic waves it is necessary to know the frequency of the source, and this may be found by a standard electrical wave-meter method. From observations on the decrement of the current maxima, mentioned above, it is possible to evaluate the coefficient of sound absorption (p. 281) for the medium, the interpretation of results, however, is not easy. A method developed by Richardson using a hot-wire to explore the stationary-wave system may simplify the procedure, although the effect of the probe itself adds a complication in a system of small dimensions.

The sonic interferometer has been used with both liquids and gases and up to frequencies of 20 Mc.p.s. with quartz oscillators, and it permits sound velocities to be determined absolutely to an accuracy better than 0·1 per cent., although the measurement of absorption coefficients is considerably less accurate. Great care has to be exercised, however, in interpreting results obtained with the interferometer, for although the reproducibility of results may be of a

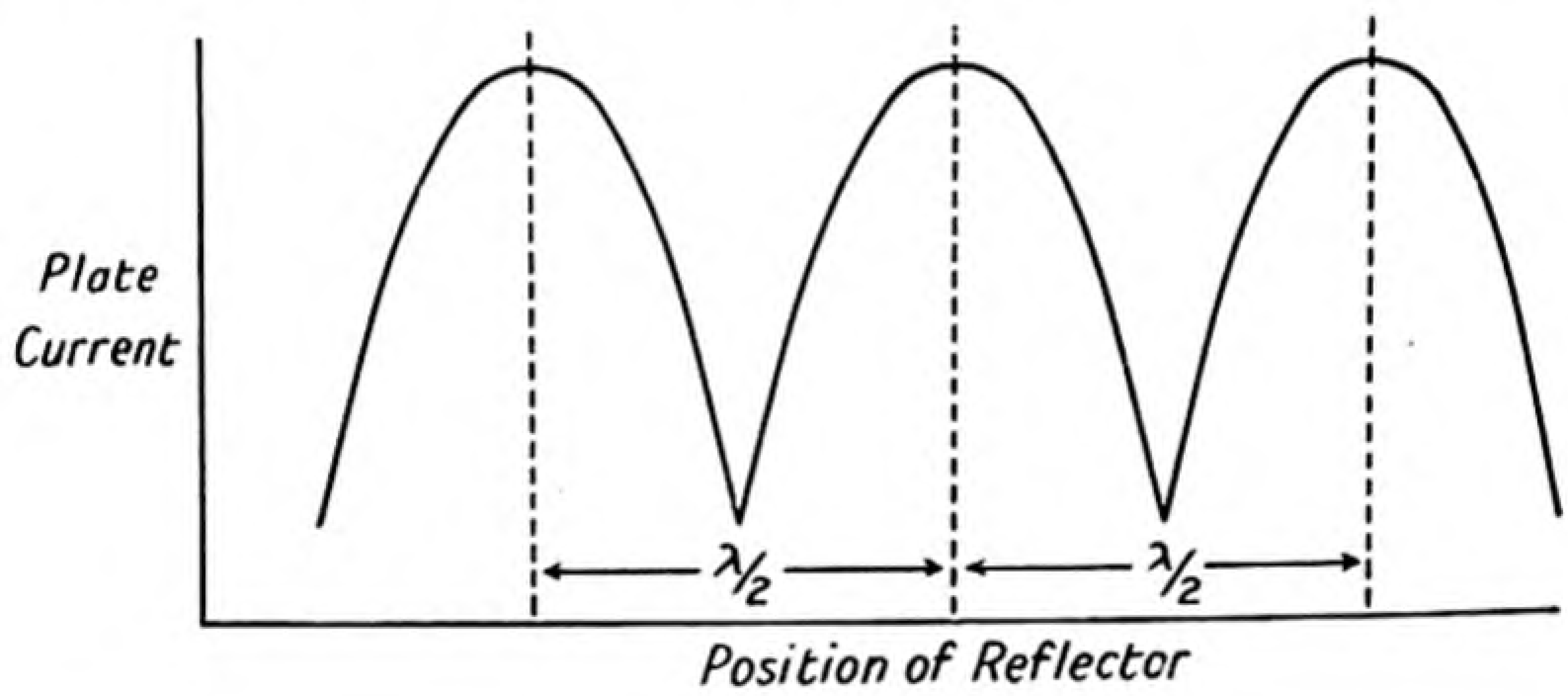


Fig. 16.15.

high order, the sound field pattern may be very complicated, owing, for example, to various parts of the oscillator not vibrating in phase with each other. Furthermore, absorption measurements may be "entangled" with heating effects especially if waves of appreciable amplitude are employed to increase the magnitudes being measured; a recent adaptation of radar pulse technique, whereby the larger amplitude can be used but persists for only a short time, promises to overcome this difficulty.

The interferometric method has also been employed with liquids, as mentioned above, and measurements performed over a wide range of temperature; one particular variation in technique which may be mentioned is the use of charcoal powder to show up the standing wave pattern. If only a small quantity of liquid is available, there is an interesting modification in procedure, based on an analogous experiment in light, namely, the determination of the refractive index of a transparent solid from the measurement of the shift of interference fringes, due to the introduction of the specimen in the path of one of the interfering light beams. The schematic arrangement of apparatus for the sound experiment is shown in Fig. 16.16*a*, and Δx is the shift of the nodal

planes due to the substitution of the liquid under test for the oil in the bakelite cell.

Let v_l=velocity of waves in the liquid,
v_o= „ „ „ oil,
λ_l=wave-length „ „ liquid,
and λ_o= „ „ „ oil.

Then if f is the frequency of the source $\frac{v_l}{\lambda_l}=f=\frac{v_o}{\lambda_o}$ or $\frac{\lambda_l}{\lambda_o}=\frac{v_l}{v_o}=\mu$, say.

Now the number of waves in the thickness d of oil $=\frac{d}{\lambda_o}$,

and „ „ „ „ „ liquid$=\frac{d}{\lambda_l}$;

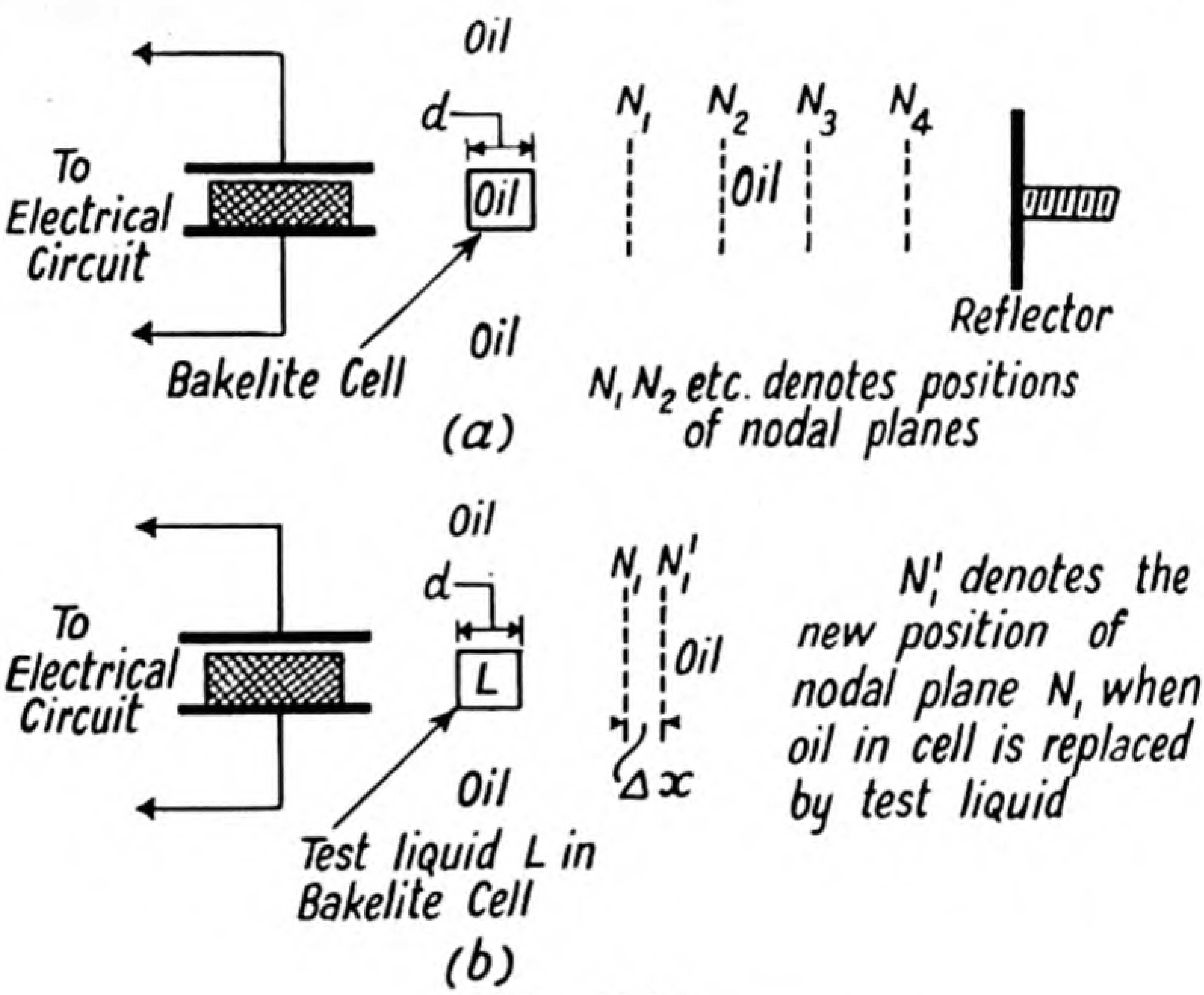


Fig. 16.16.

∴ the decrease in the number of waves due to the substitution of the liquid for the oil is given by

$$N=\frac{d}{\lambda_o}-\frac{d}{\lambda_l}=\frac{d}{\lambda_l}\left(\frac{\lambda_l}{\lambda_o}-1\right)=\frac{d}{\lambda_l}(\mu-1).$$

Now the shift of the nodal planes is measured in the oil and will be given by $\Delta x=N\lambda_o$,

i.e. $$\Delta x=d(\mu-1)\,.\,\frac{\lambda_o}{\lambda_l}=d\frac{(\mu-1)}{\mu},$$

or $$\mu=\frac{d}{d-\Delta x}.$$

Since $\mu=\frac{v_l}{v_o}$ and v_o is known, the value of v_l may be calculated when μ has been found. This method has also been adapted for the measurement of the velocity of sound in thin solid plates.

The acoustic grating

Another interesting and valuable linkage with optics technique is the suggestion, first made by Brillouin in 1925, that the periodic variations in density which are produced by *progressive* ultrasonic waves when passing through a liquid should endow the latter with a structure which, when irradiated by light, would give rise to a diffraction pattern. The problem, which is analogous to the Bragg experiments on X-ray diffraction by a crystal, is treated simply by regarding the light rays as suffering reflections at the parallel planes corresponding to the compression wave fronts (*C*,*C*, Fig. 16.17) of the sound wave; *R*,*R* denote positions of maximum rarefaction at the instant considered, and λ_s is the wave-length of the ultrasonic waves.

If *monochromatic* light of wave-length λ_l is incident at an angle i to the direction of propagation of the sound waves, then the path

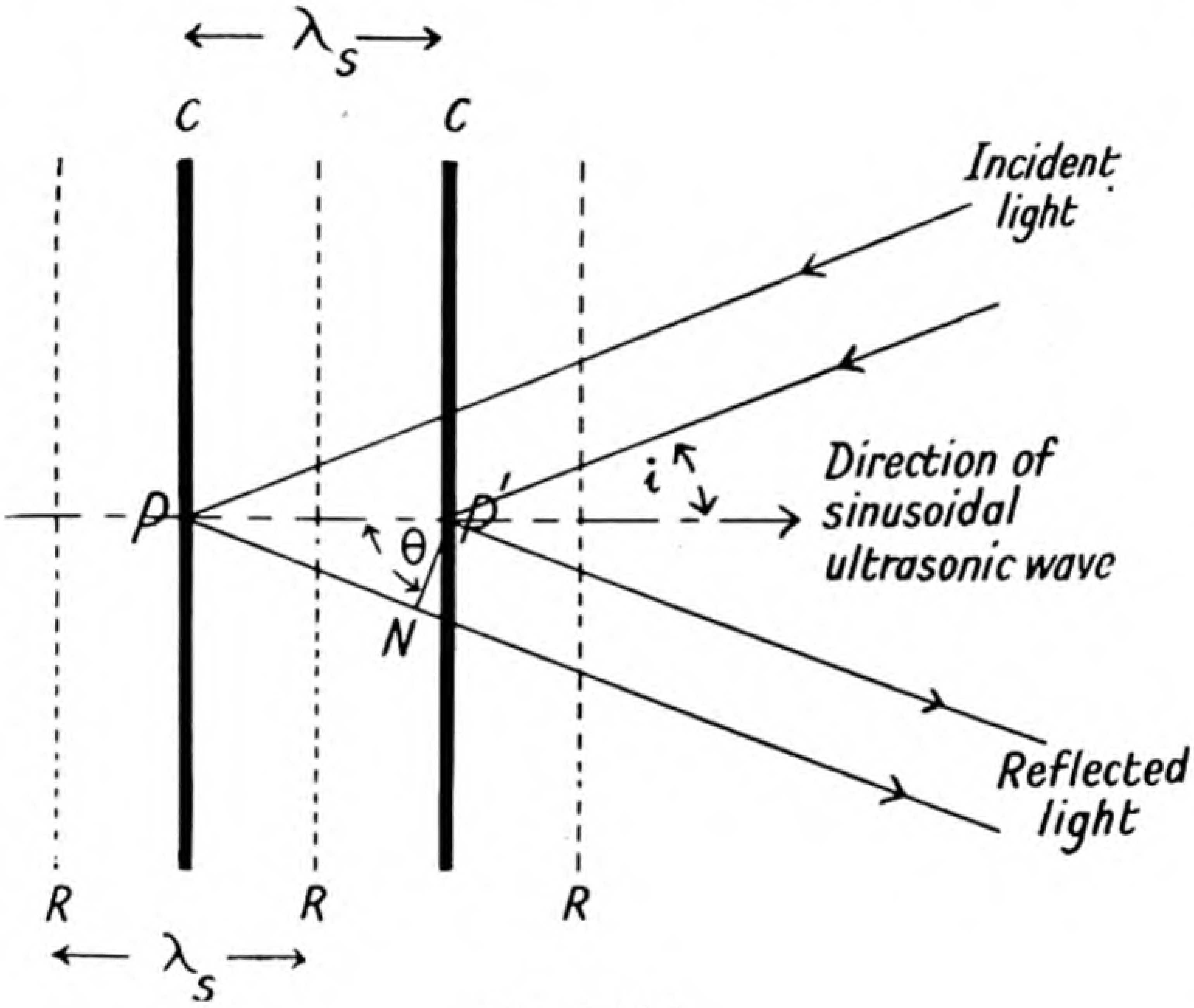


Fig. 16.17.

difference between successive reflected wave fronts is given by

$$2PN=2\lambda_s \sin \theta,$$

where $\theta=(90-i)$ is the angle between the direction of *sound* propagation and the *normal* to the *light* rays. The condition for equality of phase in the light reflected along a particular direction defined by θ is given by $2\lambda_s \sin \theta=\pm m\lambda_l$, where m is necessarily restricted to unity owing to the sinusoidal variation of density considered.

In the simplest case, therefore, two diffraction maxima will be observed, one on each side of the central undiffracted image, and it is evident from the above relation that λ_s must be very small, *i.e.* a source of sound of high frequency must be employed in order to obtain a measurable separation. Thus, with sodium light, λ_s must be of the order 10^{-3} cm. for θ to be as large as 2°. Furthermore, provided that the frequency of the ultrasonic source remains constant, then the above theory is independent of the progression of the wave fronts.

The experimental technique is simpler for standing waves than for

a progressive system, where some form of stroboscopic illumination has to be employed.

Debye and Sears, and others employing a standing wave system, used an apparatus similar to that shown diagrammatically in Fig. 16.18, the liquid cell being illuminated by parallel light in a direction normal to the direction of *propagation* of the sound waves. The positions of maximum density in the liquid are indicated (but not to scale) by dotted lines in the diagram, and their "lengths" normal to the plane of the paper constitute the lines of the grating. If the emergent light from the cell is focused on a screen it will be seen to exhibit diffraction lines evenly spaced about the central image of the slit.

Another interesting application of optical procedure is the use of a solid prism, *e.g.* of metal, immersed in a liquid to refract an incident ultrasonic beam. The technique used by Bez-Bardili follows that of Bar and Meyer, who used an opaque screen studded with evenly spaced pin-holes in front of a light source, to project beams of light parallel to the fronts of the sound waves passing through a liquid. Each pin-point will give rise to a set of diffracted images, the separation of which gives the wave-length of the sound waves in the liquid. In

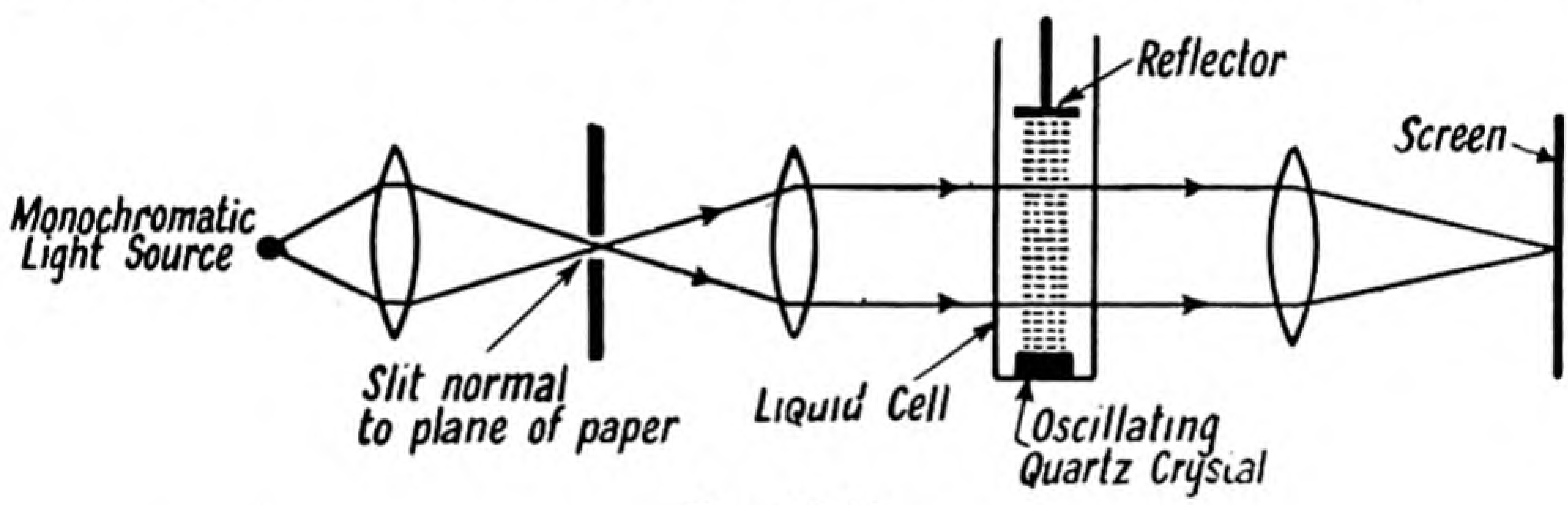


Fig. 16.18.

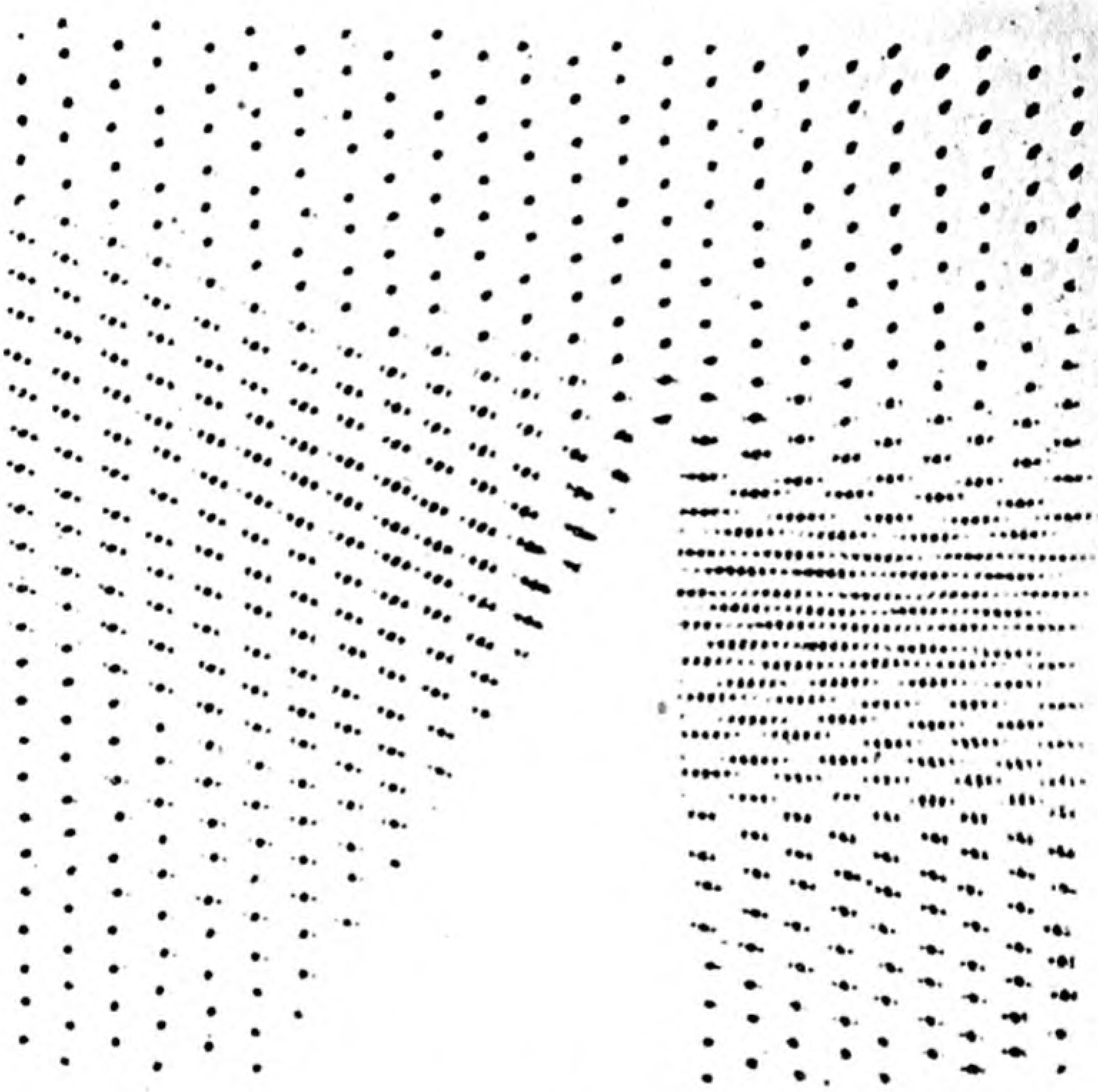
Fig. 16.19 is a prism of aluminium immersed in xylene and the incident beam is seen to be incident upon the prism from a north-westerly direction and the refracted beam is directed to the east. By measuring the angles of incidence and refraction Bez-Bardili was able to determine the refractive index of the aluminium with respect to xylene and hence the ratio of the sound-wave velocities in the two media. Since the velocity of wave propagation in the xylene is obtained from measurements on the diffracted images of the pin-points, it follows that the velocity of sound in the aluminium is calculable. Typical figures obtained at 20° C. for the sound velocity in xylene at the frequencies stated were:

Frequency (cycles/sec.)	$5{\cdot}23\times10^6$	$8{\cdot}52\times10^6$
Wave-length (cm.)	0·0255	0·0160
Velocity (metres/sec.)	1,300	1,360

From measurements using an aluminium prism the following figures were obtained:

Frequency (cycles/sec.)	$5{\cdot}23\times10^6$	8.52×10^6
Refractive index ($=\sin i/\sin r$)	0·227	0·217
Velocity of ultrasonic waves using value obtained for xylene (metres/sec.)	5,880	6,150

The acoustic grating method has been adapted to the examination of a transparent solid by creating a three-dimensional grating, and the experimental technique is rather similar to that followed in the X-ray analysis of crystal structure. The cube of transparent material is subjected to ultrasonic waves in three directions perpendicular to the faces of the cube, and corresponding systems of standing waves are set up which provide diffracting elements for light transmitted through the specimen. Now the separation between successive positions of maximum density in a standing wave-system is a measure of the wave-length of those particular types of waves, and so of their velocity which, in turn, is related to the appropriate modulus of elasticity of the medium (p. 38). It follows, therefore, that the light diffraction pattern



[*Phys. Zeit.*

Fig. 16.19.

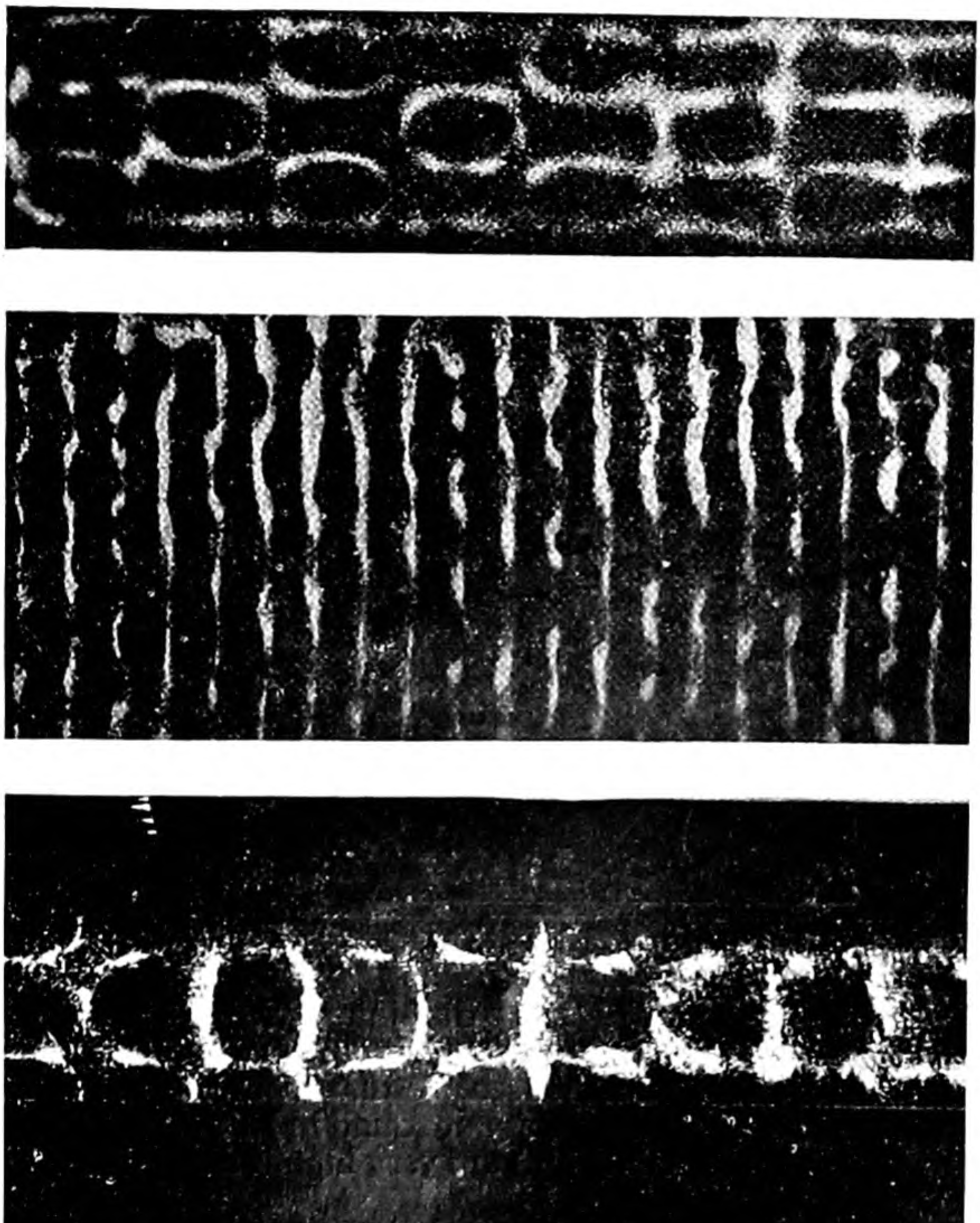
obtained will yield valuable information about the elastic constants of the material.

Transverse waves in plates, etc.

Fig. 16.20 show the sound distribution pattern obtained with sheets and rods of materials lightly covered with fine sand when excited transversely by contact at a suitable point with a magnetostrictive oscillator (a nickel rod). The nodal lines are produced by interaction between the direct waves and those reflected from the edge of the plate. The velocity C_t of these transverse waves is given by the product of frequency and inter-nodal distance and the velocity of longitudinal waves (C) may then be deduced using a formula due to Lamb, viz.:

$$C=\frac{\left(1+\frac{2\pi}{\lambda}K^2\right)^{\frac{1}{2}}}{\frac{2\pi}{\lambda}K}C_t \text{ which reduces to } C=\frac{Ct}{\frac{2\pi K}{\lambda}}=\frac{\sqrt{3}\lambda Ct}{\pi t} \text{ when } \frac{2\pi K}{\lambda} \text{ is small,}$$

cf. unity. K is the radius of gyration of the cross-section of the plate, and t is its thickness.



(a)

(b)

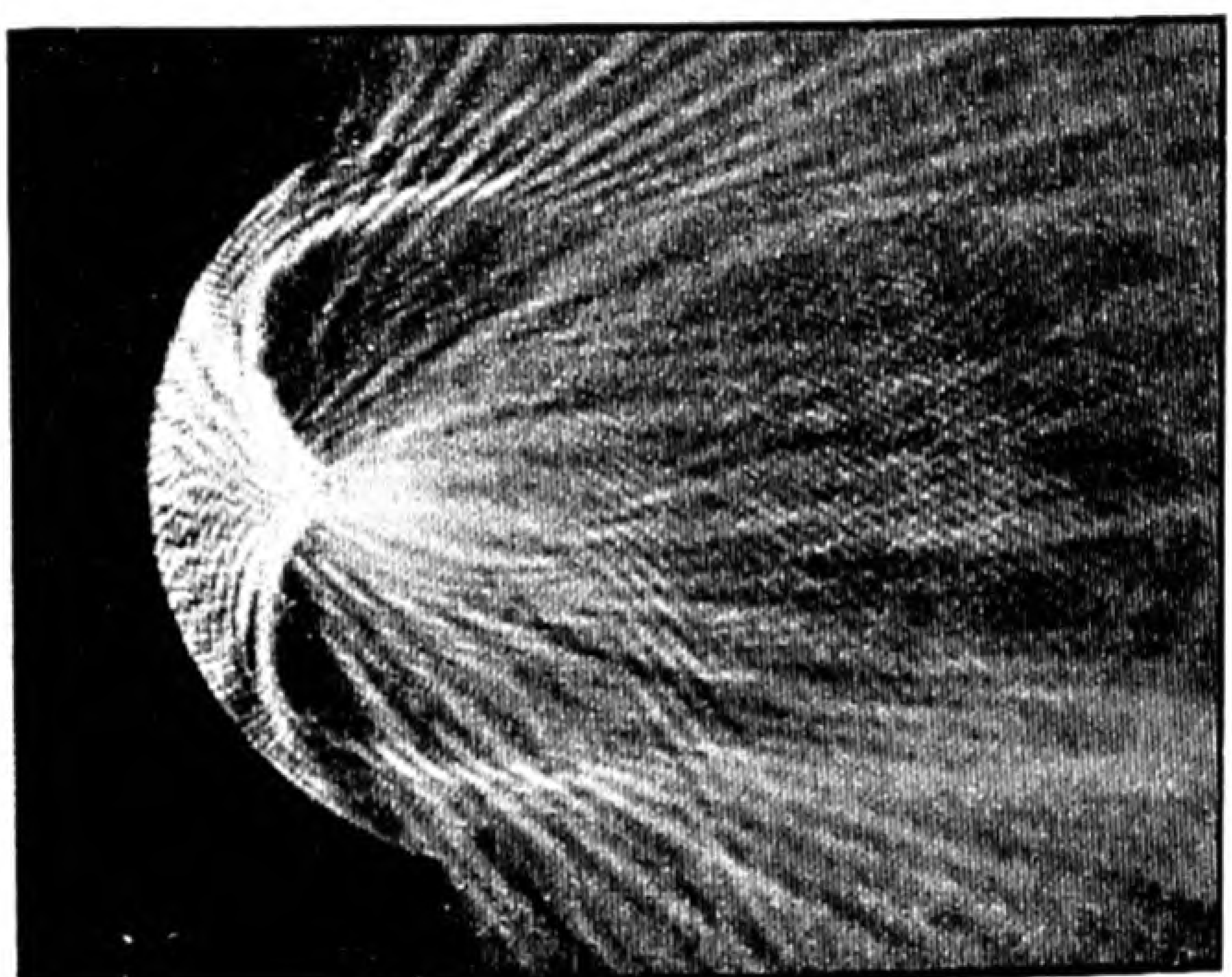
(c)

Fig. 16.20. Standing-wave patterns set up in sheet and bars by transverse high frequency vibrations. (a) Brass bar 0·102 cm. thick; (b) Brass plate 0·17 cm. thick; and (c) Polystyrene bar 0·65 cm. thick.

From Fig. 6.20, for the brass plate of thickness 0·17 cm. the measured wave-length was 2·05 cm. Hence, since the frequency of the magneto-strictive oscillator was 18,750 c.p.s. the velocity of transverse waves $C_t=18{,}750\times 2{\cdot}05=3{\cdot}8\times 10^4$ cm./sec.

Ultrasonic lenses

A disadvantage in using ultrasonics for commercial purposes is the poor efficiency of the energy conversion, and there is therefore considerable interest in methods of concentrating the high frequency radiation into a small volume where required, usually in a liquid medium. Spherical mirrors, as might be expected (Fig. 16.21), have been used to focus these ultrasonic beams and an improvement on this procedure has been the use, by Gruetzmacher, of the vibrating quartz element itself to focus the rays by shaping it in the form of a concave mirror. An easier way of achieving the same objective is to adapt the contact lens technique, as employed in optics with respect to the human eye, and use a plano-spherical or plano-cylindrical lens of a suitable material which is placed in direct contact with the vibrating quartz. It is highly desirable to minimise any loss in transference of ultrasonic energy from the crystal to the lens and so the material chosen for the latter should have an acoustic impedance as nearly equal to the



[*J. App. Phys.*

Fig. 16.21.

quartz as possible (see p. 241). Aluminium and glass come within this category, and furthermore the velocity of sound in these materials is greater than 5×10^3 metres per sec., so that their acoustical refractive indices (as measured by μ = velocity of sound in lens material $\div$ velocity of sound in surrounding medium) are very high if immersed in a liquid medium.

The loss associated with the transmission of ultrasonic waves across the boundary of two materials differing in acoustic impedance may be reduced by the insertion of plane parallel plates of a material possessing an intermediate value R of acoustic impedance, *i.e.* $R_1 > R > R_2$, where the suffixes 1 and 2 refer to the two materials to be "matched." The problem is analogous to the optical one whereby reflection from glass is reduced by coating it with a transparent thin film of cryolite.

Now it was shown by Rayleigh that for perpendicular incidence the fraction W_r of incident sound energy which is reflected at the interface between two media is given by the equation:

$$W_r = \frac{\left(\frac{R_1}{R_2} - \frac{R_2}{R_1}\right)^2}{4\cot^2\left(\frac{2\pi}{\lambda}\right)d_2 + \left(\frac{R_1}{R_2} + \frac{R_2}{R_1}\right)^2} \quad . \; . \; . \; . \; . \quad (1)$$

where d_2 is the thickness of the second medium. It becomes evident that $W_r = 0$ when

$$d_2 = \frac{\lambda}{2\pi}(n.\pi) = \frac{n\lambda}{2},$$

n being any integer 0, 1, 2, etc. Under such conditions the plate is acoustically transparent for the particular frequency of sound wave associated with the wave-length λ, and the phenomenon is analogous to the corresponding optical reflection at a thin film. The acoustical case did not receive experimental attention until the generation of wave-lengths comparable with small values of d was common, and the first experiments with ultrasonic waves were not carried out until 1925 by Boyle and Lehmann, who used a torsion pendulum to measure the energy density. Incidentally, by employing a fixed thickness and varying the frequency of the source until maximum sound is transmitted a convenient means is provided of determining the velocity of sound in the material. Any ambiguity due to uncertainty in n is avoided by a previous approximate calculation of the velocity; this method was originally suggested by Rayleigh.

On further reference to the expression (1) it is seen that the transmission through the second medium is zero, *i.e.* W_r is a maximum, when

$$d_2 = \frac{\lambda}{2\pi}\left(\frac{2n+1}{2}\right)\frac{\pi}{2} = (2n+1)\frac{\lambda}{4},$$

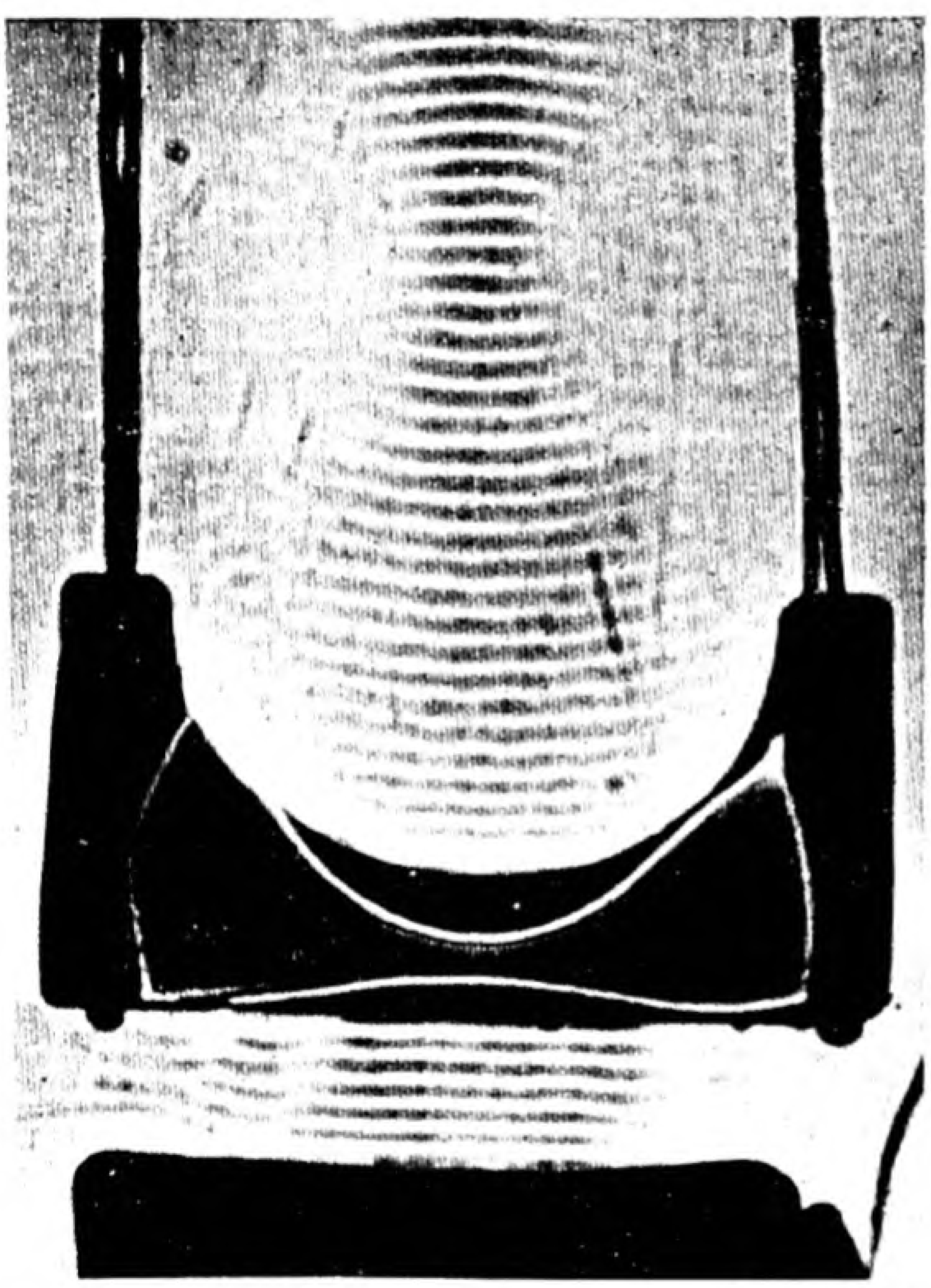
where $n = 0, 1, 2$, etc. Use is made of this effect to restrict the propagation of ultrasonic energy to one direction by mounting the generating quartz crystal on an air cushion of thickness $d = \frac{\lambda}{4} = \frac{C}{4f}$, where C is the velocity of sound in air and f is the frequency of oscillation.

A mathematical consideration of the problem of the transmission plate inserted between the two media to be "matched" indicates that $R = \sqrt{R_1 R_2}$ for maximum transmission (*cf.* optical case $\mu = \sqrt{\mu_1 \mu_2}$). A further improvement is effected by increasing the number of insertion plates and if R, R', R'', etc., represent the impedances of the various intermediate layers then it is found that together with R_1 and R_2 they should form a geometric series such that the quotient is given by $\left(\frac{R_2}{R_1}\right)^{1/n}$, where there are $(n-1)$ intermediate layers. Finally, the transmission will be further increased if the thicknesses of the respective plates satisfy the conditions for a minimum value of W_r in expression (1).

Fig. 16.22 shows the use of a plano-cylindrical polymethyl methacrylate lens (radius of curvature 14 mm.) suspended in paraffin to focus ultrasonic radiation from a quartz crystal. The axis of the lens is parallel to the central beam of light used to project the striae set up by the standing ultrasonic wave system.

The use of ultrasonics

So far, the subject of ultrasonics has probably appeared to the reader as merely a natural extension of the frequency range of audio-acoustics, but it has a greater significance, for further consideration will indicate that at the higher frequencies the periods of vibration of the waves actually become comparable with the periods of the molecules of the medium. The experimenter in the field of molecular physics, previously restricted to the use of light or of intense electric fields as instruments of investigation, is thus provided with a new and powerful "mechanical tool." A particular field of enquiry, leading to valuable information about the constitution of complex molecules, is provided by the study of the specific heats of gases, especially in regard to the relative importance of the vibrational and rotational frequencies of the molecules. The approach to the solution of this problem lies in the investigation of the velocity (and absorption) of sound in a gas at different frequencies and temperatures, and should become evident after the following considerations.



[*J. Sci. Inst.*

Fig. 16.22.

In a polyatomic gas the atoms in a molecule may oscillate with respect to one another, and even the whole molecule may be rotating about one or more axes. Such energy of motion is known as the *internal energy* of the molecule, and it follows that in collisions between complex molecules there is liable to be an interchange between the internal and translational energies. Now there is a theorem in statistical mechanics, due to Boltzmann, known as the principle of *equipartition of energy*, which states that in a system consisting of a very large number of particles in motion the *average* energy of *each* degree of freedom of a particle is the same. The number of degrees of freedom correspond to the number of coordinates required to define the position of a particle, so that a molecule of a monatomic gas which possesses only translational energy is said to have 3 degrees of freedom. According to the kinetic theory of gases, the average translational energy of *any* molecule is $=\frac{3}{2}\frac{RT}{N}=\frac{3}{2}kT$, where T is the absolute temperature, R is the gas constant, N is the number of molecules per gramme-molecule, and k is Boltzmann's constant.

It follows that the mean kinetic energy for *each degree of freedom* $=\frac{3}{2}kT\div 3=\frac{kT}{2}$, and hence for a gas molecule possessing *n internal* degrees of freedom, the total average energy per molecule is $\frac{3}{2}kT+\frac{nkT}{2}$ $=\frac{(3+n)}{2}kT$. The change of this average energy per molecule for one degree absolute change in temperature will therefore be given by $\left(\frac{3+n}{2}\right)k$, and hence for N molecules by $\left(\frac{3+n}{2}\right)R$. It follows that the specific heat (per gramme-molecule) of a gas at constant volume $c_v=\left(\frac{3+n}{2}\right)R$, and since the corresponding specific heat at constant pressure for a perfect gas $c_p=c_v+R$, then $c_p=\left(\frac{5+n}{2}\right)R$. But the ratio of specific heats is defined by $\gamma=\frac{c_p}{c_v}$, and therefore $\gamma=\frac{5+n}{3+n}$.

In the case of a gas known to be diatomic, such as hydrogen, the value of γ found by experiment is $\frac{7}{5}$, which would make $n=2$. In the early days of atomic theory this result was interpreted as meaning that the two atoms were arranged in a dumb-bell type of structure, permitting of two axes of rotation, but for a completely satisfactory explanation the classical system of dynamics has had to be discarded in favour of the quantum-dynamics.

Now returning to the question of the interchange of translational and internal energies of the complex gas molecule, it should be conceivable that the exchange will take a finite time, dependent upon the different vibrational degrees of freedom. Furthermore, it should be evident that when this "relaxation time," as it is termed, for a particular degree of freedom becomes comparable with that of the rapid oscillations of pressure in the supersonic wave, then these internal molecular motions will be unable to follow completely the adiabatic changes of temperature. In other words, c_v assumes a value less than $\left(\frac{3+n}{2}\right)R$, and ultimately, as the ultrasonic frequency is indefinitely increased, approaches the value (for a perfect gas) of $\frac{3}{2}R$. It follows that $\gamma=\frac{5+n}{3+n}=1+\frac{2}{3+n}$ will increase at these critical frequencies, and correspondingly also the velocity of sound (v) since

$$v=\sqrt{\gamma\frac{P}{\rho}},\ \text{(p. 38).}$$

From measurements of this change of velocity with frequency it is now evident how a measure can be obtained of the relative contributions of the translational and internal energies of the gas molecules to the specific heat of the gas. This sound dispersion was first found experimentally by Pierce using his acoustic interferometer, and his results showed that the velocity of sound in carbon dioxide increased

from a value of 258·8 m. per sec. at 42,000 c.p.s. to 260·2 m. per sec. at 206,000 c.p.s.

Again, in the case of a liquid, by measuring the velocity (v) of ultrasonic waves in the medium, a value of the adiabatic bulk modulus (K_a) may be found, since $v=\sqrt{\frac{K_a}{\rho}}$, where ρ is the density. If the isothermal modulus K_i is found by a static experiment, the ratio of the specific heats of the liquid may be evaluated.

When the ultrasonic waves in the liquid are propagated with considerable intensity, the medium becomes heated, but what is perhaps a more important consequence is the formation of bubbles of gas or vapour within the fluid, due probably to the high particle accelerations in an ultrasonic field. This effect is known as *cavitation*, and it is the agency upon which a number of processes or phenomena are dependent, such as the emulsification of one liquid in another. Wood and Loomis showed, for example, that intense ultrasonic radiation transformed immiscible liquids like water and mercury into quite stable emulsions. This process of ultrasonic emulsification has been applied to the quality improvement of such industrial products as photographic plates. Ultrasonics have quite a reverse effect on dispersed systems in gases, *e.g.* mist or dust, and the passage of the high frequency waves through these aerosols, as they are termed, tends to bring about the *coagulation* of the particles. The phenomenon (Fig. 16.23), as pointed out by Andrade, results from the fact that spheres at rest in a vibrating medium mutually repel one another if their line of centres is parallel, but attract if it is normal, to the vibration vector. This force of repulsion or attraction varies, directly as the square of maximum velocity of the vibrating fluid, and inversely as $(\text{distance})^4$.

The latest types of electronic calculating machines incorporate a new method of storing information, which depends upon the use of ultrasonics and the conversion of alternating E.M.F.s into sound waves and vice-versa. The information to be stored is applied, in the form of electrical pulses, to a quartz crystal situated at one end of a six-foot length glass tube containing mercury, in which high frequency sound waves are generated by the vibrating crystal. These waves are received by a second quartz crystal, situated at the other end of the mercury column, and are transformed back again into electrical impulses, amplified, "cleaned," and again applied to the "sender" quartz crystal. The different numbers and instructions, represented by various patterns of signals, are thus continuously circulated around the circuit and the use of the ultrasonic transformation is to slow down the circulation, for the pulses would travel too rapidly in an "all-electric" circuit. The information which is required to be transferred to the operating circuits of the calculating machine may be extracted from the storage circuit at will when the corresponding signals are leaving the mercury column. Each mercury tube can deal with 16 numbers of 10 decimal places, circulating behind one another.

One further application will be mentioned, and it depends upon the fact, already stated (p. 106), that sound waves travelling in one medium will, in general, suffer appreciable reflection on meeting a second

medium. Suppose now that this latter medium exists as a flaw, *i.e.* a hollow, crack or foreign impurity, in the first medium which is the material under test, then it follows that the intensity of the transmitted beam through this material will be reduced. Hence by using some means of observing the reduction in intensity of the incident ultrasonic beam, it is possible to test the degree of homogeneity, for example, of a metal casting. The use of ultrasonics in depth finding has already been mentioned, it will suffice to say here that magnetostrictive transmitters are usually employed, and it is the time interval between the transmission and the return of the reflected signal which is observed.

Recent work in liquids and solids

Viscous and shear elasticity measurements in liquid have recently been made by Mason and his co-workers in America using crystals (quartz and ammonium dihydrogen phosphate) vibrating in shear. When a crystal is vibrating in a purely torsional mode at a high frequency all the motion is tangential to the surface and highly attenuated viscous waves can be set up in the surrounding fluid medium. It may be shown that the propagation constant $\Gamma=(1+j)\sqrt{\frac{\pi f\rho}{\eta}}$, where f is the frequency of vibration, and η and ρ are respectively the viscosity and density of the fluid medium. In the case of carbon tetrachloride, for example ($\rho=1{\cdot}595$ gm./c.c. and $\eta=0{\cdot}01$ poise approx.), calculation shows that at 14 kc.p.s. the attenuation is 2,700 nepers/cm., so that the shearing stress becomes inappreciable after a *few thousandths of a centimetre* from the crystal surface. Although such high attenuation of the waves prevents the examination of their wave propagation properties, yet they produce a *loading* effect on the vibrating crystal which can be measured as an increase of the resonant resistance and a decrease of the resonant frequency of the crystal from its value in vacuum. It should be noted that only a small quantity of fluid is required to make such measurements. For very viscous liquids the resistance component of the viscosity becomes progressively larger than the reactance at high frequencies, which may be explained by assuming the liquid to possess a shear electricity as well as a shear viscosity.

An interesting effect which was initially predicted from the quantum theory of liquids by Landau and Tisza, and subsequently verified experimentally by Peshkov, is the fact that waves can be propagated through helium II, the low temperature phase of liquid helium with two distinct velocities. The ordinary pressure wave has a velocity of approximately 240 metres/sec. and may be transmitted and detected in the usual way. The other wave motion, termed the "second sound," is essentially the propagation of a temperature wave, whose velocity is strongly dependent on temperature, having a maximum value at $1{\cdot}6°$ K. of about 20 metres/sec. This wave is generated by means of an alternating current heater and detected thermally, for example, by a phospher-bronze resistance thermometer.

The absorption in amorphous solids and single crystals is generally less than that in liquids, but the presence of anisotropy brings about

an increased absorption in solids. This increase is due to two distinct processes: (1) the scattering of the waves in the Rayleigh manner, the magnitude being proportional to the inverse fourth power of the wave-length, *i.e.* to the fourth power of the frequency; and (2) the thermal relaxation losses consequent upon the inhomogeneous heating and cooling of the substance during the passing of the sound wave. In the range above one megacycle the general features of absorption in poly-crystalline materials are explained by the Rayleigh formula, which, however, fails when the acoustic wave-length becomes of the same order, or less, than the size of the grains of the material.

The mechanism of the second type of absorption has been simply explained by Zener, who considers the thermal conditions operating during the transverse vibrations of a reed. In bending the outer side of the reed becomes cooled and the inner is heated, a state of affairs which is reversed in the other half of the cycle. For very *slow* vibrations this transfer of mechanical energy into heat can take place reversibly, and under such isothermal conditions there is an absence of internal friction. The same result will occur at very high frequencies, where the mechanical vibrations are too rapid for heat transfer to take place and the conditions are adiabatic. It is evident that at frequencies between these extreme limits, however, there will be a loss of energy due to a transfer of heat. Zener has derived an expression for the thermoelastic "Q" of a reed vibrating transversely in terms of the thermoelastic relaxation time $\left(\frac{1}{f_o}\right)$, viz.

$$\text{"Q"} = \frac{E_s}{E_s - E_\theta} \cdot \left(\frac{f_o}{f} + \frac{f}{f_o}\right),$$

where f is the frequency of vibration, and E_s and E_θ are respectively the adiabatic and isothermal values of Young's modulus. The relaxation frequency may be calculated from the expression $f_o = \frac{\pi D}{2d^2}$, where D is the coefficient of *thermal diffusivity* of the material and d is the thickness of the reed in its plane of vibration. For an aluminium reed of thickness 10^{-3} cm., the value of f_o is 1·6 Mc.p.s.

In making tests on solids over a frequency range it is desirable for the same specimen to be employed throughout. One method which has recently been employed by Parfitt is to measure, by a form of condenser microphone, the amplitude response of a rod lightly clamped at its centre, when excited electrically at its fundamental and overtone frequencies. In this manner the twelfth harmonic of a fundamental frequency of 5 kc.p.s. has been obtained. By applying a range of frequencies in the neighbourhood of each resonant frequency and measuring the responses, a response-frequency curve may be drawn from which the "Q" of the rod can be calculated (see p. 235). It follows that 1/"Q" will be a measure of the internal damping of the specimen at the appropriate frequency. Resonance methods are, of course, less accurate when the damping is large.

Ultrasonics in nature

Although ultrasonic sound is above the audible range of the human being, yet many animals are obviously sensitive to these high frequency vibrations and it has been reported from America that they have been utilised as a means of protecting reservoirs from pollution by sea gulls. Most interesting experiments have been performed in America by Pierce, Griffin and Galambos, and by Hartridge in this country, in connection with bats, whose ability to fly at speed in and out of trees in darkness has always been somewhat of a mystery. The bat appears to make use of audio-location akin to the technique of radar. In other words, by sending out pulses of ultrasonic waves the bat is able to locate an object in its path from the time and direction of the echo received.

Three distinct types of sound emitted by bats have been recognised:

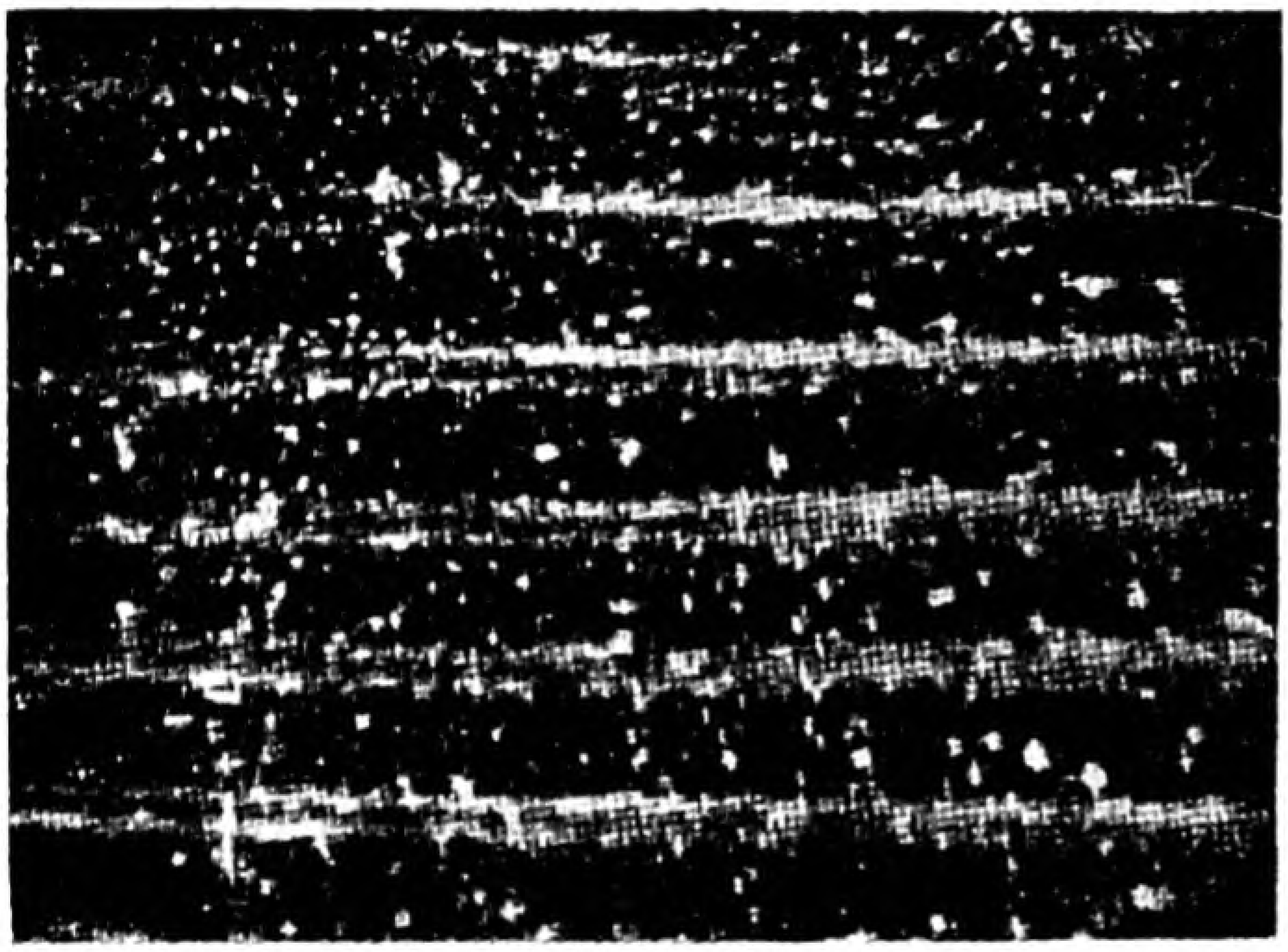
(1) A signalling tone of about 7,000 c.p.s. which lasts for 0·25 sec. or less and may be repeated over and over again. It can be produced with, or independently of, the ultrasonic tone.

(2) The ultrasonic cry, which ranges from 30 to 70 kc.p.s., but usually lies between 40 and 55 kc.p.s., and may be emitted as a single pulse, lasting 0·10 to 0·15 sec., or as a sequence of such pulses. A bat may produce 5 to 10 ultrasonic cries per second when at rest preparing for flight, but the rate is increased to 20 or 30 per second in flight through unobstructed space. When, however, obstacles are in its immediate path the bat increases the rate to 50 or 60 per second for short periods which drops back to 30 when the obstacle has been successfully negotiated.

(3) A rather feeble click, which is always accompanied by the ultrasonic tone; when the bat is in flight the rapid repetition of this click becomes recognisable as a buzz.

A frequency of around 50 kc.p.s. for the ultrasonic tone is about the optimum value to be expected from purely physical deductions, for, although the intensity of sound reflected back from a small object increases with the frequency of the waves, the effect of attenuation also increases, but with the square of the frequency. This limitation of resolving power of the acoustical system is analogous to the case of the optical microscope in which the advantage gained in resolving power by using the shorter wave-lengths of the ultra-violet are finally limited by their increasing absorption in the material of the optical parts. The bat has a potential power for resolving objects in its neighbourhood which is comparable to that for radar, since an ultrasonic frequency of 50 kc.p.s. corresponds to a wave-length of 0·7 cm., whereas for radar the operation frequency of 30,000 Mc.p.s. means a wave-length of 1 cm. In addition, by virtue of its two ears as receivers, it has an additional advantage of stereophonic perception. The mechanism of the generation of these waves is not yet fully understood, but Hartridge comes to the conclusion that it must take place through the nose, which is also used for breathing. He assumes that the bat breathes only slowly, say 2 or 3 times per second, and that it phonates during both expiration and inspiration just as a skylark appears to

sing continuously and a cat purr during the complete breathing cycle. The various arguments involved cannot be considered here, but there remains one further point of interest, namely, the possibility of the returning echo overlapping with the emitted sound. This may be avoided if the duration of each cry is reduced as the object is approached, which may conceivably occur if the rate of production of tones increases as already stated. The problem of direct sound reaching the ears of the bat is apparently avoided, in one group of bats by the shaping of the snout to give rise to a well-defined beam in a forward direction, and in the other type by the existence of what is known as the tragus in front of the external auditory meatus and which would appear to function as a screen to the direct sound. The possibilities of obstacle detection by ultrasonic means are being thoroughly investigated with respect to providing a means of guiding blind people and as an aid to road transport in foggy weather.



[*Parker and Andrade.*

Fig. 16.23. Standing wave system in an enclosed air column, resonating at 220 Kc/s marked, on the walls of the glass tube, by double lines of coagulated particles of MgO smoke.

Submarine detection or depth-finding ultrasonic apparatus has been used for the location of fish shoals. Small fish may even be killed in a strong ultrasonic beam. It appears that dolphins are sensitive to high pitched notes for they have been observed to disappear from the proximity of a ship when its ultrasonic echo-sounding apparatus has been switched on.

A marked biological action has been observed when living specimens, plants and various organisms, have been subjected to ultrasonic radiation, and in this direction an interesting application is its use for the destruction of bacteria in the sterilisation of milk. The bacteria cells in a fluid medium explode quickly on exposure to ultrasonic radiation and the antigens, normally difficult to extract, are liberated.

For further reading

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APPENDIX 1

More Exact Expression for the Period of a Simple Pendulum

Rewriting equation (2.15)

$$\ddot{\theta}=-\frac{g}{l}\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{A1.1})$$

The first integration of this equation may be made by multiplying each side by $\dot{\theta}dt$.

Then
$$\int\ddot{\theta}\,\dot{\theta}.dt=-\frac{g}{l}\int\sin\theta.d\theta$$

or
$$\frac{1}{2}\dot{\theta}^2=\frac{g}{l}\cos\theta+C \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{A1.2})$$

If the initial condition is $\dot{\theta}=0$ for $\theta=\alpha$, where α is the angular *amplitude* at $t=0$, then on substitution in (A1.2) the value of the constant is seen to be $\frac{-g}{l}\cos\alpha$.

Equation (A1.2) now becomes

$$\frac{1}{2}\dot{\theta}^2=\frac{g}{l}(\cos\theta-\cos\alpha) \quad . \quad . \quad . \quad . \quad . \quad (\text{A1.3})$$

or
$$\frac{d\theta}{\sqrt{\cos\theta-\cos\alpha}}=\sqrt{\frac{2g}{l}}.dt \quad . \quad . \quad . \quad . \quad (\text{A1.3}a)$$

But $\cos\theta=1-2\sin^2\frac{\theta}{2}$, hence (A1.3*a*) may be rewritten as

$$\frac{d\left(\frac{\theta}{2}\right)}{\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}}=\sqrt{\frac{g}{l}}\,dt \quad . \quad . \quad . \quad . \quad . \quad (\text{A1.4})$$

In order to integrate this equation the following substitution is made, viz. $\sin\frac{\theta}{2}=\sin\frac{\alpha}{2}\sin\phi$, a justifiable procedure as θ is never greater than α. Then it follows that $\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}=\sin\frac{\alpha}{2}\cos\phi$ $=\beta\cos\phi$, say, where β is a constant for a limited motion. Furthermore by differentiating the expression $\sin\frac{\theta}{2}=\sin\frac{\alpha}{2}\sin\phi$ it is easily verified that $d\left(\frac{\theta}{2}\right)=\frac{\beta\cos\phi}{\sqrt{1-\beta^2\sin^2\phi}}d\phi$, and on substitution in (A1.4) the the following expression is obtained.

$$\int\frac{d\phi}{\sqrt{1-\beta^2\sin^2\phi}}=\int\sqrt{\frac{g}{l}}dt \quad . \quad . \quad . \quad . \quad . \quad (\text{A1.5})$$

Now $\theta=\alpha$ at $t=0$ and $\theta=0$ at $t=\frac{T}{4}$, where T is the periodic time of the motion.

Hence corresponding limits for integration of L.H.S. of equation (A1.5) will be given by $\sin\phi=1$, *i.e.* $\phi=\frac{\pi}{2}$, for $\theta=\alpha$, and $\sin\phi=0$, *i.e.* $\phi=0$, for $\theta=0$.

Equation (A1.5) now becomes

$$T=4\sqrt{\frac{l}{g}}\int_{\frac{\pi}{2}}^{0}\frac{d\phi}{\sqrt{1-\beta^2\sin^2\phi}} \quad . \quad . \quad . \quad (A1.6)$$

The expression under the integral sign is known as an elliptic integral and it may be evaluated by means of mathematical tables of such integrals. For example, if α is as large as 20°, then $\beta=\sin 10°=0{\cdot}1736$ and T is found to be $6{\cdot}331\sqrt{\frac{l}{g}}$. For very small angles $\sin\phi=0$ and on simplification equation (A1.6) yields the approximate formula, already proved, $T=2\pi\sqrt{\frac{l}{g}}=6{\cdot}283\sqrt{\frac{l}{g}}$. Hence the error for such a large angle of swing as 20° is less than one per cent.

An approximate solution of equation (6) which suffices for most purposes, is derived by expanding the integrand as a series and integrating term by term.

Thus $$T=4\sqrt{\frac{l}{g}}\int_{\frac{\pi}{2}}^{0}\left(1+\frac{\beta^2}{2}\sin^2\phi+\frac{3}{8}\beta^4\sin^4\phi+\ .\ .\ .\right)d\phi \quad (A1.7)$$

Considering only the first two terms of the integrand equation (A1.7) becomes

$$T=4\sqrt{\frac{l}{g}}\cdot\left[\phi+\frac{\beta^2}{4}\left(\phi-\frac{1}{2}\sin 2\phi\right)\right]_{\frac{\pi}{2}}^{0}$$

$$=2\pi\sqrt{\frac{l}{g}}\left[1+\frac{\beta^2}{4}\right]$$

$$=2\pi\sqrt{\frac{l}{g}}\left[1+\frac{\sin^2\frac{\alpha}{2}}{4}\right] \quad . \quad . \quad . \quad . \quad (A1.8)$$

If α is small and is in radians the above expression assumes the simplified form

$$T=2\pi\sqrt{\frac{l}{g}}\cdot\left[1+\frac{\alpha^2}{16}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (A1.9)$$

In practice, due to damping, the angular amplitude falls from α_o to α_n, say, while n periods are being observed and if $\alpha_o-\alpha_n$ is small expression (A1.9) may be written as

$$T=2\pi\sqrt{\frac{l}{g}}\left[1+\frac{\alpha_o\alpha_n}{16}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (A1.10)$$

APPENDIX 2

Damped Harmonic Motion

In discussing simple harmonic motion the non-existence of any frictional forces was presumed, a state of affairs which is highly improbable in nature. Work has to be performed in overcoming any frictional resistance and so the original store of energy in the vibrating system becomes gradually dissipated as heat in the surrounding medium. As a consequence the amplitude of the vibrations will decrease continuously until the motion ceases This decay of amplitude is known as damping and the motion is referred to as *damped harmonic motion.* A pendulum vibrating in air is only lightly damped, but in water the damping would be considerable, and in treacle it might not even oscillate but merely return to its rest position after an initial displacement. Unlike the frictional force between two solids, the friction experienced by a body moving in a fluid medium is dependent upon the fluid contact area but independent of the nature of the surface of the body provided it is smooth. For low velocities the frictional force opposing a solid body moving through a fluid may be taken as approximately proportional to the first power of the velocity. Hence if R be the frictional force per unit velocity the resistance force experienced by a body moving with velocity $\frac{dx}{dt}$ is given by $R\frac{dx}{dt}$.

Equation (13*a*) now becomes modified as follows:

$$M\frac{d^2x}{dt^2}+R\frac{dx}{dt}+Sx=0 \quad . \quad . \quad . \quad . \quad . \quad (A2.1)$$

where S is the restoring force per unit displacement.

This equation is conveniently written as

$$\ddot{x}+r\dot{x}+sx=0 \quad . \quad . \quad . \quad . \quad . \quad (A2.2)$$

where $r=\frac{R}{M}$ and $s=\frac{S}{M}$.

This second order differential equation may be solved by trying $x=Ae^{\lambda t}$ as a solution.

On substitution $(\lambda^2+r\lambda+s)Ae^{\lambda t}=0$, and the roots of the quadratic are $\lambda_1=-\frac{r}{2}+\sqrt{\frac{r^2}{4}-s}$ and $\lambda_2=-\frac{r}{2}-\sqrt{\frac{r^2}{4}-s}$.

The general solution of the differential equation is given by the sum of the two separate solutions corresponding to the roots λ_1 and λ_2, i.e.

$$x=A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t} \quad . \quad . \quad . \quad . \quad . \quad (A2.3)$$

where A_1 and A_2 are constants which can be evaluated from the defining conditions of the motion. Hence on substitution

$$x=e^{-\frac{rt}{2}}\left[A_1e^{+\sqrt{\frac{r^2}{4}-s}.t}+A_2e^{-\sqrt{\frac{r^2}{4}-s}.t}\right] \quad . \quad (A2.4)$$

For any particular system the physical nature of the resulting motion

will depend upon the relative magnitudes of r and s, *i.e.* of $\frac{R}{M}$ and $\frac{S}{M}$ respectively.

Four different cases may be recognised.

(1) $R=0$, *i.e.* frictional resistance is absent, this case corresponding to the S.H.M. already considered. The solution (A2.4) above reduces to

$$x=A_1e^{j\sqrt{s}.t}+A_2e^{-j\sqrt{s}.t} \quad . \quad . \quad . \quad . \quad (A2.5)$$

where $j=\sqrt{-1}$.

If the exponential terms are replaced by their trigonometrical equivalents equation (A2.5) may be rewritten as

$$\begin{aligned} x&=A_1(\cos\sqrt{s}.t+j\sin\sqrt{s}.t)+A_2(\cos\sqrt{s}.t-j\sin\sqrt{s}.t)\\ &=C\cos\sqrt{s}.t+D\sin\sqrt{s}.t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A2.6)\end{aligned}$$

where C and D are constants.

This solution may be rewritten in a number of alternative ways, as follows, since from (A2.6)

$$x=\sqrt{C^2+D^2}\left\{\frac{C}{\sqrt{C^2+D^2}}\cos\sqrt{s}.t+\frac{D}{\sqrt{C^2+D^2}}\sin\sqrt{s}.t\right\}$$

whence

$$\left.\begin{aligned} &(a)\quad x=A_0\sin(\sqrt{s}.t-\alpha),\ \text{where}\ \tan\alpha=-\frac{C}{D}\\ &(b)\quad x=A_0\sin(\sqrt{s}.t+\beta),\ \text{where}\ \tan\beta=\frac{C}{D}\\ &(c)\quad x=A_0\cos(\sqrt{s}.t+\gamma),\ \text{where}\ \tan\gamma=-\frac{D}{C}\\ &(d)\quad x=A_0\cos(\sqrt{s}.t-\epsilon),\ \text{where}\ \tan\epsilon=\frac{D}{C}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (A2.7)$$

A_0 is a constant equal to $\sqrt{C^2+D^2}$ and therefore involving A_1 and A_2. The period T_0 of this undamped motion is given by

$$T_0=\frac{2\pi}{\sqrt{s}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A2.8)$$

(2) $\frac{r^2}{4}<s$, *i.e.* the damping is small. Hence following (A2.4) the solution becomes

$$\begin{aligned} x&=e^{-\frac{rt}{2}}\left[A_1e^{j\sqrt{s-\frac{r^2}{4}}.t}+A_2e^{-j\sqrt{s-\frac{r^2}{4}}.t}\right]\\ &=e^{-\frac{rt}{2}}\left[C\cos\sqrt{s-\frac{r^2}{4}}.t+D\sin\sqrt{s-\frac{r^2}{4}}.t\right]\end{aligned}$$

or $$x=A_0e^{-\frac{rt}{2}}\sin\left(\sqrt{s-\frac{r^2}{4}}.t-\alpha\right) \quad . \quad . \quad . \quad . \quad . \quad (A2.9)$$

where $\tan\alpha=-\frac{C}{D}$ and A_0 involves A_1 and A_2. This equation

represents a simple harmonic motion, of period $T=\dfrac{2\pi}{\sqrt{s-\dfrac{r^2}{4}}}$, in which the amplitude of motion decays exponentially with time. It is evident that both the damping and period decrease as r becomes smaller. An expression for the variation of period with damping in terms of the undamped period (T_0), is easily deduced from the expression for the logarithmic decrement (δ) given on p. 222. Since $\delta=\dfrac{R}{M}\cdot\dfrac{T}{2}=r\dfrac{T}{2}$ it follows that

$$\frac{4\pi^2}{T^2}=s-\frac{r^2}{4}=\frac{4\pi^2}{T_0^2}-\frac{4\delta^2}{4T^2} \text{ or } T=T_0\sqrt{1+\frac{\delta^2}{4\pi^2}}=T_0\left(1+\frac{1}{8}\frac{\delta^2}{\pi^2}\right) \text{approximately.}$$

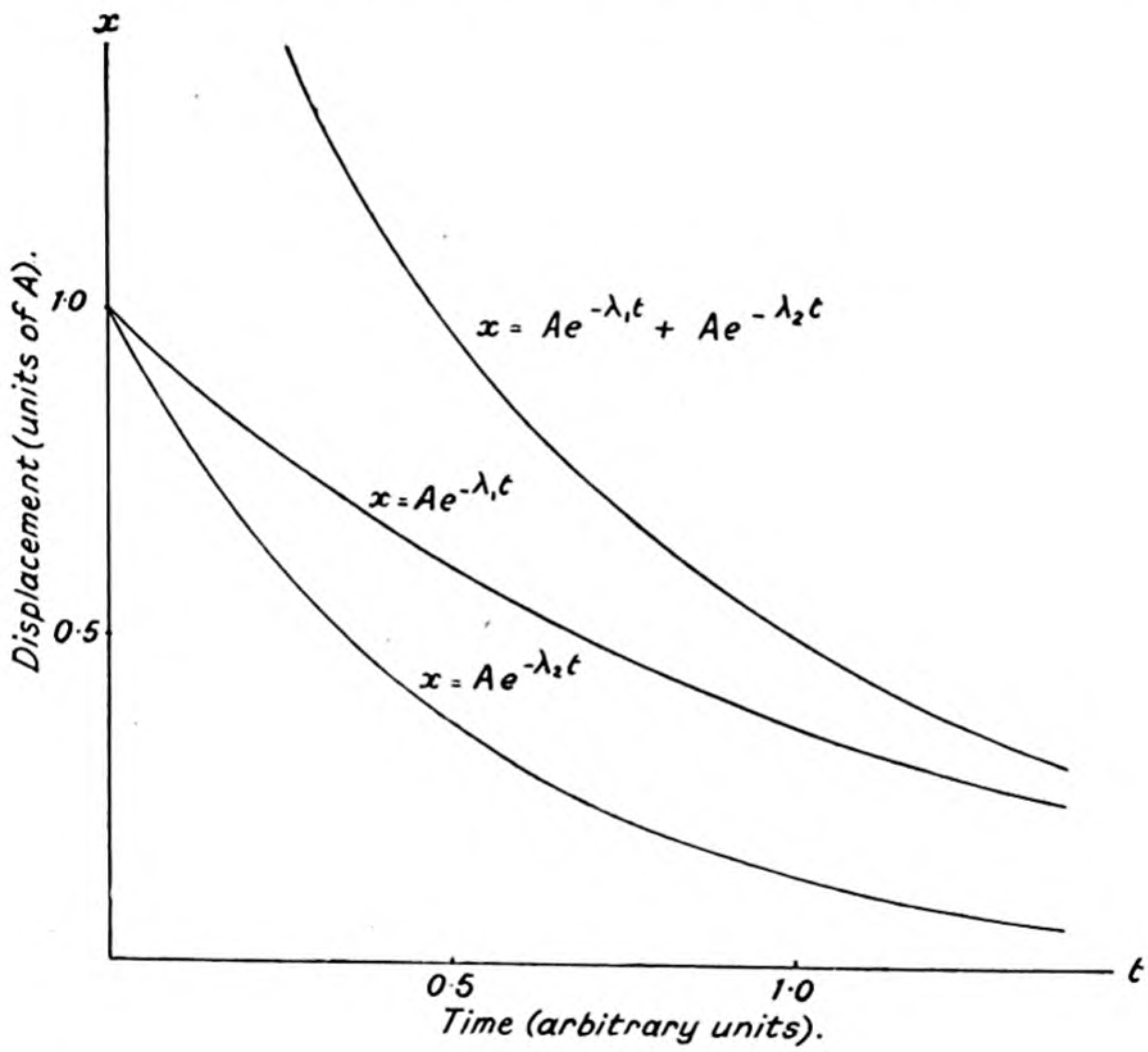


Fig. A2.1.

(3) $\dfrac{r^2}{4}>s$. This is the case of *over-damping* or *aperiodic* motion and the solution of the equation becomes

$$x=A_1e^{\left[-\frac{r}{2}+\sqrt{\frac{r^2}{4}-s}\right]t}+A_2e^{\left[-\frac{r}{2}-\sqrt{\frac{r^2}{4}-s}\right]t} \quad . \quad \text{(A2.10)}$$

Both exponential indices are seen to be negative and it is easily verified that $\dfrac{dx}{dt}$ is always negative and never changes sign. Fig. A2.1 shows a plot of displacement against time for the particular case where the initial displacements are equal, i.e. $A_1=A_2=A$. The rate of decay

becomes more rapid as r diminishes and is most rapid in the limiting case, viz. $\frac{r^2}{4}=s$.

(4) $\frac{r^2}{4}=s$. This case, known as *critical damping*, is of practical importance since the displaced system comes to rest in the minimum time. The solution of the equation of motion is indeterminate if derived from the above working, *i.e.* $\lambda_1=\lambda_2$. But the general solution must contain two arbitrary constants so the substitution $x=ue^{-\frac{rt}{2}}$ is tried as a solution, where u is an undetermined function of t. Substitution in (A2.2) leads to the equation $\frac{d^2u}{dt^2}=0$, which on integration gives the solution $u=C_1+C_2t$, where C_1 and C_2 are arbitrary constants.

Hence the solution of the general equation in the case of critical damping is

$$x=(C_1+C_2t)e^{-\frac{rt}{2}} \quad . \quad . \quad . \quad . \quad (A2.11)$$

The expression in the bracket increases with time but at a slower rate than the decrease in displacement due to the exponential factor.

APPENDIX 3

Linear Damped Harmonic Motion and its Representation by Means of a Logarithmic Spiral

Simple undamped harmonic motion has been shown to be represented by the projection of the uniform circular motion of a particle on a diameter of the circle. In the case of *small* damping, since the amplitude of motion is continually decreasing it is natural to presume that the reference circle of S.H.M. will be replaced by a reference spiral, which by the nature of the constancy of the *ratio* of successive amplitudes to one another is logarithmic in type.

A geometrical feature of the logarithmic spiral is that the angle vPR in Fig. A3.1) between the radius vector (OP) and the tangent (Pv) drawn to the curve at any point P is a constant and is equal to $\left(\frac{\pi}{2}+\theta\right)$ in Fig. A3.1.

The problem is to show that the projection Q of P on the axis of reference Ox represents a linear damped harmonic motion, when the particle P moves with a variable speed along the spiral such that the angular velocity $\omega\left(=\frac{d\phi}{dt}\right)$ of the radius vector is constant.

Let v be the velocity of P at a particular instant then its components along and perpendicular to the radius vector are given by $-\frac{dr}{dt}$ and $\frac{rd\phi}{dt}$ respectively.

But $\frac{\pi}{2}+\theta=$constant, *i.e.* $\tan\theta=$constant, and hence

$$\frac{-\frac{dr}{dt}}{r\frac{d\phi}{dt}}=-\frac{1}{r}\cdot\frac{dr}{d\phi}=\tan\theta=C, \text{ say} \quad . \quad . \quad . \quad . \quad . \quad \text{(A3.1)}$$

Integrating this equation,

$$\log r=-C\phi+A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A3.2)}$$

The constant A is evaluated by applying the initial conditions that at $t=0$, $r=r_0$ and $\phi=0$, when it follows that $A=\log r_0$. Hence equation (A3.2) becomes

$$r=r_0e^{-C\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A3.3)}$$

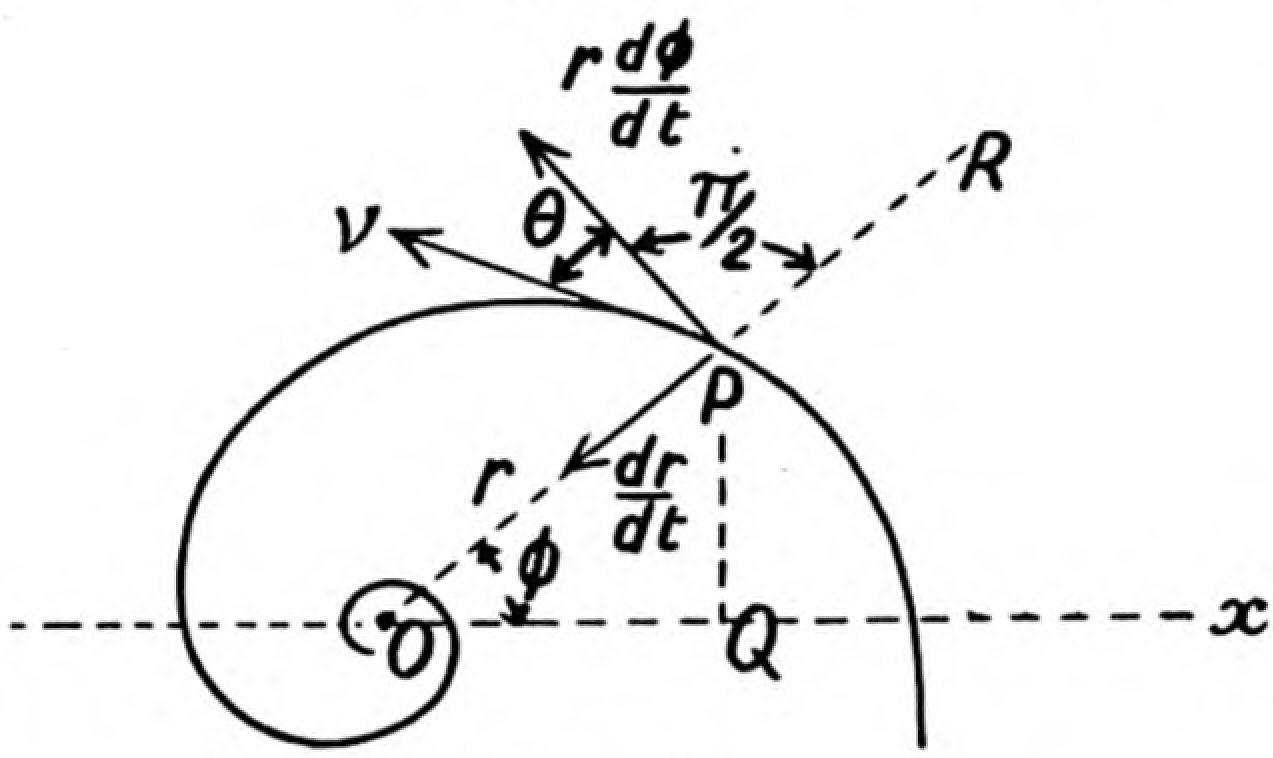


Fig. A3.1.

To find the acceleration of the projection Q it is noted that its displacement $OQ=x=r\cos\phi$;

$$\therefore \text{ the velocity of } Q=\frac{dx}{dt}=(\cos\phi)\frac{dr}{dt}-(r\sin\phi)\frac{d\phi}{dt}$$

$$=(\cos\phi).-r\frac{d\phi}{dt}.C-(r\sin\phi)\frac{d\phi}{dt}$$

$$=-x\omega C-(r\sin\phi)\omega \quad . \quad . \quad . \quad . \quad . \quad \text{(A3.4)}$$

since from (A3.3)

$$\frac{dr}{dt}=-Cr_0e^{-C\phi}.\frac{d\phi}{dt}=-Cr\frac{d\phi}{dt} \quad . \quad . \quad . \quad . \quad \text{(A3.5)}$$

The acceleration of Q will be given by the second derivative of x hence from (A3.4)

$$\frac{d^2x}{dt^2}=-C\omega\frac{dx}{dt}-\omega\sin\phi\frac{dr}{dt}-\omega^2r\cos\phi$$

i.e.

$$\frac{d^2x}{dt^2}=-\omega\left\{C\frac{dx}{dt}+\sin\phi.\frac{dr}{dt}+\omega x\right\} \quad . \quad . \quad . \quad \text{(A3.6)}$$

Substituting for $\sin\phi.\frac{dr}{dt}$ from (A3.4) and (A3.5) the above equation on rearrangement becomes

$$\frac{d^2x}{dt^2}+2\omega C\frac{dx}{dt}+\omega^2(1+C^2)x=0 \quad . \quad . \quad . \quad \text{(A3.7)}$$

The coefficients of x and $\frac{dx}{dt}$ in the above equation are constants, hence it may be written in the form

$$\frac{d^2x}{dt^2}+\alpha\frac{dx}{dt}+\beta x=0$$

which is immediately identifiable as the general form of equation representing a lightly damped harmonic motion, as $\frac{\alpha^2}{4}<\beta$.

APPENDIX 4

Forced Vibrations

Consider the equation $M\ddot{x}+R\dot{x}+Sx=F\sin ft$ which represents the motion of a system under a maintained external periodic force of frequency $\frac{f}{2\pi}$ (see p. 221). Rewrite this equation as $\ddot{x}+\alpha\dot{x}+\beta x=A\sin ft$ and try as a solution $x=B\sin(ft-\phi)$.
Then $\dot{x}=Bf\cos(ft-\phi)$ and $\ddot{x}=-Bf^2\sin(ft-\phi)$. Substituting in the original equation

$$-Bf^2\sin(ft-\phi)+\alpha Bf\cos(ft-\phi)+\beta B\sin(ft-\phi)=A\sin ft$$

or $$B(-f^2\sin ft\cos\phi+f^2\cos ft\sin\phi+\alpha f\cos ft\cos\phi+\alpha f\sin ft\sin\phi$$
$$+\beta\sin ft\cos\phi-\beta\cos ft\sin\phi)=A\sin ft.$$

$$[A+B(f^2\cos\phi-\alpha f\sin\phi-\beta\cos\phi)]\sin ft$$
$$+B[-f^2\sin\phi-\alpha f\cos\phi+\beta\sin\phi]\cos ft=0$$

which is satisfied for all times provided coefficients of $\sin ft$ and $\cos ft$ vanish.

When the coefficient of $\cos ft$ is equated to zero

$$\beta\sin\phi-\alpha f\cos\phi-f^2\sin\phi=0$$

or $$\tan\phi=\frac{\alpha f}{\beta-f^2},\ \sin\phi=\frac{\alpha f}{\sqrt{(\beta-f^2)^2+\alpha^2f^2}}$$

$$\cos\phi=\frac{(\beta-f^2)}{\sqrt{(\beta-f^2)^2+\alpha^2f^2}}$$

and on equating the coefficient of $\sin ft$ to zero it follows that

$$B=\frac{A}{\beta\cos\phi+\alpha f\sin\phi-f^2\cos\phi}$$

$$=A\frac{\sqrt{(\beta-f^2)^2+\alpha^2f^2}}{[(\beta-f^2)^2+\alpha^2f^2]}$$

whence $$x=\frac{A}{\sqrt{(\beta-f^2)^2+\alpha^2f^2}}\sin(ft-\phi)$$

where $$\tan\phi=\frac{\alpha f}{\beta-f^2}\quad\ldots\ldots\ldots\ldots\quad\text{(A4.1)}$$

The denominator of this expression is a measure of the magnitude of the mechanical impedance Z of the system which may be written in

complex notation (see p. 378) as $Z=Z_1+jZ_2$ where the real part Z_1 is known as the mechanical resistance and jZ_2 as the mechanical reactance (cf. corresponding electrical quantities).

In an electrical circuit it should be noted an impressed E.M.F. $E \sin ft$ would give rise to a current

$$i=\frac{E \sin (ft-\phi)}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}} \quad . \quad . \quad . \quad . \quad . \quad \text{(A4.2)}$$

which is analogous to the mechanical case if it is noted that i corresponds to $\dot{x}$, and

$$\dot{x}=f\frac{A \cos (ft-\phi)}{\sqrt{(\beta-f^2)^2+a^2f^2}}=\frac{A \cos (ft-\phi)}{\sqrt{a^2+\left(f-\frac{\beta}{f}\right)^2}}$$

$$=\frac{F \cos (ft-\phi)}{\sqrt{R^2+\left(fM-\frac{S}{f}\right)^2}} \quad . \quad . \quad . \quad \text{(A4.3)}$$

APPENDIX 5

Coupled Vibrations

Any mechanical vibrating system will communicate its motion to a varying extent upon any other such system in contact with it. Two separate systems which are linked together in some manner so that neither is able to perform it own natural or free vibration without being affected by the other are referred to as a *coupled* system. When only a slight mutual interference exists between the systems they are said to be *loosely coupled*, but when their influence upon each other is considerable, the condition of coupling is said to be *tight* or *close*.

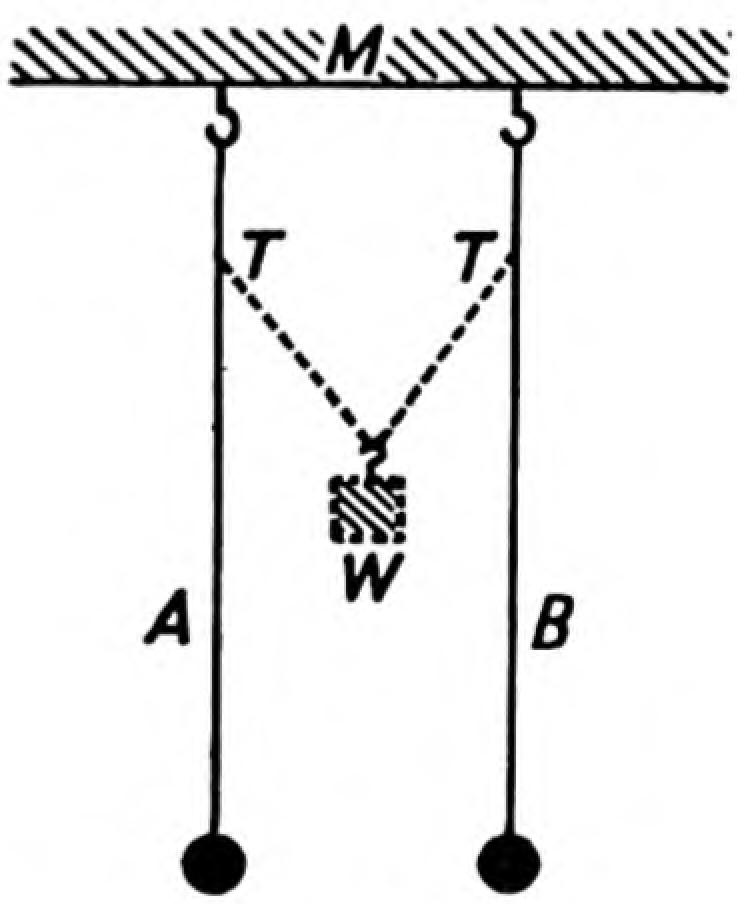


Fig. A5.1.

A simple illustration of a coupled system is provided by two simple pendulums A and B (Fig. A5.1) suspended from the same wooden beam M, which is capable of acting as a weak link in transmitting energy between the pendulums when set into vibration. If A and B consist of heavy lead balls at the ends of two cords, each about 2 yd. long and 12 to 18 in. apart then, on moving, say, A perpendicular to their common plane of equilibrium, it will be found that after a somewhat longish time the pendulum B begins to show appreciable movement. This *rate* of energy transfer is increased by *tightening* the coupling between A and B, for example, by attaching a thread TT and weight W (shown dotted in Fig. A5.1), and the lower the points of attachment of this thread the closer is the degree of coupling. It will also

be noted that as the amplitude of B's motion gradually builds up, that of A correspondingly decreases until this pendulum is momentarily at rest in its undisplaced position. The transfer of energy now takes place in the opposite direction, and the amplitude of B gradually diminishes to zero, while that of A correspondingly builds up again to a maximum. This cycle of events would be repeated indefinitely in an undamped system, but in practice, of course, the successive maximum amplitudes gradually get smaller as indicated in Fig. A5.2.

This energy interchange between the pendulums will involve some change in their natural or free periods, for each is now *loaded* to a certain extent due to the presence of the other, and one way of regarding the action is from the aspect of forced vibrations in which the resonator has a strong reaction upon the exciter. It is shown later that if n_1

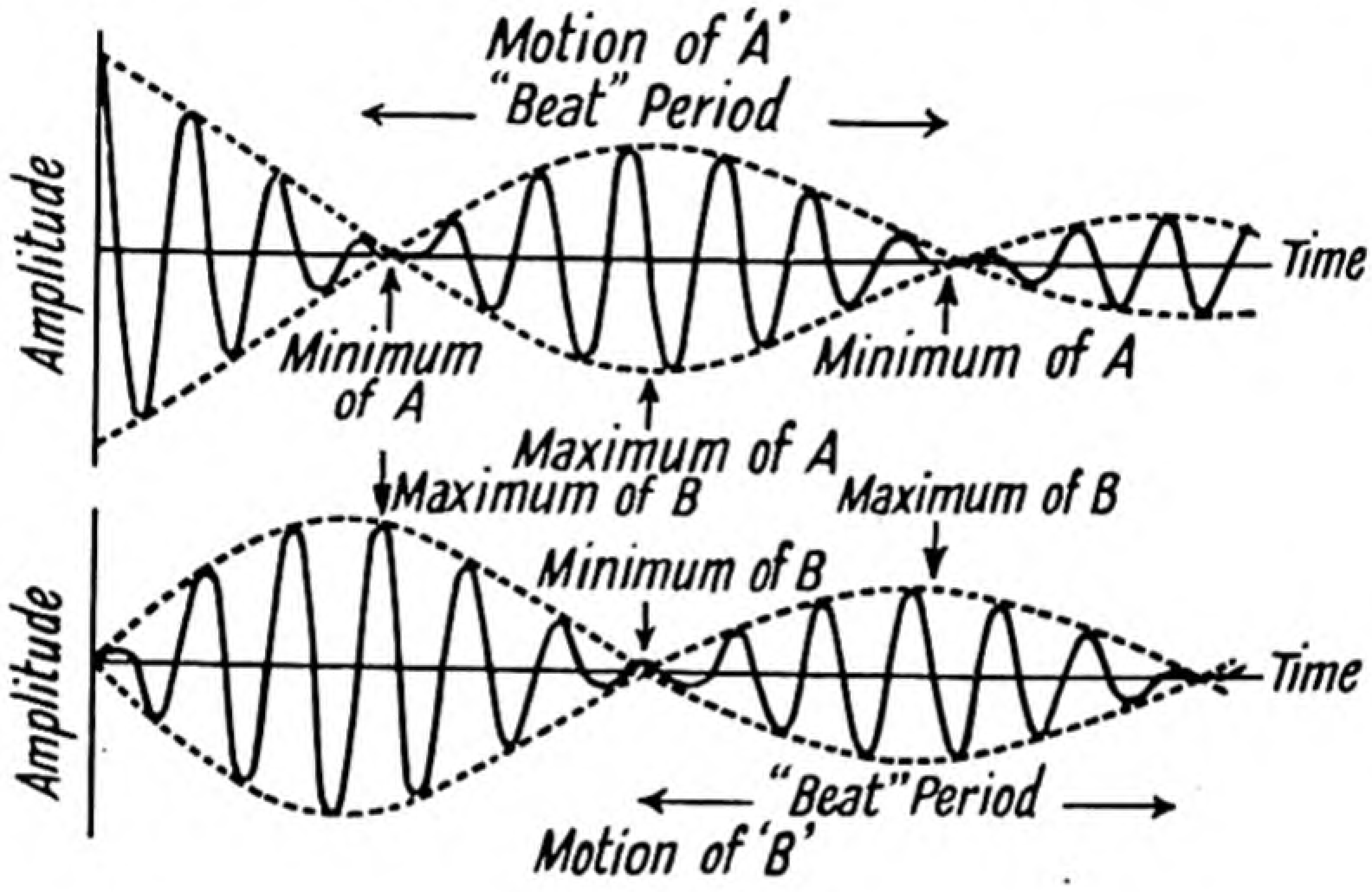


Fig. A5.2.

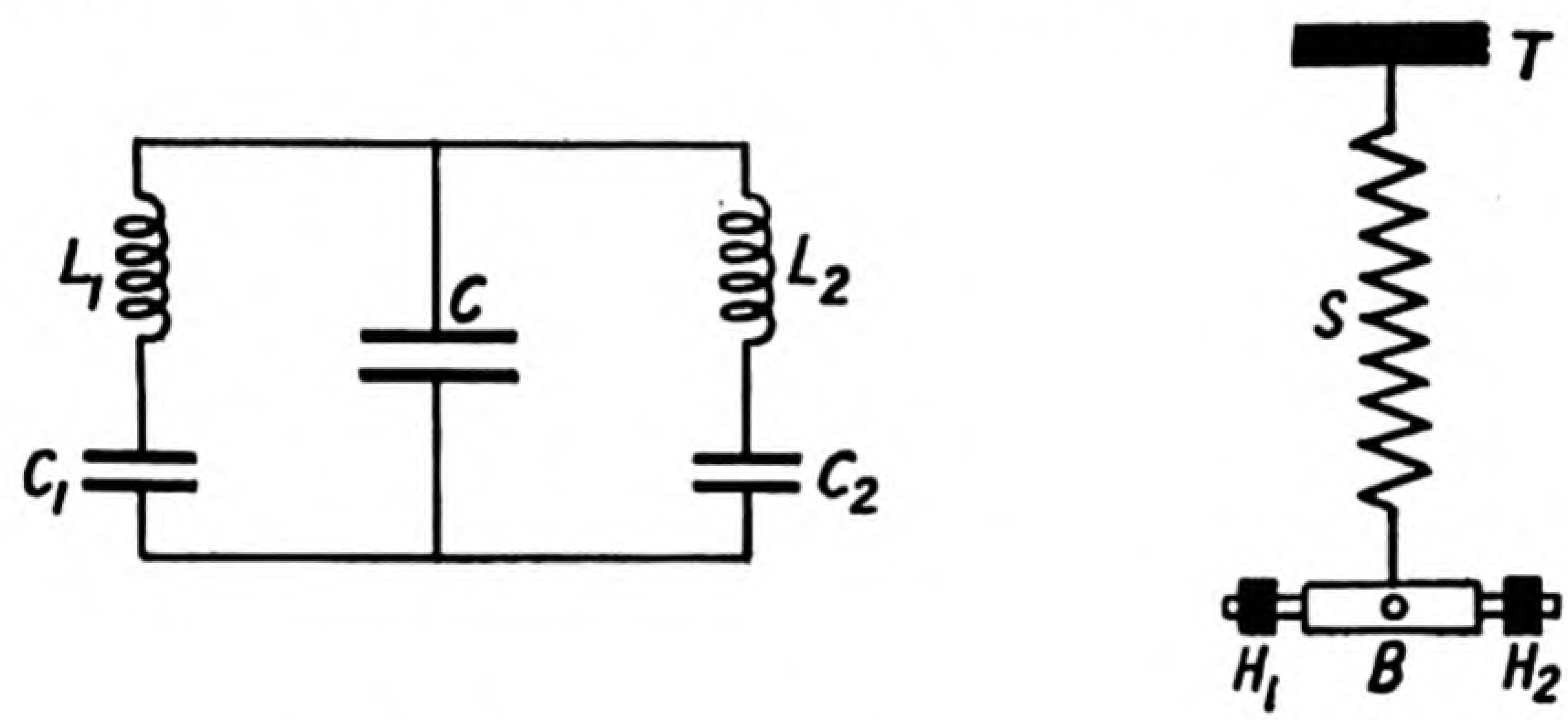
and n_2 are the free frequencies of A and B respectively, then the coupled system will possess two frequencies N_1 and N_2 which are given by $N_1 = \dfrac{n}{\sqrt{1-K}}$ and $N_2 = \dfrac{n}{\sqrt{1+K}}$ for the particular case where $n_1 = n_2 = n$, K being the coefficient of coupling. N_1 and N_2 are thus respectively greater and less than the natural frequencies of the component systems, and hence their superposition will give rise to "beats" as made evident by Fig. A5.2. The frequency of the "beats" is given by

$$N_1 - N_2 = n\left[\frac{1}{\sqrt{1-K}} - \frac{1}{\sqrt{1+K}}\right],$$

which reduces to nK for very small values of K, thus verifying a statement made above that the rate of energy transfer, proportional to the frequency, is smaller the looser the coupling. On the other hand, the frequency of the beats will *increase* with the tightness of coupling.

Two further examples of coupled systems are shown in Fig. A5.3 *a* and *b*, the former indicates how two electrical circuits can be coupled together by means of a condenser C, the larger this capacitance the weaker being the electrical linkage between the circuits. The other example is a device known as Wilberforce's spring, indicated by S

and fixed at the upper end T. It is essentially a system having two degrees of freedom, one of which is associated with a to-and-fro rotational motion about the vertical axis and the other an up-and-down motion. Now if a coiled spring, *i.e.* one in which the turns can no longer be considered as flat, is stretched, then a small torque is also simultaneously exerted, and vice-versa. Hence, on the release of such a spring the subsequent motion will depend upon both Young's modulus and the modulus of rigidity of the material of the spring. By adjustment of the two screws H_1 and H_2 it is possible to vary the moment of inertia of the suspended system B about the vertical axis of symmetry, until finally the period of vibration for rotational and up-and-down motions are approximately identical. When this is the case and the spring is set into up-and-down vibrations its motion will be found to gradually change until only a rotational motion occurs, and this in turn will give way to up-and-down vibrations, and so on. The sole coupling is brought about by the above-mentioned fact that torques and pulls are not inseparable in the case of an inclined spring.



(*a*) (*b*)

Fig. A5.3.

Theory of coupled vibrations

As a first approximation the loading of one of the component systems (A or B) is to be regarded as being equivalent to an added force due to the mass acceleration of the other. Furthermore, damping or frictional forces are neglected.

Then if m_1 and m_2 and y_1 and y_2 are respectively the masses and the instantaneous displacements of the component systems it follows that

$$m_1\frac{d^2y_1}{dt^2}+M\frac{d^2y_2}{dt^2}+\beta_1y_1=0 \quad . \quad . \quad . \quad (A5.1)$$

and

$$m_2\frac{d^2y_2}{dt^2}+M\frac{d^2y_1}{dt^2}+\beta_2y_2=0 \quad . \quad . \quad . \quad (A5.2)$$

where β_1 and β_2 are the respective restoring forces per unit displacement and M is the "coupling mass" between the two systems as defined by the coefficient of coupling K, viz. $K^2=\frac{M^2}{m_1m_2}$. This expression is analogous to that employed for electrical circuits, i.e. $K^2=\frac{M^2}{L_1L_2}$, where

L_1 and L_2 are the respective self-inductances of the component circuits and M their coefficient of mutual induction. The electrical equations corresponding to (A5.1) and (A5.2) are (q and c being the electric charge and capacitance respectively)

$$L_1\frac{d^2q_1}{dt^2}+M\frac{d^2q_2}{dt^2}+\frac{q_1}{c_1}=0 \text{ and } L_2\frac{d^2q_2}{dt^2}+M\frac{d^2q_1}{dt^2}+\frac{q_2}{c_2}=0.$$

Returning to equations (A5.1) and (A5.2) and differentiating each twice it follows that

$$m_1\frac{d^4y_1}{dt^4}+M\frac{d^4y_2}{dt^4}+\beta_1\frac{d^2y_1}{dt^2}=0 \quad . \quad . \quad . \quad \text{(A5.3)}$$

and

$$m_2\frac{d^4y_2}{dt^4}+M\frac{d^4y_1}{dt^4}+\beta_2\frac{d^2y_2}{dt^2}=0 \quad . \quad . \quad . \quad \text{(A5.4)}$$

If $n_1=\frac{\omega_1}{2\pi}$ and $n_2=\frac{\omega_2}{2\pi}$ are the natural frequencies of each system when free then equations (A5.1), (A5.2), (A5.3) and (A5.4) may be rewritten as follows:

$$\frac{d^2y_1}{dt^2}+\frac{M}{m_1}\frac{d^2y_2}{dt^2}+\omega_1{}^2y_1=0 \quad . \quad . \quad . \quad . \quad \text{(A5.5)}$$

$$\frac{d^2y_2}{dt^2}+\frac{M}{m_2}\frac{d^2y_1}{dt^2}+\omega_2{}^2y_2=0 \quad . \quad . \quad . \quad . \quad \text{(A5.6)}$$

$$\frac{d^4y_1}{dt^4}+\frac{M}{m_1}\frac{d^4y_2}{dt^4}+\omega_1{}^2\frac{d^2y_1}{dt^2}=0 \quad . \quad . \quad . \quad \text{(A5.7)}$$

$$\frac{d^4y_2}{dt^4}+\frac{M}{m_2}\frac{d^4y_1}{dt^4}+\omega_2{}^2\frac{d^2y_2}{dt^2}=0 \quad . \quad . \quad . \quad \text{(A5.8)}$$

Substitute in (A5.7) the value of $\frac{d^4y_2}{dt^4}$ from (A5.8) and then the value of $\frac{d^2y_2}{dt^2}$ from (A5.5). The following equation is obtained:

$$(1-K^2)\frac{d^4y_1}{dt^4}+(\omega_1{}^2+\omega_2{}^2)\frac{d^2y_1}{dt^2}+\omega_1{}^2\omega_2{}^2y_1=0 \quad \text{(A5.9)}$$

If it is now assumed that the coupled system takes on a S.H.M. motion of frequency $N=\frac{\Omega}{2\pi}$, then

$$\frac{d^2y_1}{dt^2}+\Omega^2y_1=0 \text{ and } \frac{d^4y_1}{dt^4}-\Omega^4y_1=0 \quad . \quad . \quad \text{(A5.10)}$$

From (9) and (10) it follows that

$$(1-K^2)N^4-(n_1{}^2+n_2{}^2)N^2+n_1{}^2n_2{}^2=0;$$

and

$$\therefore\ N^2=\frac{(n_1{}^2+n_2{}^2)\pm\sqrt{(n_1{}^2-n_2{}^2)^2+4n_1{}^2n_2{}^2K^2}}{2(1-K^2)} \quad \text{(A5.11)}$$

In the particular case where $n_1=n_2=n$, (A5.11) becomes

$$N^2=\frac{n^2(1\pm K)}{1-K^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A5.12)}$$

Hence the two frequencies satisfying (A5.12) are

$$N_1 = \frac{\Omega_1}{2\pi} \quad \text{and} \quad N_2 = \frac{\Omega_2}{2\pi},$$

where
$$N_1 = \frac{n}{\sqrt{1-K}} \quad \text{and} \quad N_2 = \frac{n}{\sqrt{1+K}}.$$

It would seem, therefore, that each particle of the complex system can be excited at both frequencies simultaneously, and hence the resultant vibration can be expressed as $y = y_1 + y_2 = \sin \Omega_1 t + \sin \Omega_2 t$, assuming for simplicity unit amplitudes for each vibration. It follows that $y = \left[2 \cos \left(\frac{\Omega_1 - \Omega_2}{2}\right) t\right] \sin \left(\frac{\Omega_1 + \Omega_2}{2}\right) t$ which indicates a rapid vibration of frequency $\left(\frac{N_1 + N_2}{2}\right)$ in which the amplitude varies at a slower rate given by the frequency $\left(\frac{N_1 - N_2}{2}\right)$.

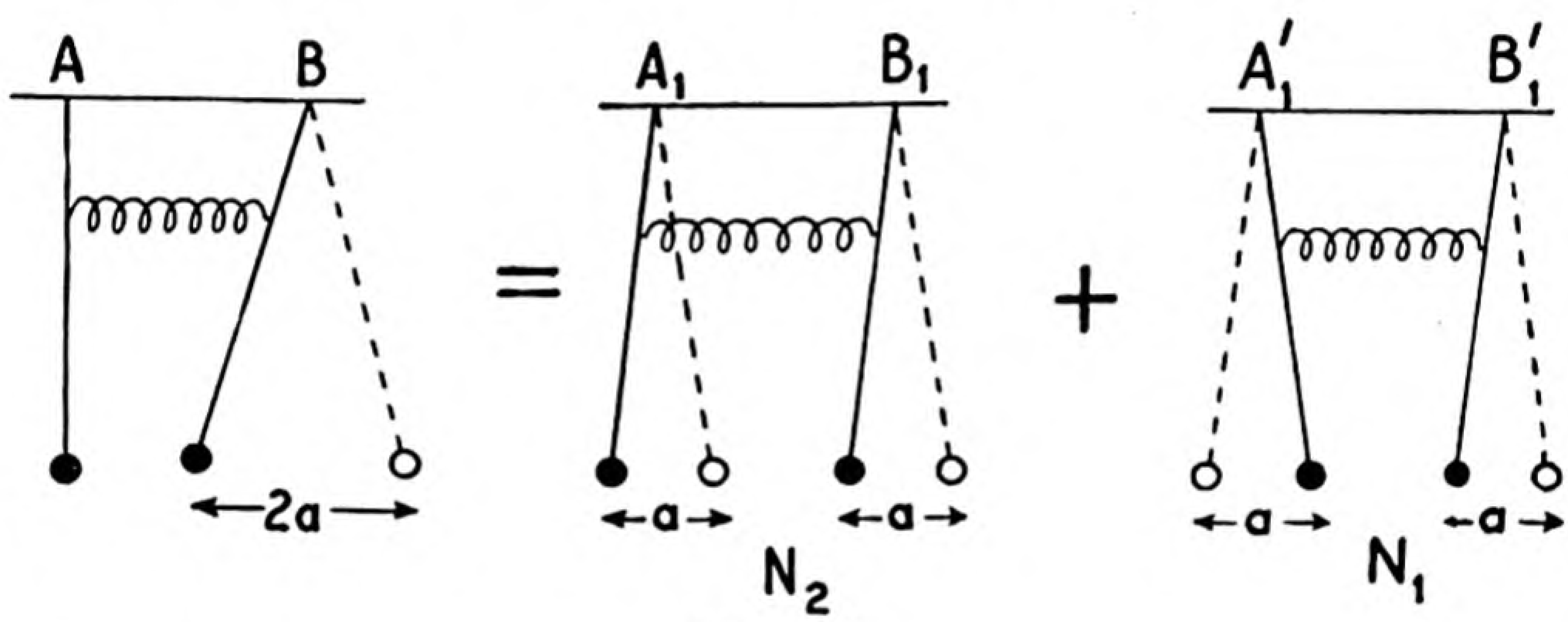


Fig. A5.4.

The above results may be conveniently applied to the system of two pendulums (A and B) in Fig. A5.4, each pendulum now possessing the same natural frequency n. If pendulum B is momentarily displaced, A remaining stationary, then the immediate motion of the system may be regarded as equivalent to the two partial motions shown in right-hand part of Fig. A5.4. In the *symmetric* N_2 mode the pendulums vibrate

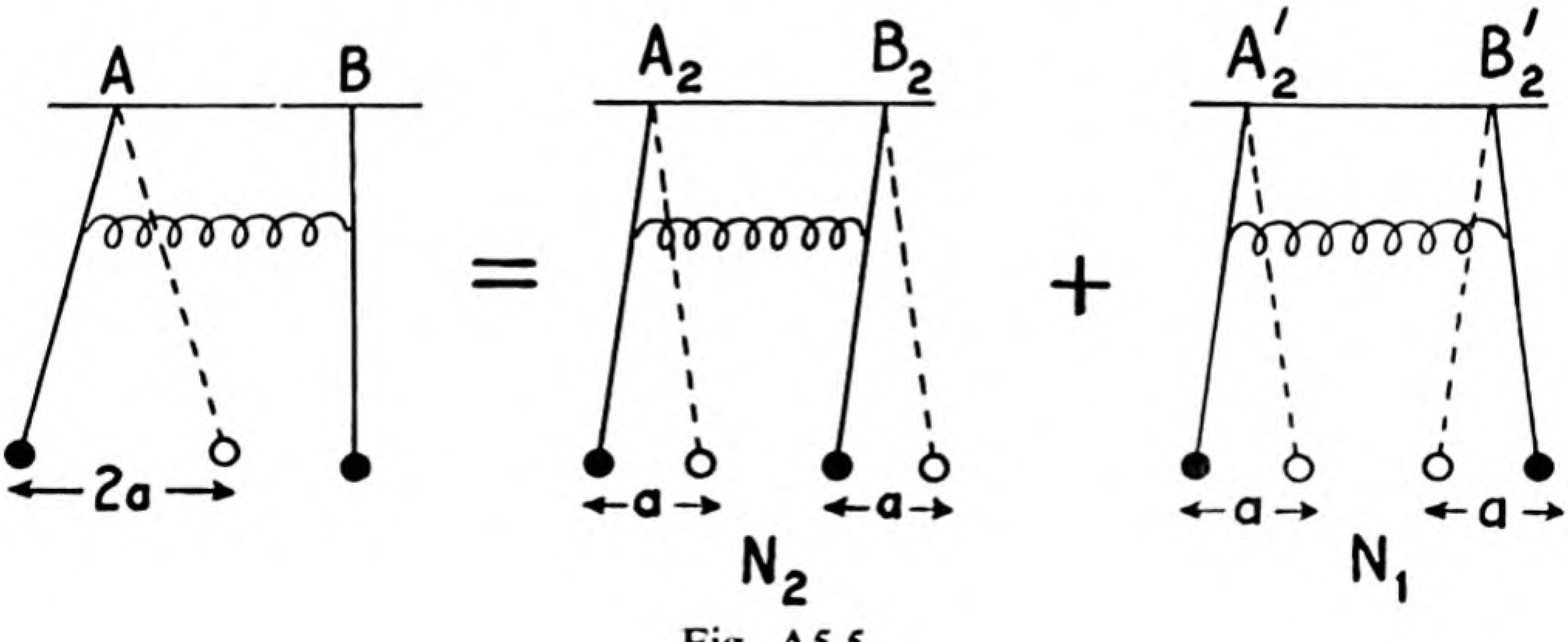


Fig. A5.5.

in phase with each other, *i.e.* the restoring force is less than for either vibrating alone, and hence $N_2<n$. On the other hand, for the *anti-symmetric* N_1 mode the pendulums are out of phase and tend to act against each other so that the restoring force is increased; consequently for this mode $N_1>n$. It follows since $N_1>N_2$ that the anti-symmetric will gain on the symmetric mode, and when it has advanced by π radians the state of affairs will be as given in Fig. A5.5, where pendulum B is now momentarily stationary. The phenomenon will be repeated again and again until the damping of the system brings the motion to rest.

An interesting example of the superposition of two modes of vibration, as above, occurs in the case of the "excited" helium atom. The actual state of this atomic system can be regarded as the sum of the symmetrical and anti-symmetrical modes of motion of the two electrons of the helium atom.

APPENDIX 6

Relaxation Oscillations

When a linear vibrating system possesses a negative damping coefficient it is evident from an inspection of equation (A2.1) and its solution, that its amplitude builds up to an infinite value, the sign of the exponential coefficient now being positive. Such a condition is physically untenable and in actual practice there is always a limit above which the damping assumes a positive value. In other words, the damping is non-linear in character and for a system of one degree of freedom the simplest way of expressing the equation of motion is

$$M\frac{d^2x}{dt^2}-(R_1-R_2x^2)\frac{dx}{dt}+Sx=0 \quad . \quad . \quad . \quad . \quad \text{(A6.1)}$$

where M, S, x and t have their previous significance and R_1 and R_2 are two resistance coefficients. It follows from the above equation that the damping is momentarily zero when the amplitude x_0 attains a value given by $R_1-R_2x_0{}^2=0$ or $x_0=\sqrt{\frac{R_1}{R_2}}$; for values of amplitude greater or less than x_0 the damping is respectively positive or negative. Hence when the displacement (x) is zero the system is unstable and the displacement increases aperiodically until it reaches the value $\sqrt{\frac{R_1}{R_2}}$, at which point the damping changes sign and eventually the system returns *towards* its origin.

Rewriting equation (A6.1) and substituting $y=\dfrac{x}{\sqrt{\frac{R_1}{R_2}}}$

$$\ddot{x}-\left(\frac{R_1-R_2x^2}{M}\right)\dot{x}+\frac{S}{M}x=0$$

or

$$\ddot{y}-\frac{R_1}{M}(1-y^2)\dot{y}+\frac{S}{M}y=0 \quad . \quad . \quad . \quad . \quad \text{(A6.2)}$$

Now the natural frequency $\left(\frac{\omega_0}{2\pi}\right)$ of the *undamped* system is given by $\omega_0{}^2=\frac{S}{M}=\frac{4\pi^2}{T_0{}^2}$ where T_0 is undamped period, therefore equation (A6.2) may be written as

$$\ddot{y}-\frac{R_1}{M}(1-y^2)\dot{y}+\omega_0{}^2y=0 \quad . \quad . \quad . \quad . \quad . \quad \text{(A6.3)}$$

Furthermore, if the time is expressed in units of $\frac{T_0}{2\pi}$ then τ the new time scale is given by $\tau=\frac{t}{\frac{T_0}{2\pi}}$ where t and T_0 are measured in seconds, and so equation (A6.3) now takes the form

$$\frac{d^2y}{d\tau^2}-a(1-y^2)\frac{dy}{d\tau}+y=0.$$

For the cases where $a=\frac{R_1}{M\omega_0}$ is very small, *i.e.* $<<1$, $\frac{d^2y}{d\tau^2}+y=0$, and hence $y=A\sin\tau$ is a particular solution. The motion is thus approximately harmonic of natural frequency $\frac{\omega_0}{2\pi}$ and hence $x=A\sqrt{\frac{R_1}{R_2}}\sin\omega_0t=x_0\sin\omega_0t$, and $\therefore\ \dot{x}=x_0\omega_0\cos\omega_0t$. Hence the maximum negative damping force, which occurs when $x=0$, is given by $R_1\dot{x}_{\max}=R_1x_0\omega_0$. Now the maximum value of the restoring force is $Sx_{\max}=Sx_0=Mx_0\omega_0{}^2$, therefore the ratio

$$\frac{\text{Maximum negative damping force}}{\text{Maximum restoring force}}=\frac{R_1x_0\omega_0}{Mx_0\omega_0{}^2}=\frac{R_1}{M\omega_0}=a.$$

The final amplitude x_0' attained by such a vibrating system of small a may be deduced by equating the energy introduced into the system by the negative damping force during the middle part of the "oscillation path" to the energy dissipated by the positive damping force at the ends of its path. Expressed mathematically it means

$$\int_0^{2\pi}\left[a(1-y^2)\left(\frac{dy}{d\tau}\right)^2\right]d\tau=0$$

where $y=A\sin\tau$, and the result obtained can be shown to give $A^2=4$ or numerically $x_0'=2\sqrt{\frac{R_1}{R_2}}$, *i.e.* $2x_0$.

Among the examples of relaxation oscillations in mechanical systems quoted by Pohl are the flapping of a flag in the wind, the scratching of a knife on a plate, the heart-beat, the aeolian harp, etc., while in electrical systems such oscillations are the basis of the action of the multivibrator, the blocking oscillator, the neon-lamp flashing circuit, etc. Reference is made to the latter (on p. 261) in connection

with the time-base of a cathode-ray oscillograph, and the approximate period of these oscillations, or flashings, may be deduced as follows. Using the notation on p. 261, the potential difference V between the condenser plate at any instant t after discharge of the neon lamp is given by $(V-V_E)=(E-V_E)(1-e^{-\frac{t}{CR}})$, where E, C and R are respectively, the E.M.F. of the electric supply, the capacitance and the high resistance in the series circuit. V_E is the potential difference across the neon tube when the discharge ceases and if V_S is the sparking potential of the gas then the discharge will recommence t_S seconds after the completion of the previous discharge where $t_S=(CR).\log_e\left(\frac{E-V_E}{E-V_S}\right)$. Neglecting the time of discharge (t_f in Fig. 13.28) then it follows from the above analysis that the period of these neon lamp relaxation oscillations is proportional to the product of capacitance and resistance. By the analogy of R and $\frac{1}{S}$ of a mechanical system with R and C respectively of the corresponding electrical system, then it follows that the period of the relaxation oscillations of the mechanical system is proportional to $\frac{R}{S}$. In this case $\alpha>>1$ and the assumption of a harmonic motion is no longer justifiable, and the motion, in fact, will contain a large number of higher harmonics as exemplified by the multivibrator (see Problem).

Suppose a small harmonic E.M.F. be injected into the multivibrator or similar relaxation oscillator circuit, and also let the frequency ω_r of this source be approximately equal to n times the fundamental frequency of the oscillator where n is an integer. Since these relaxation oscillations contain a large number of harmonics it follows that the nth harmonic will become excited by this external A.C. source and consequently the other harmonics also will be "pulled-in." The whole relaxation oscillation will therefore be excited at a frequency which is a *sub-multiple* of the injected frequency. This phenomenon forms the basis of automatic synchronisation and is known as "frequency demultiplication" or "sub-harmonic resonance"; values of n up to 200 have been realised.

Now the period of the mechanical relaxation oscillation has been shown to be proportional to $\frac{R}{S}$ and since $\omega_0=\sqrt{\frac{S}{M}}$ and R_1 may be associated with R, it follows that the period is proportional to $\frac{a}{\omega_0}$.

The outstanding features of relaxation oscillators as compared with ordinary harmonic oscillators may be summarised as follows:—(*a*) the period is determined by a relaxation time or time constant which is dependent only on elastic and resistance forces, (*b*) the waveform, especially on the steep "relaxing" part, contains a large number of harmonics of appreciable magnitude, and (*c*) the amplitude does not exhibit the phenomenon of resonance so that a relaxation oscillation may easily be brought into synchronisation with an external periodic force over a frequency range equal to an octave.

APPENDIX 7

Fourier Analysis

Ohm's law of acoustics states that all *musical* tones are periodic functions and that the ear recognises pendular vibrations alone as *simple* tones. Furthermore, it points out that all varieties of tone quality (or colour) are due to particular combinations of a larger or smaller number of simple tones of *commensurable* frequencies, and so a complex musical tone (or a composite mixture of musical tones) should be capable of being *analysed* into a sum of simple tones. The tone having the lowest frequency is termed the *fundamental*, and the other components are known as *overtones*.

The theorem which is at the basis of all *wave-form analysis* is that due to *Fourier*, who discovered it while working on the mathematical theory of heat conduction although Lagrange had almost anticipated it earlier when investigating the problem of vibrating strings. Now any periodic function of time, say $f(t)$, is understood to mean one which repeats itself in successive equal intervals of time. This time interval is known as the periodic time T, and the above condition implies that $f(t)=f(t+nT)$ where n is any positive or negative integer.

Fourier's Theorem shows that *any periodic* curve, however complicated, can always be reproduced by compounding a definite series of simple sinusoidal curves having frequencies in the ratios 1, 2, 3, etc., times that of the given curve. Hence the resultant displacement y of any particle in the wave represented by a complex periodic vibration is given by

$$y=f(t)=A_0+a_1\cos(\omega t+\xi_1)+a_2\cos(2\omega t+\xi_2)+\ldots+\ldots+a_r\cos(r\omega t+\xi_r) \quad . \quad . \quad . \quad \text{(A7.1)}$$

or
$$y=A_0+A_1\cos\omega t+A_2\cos 2\,\omega t+\ldots+A_r\cos(r\omega t)\ldots+B_1\sin\omega t+B_2\sin 2\omega t+\ldots+B_r\sin(r\omega t) \quad . \quad . \quad \text{(A7.2)}$$

where $A_1=a_1\cos\xi_1$, $A_2=a_2\cos\xi_2$, etc., $B_1=-a_1\sin\xi_1$, $B_2=-a_2\sin\xi_2$, etc., ξ_1, ξ_2, etc., are phase angles, and A_1, A_2, $\ldots$, B_1, B_2, $\ldots$, etc., represent the amplitudes of the various fundamental and harmonic terms, the fundamental frequency being given by $\frac{\omega}{2\pi}$. A_0 is a constant term and represents the mean level of the ordinates, *e.g.* a D.C. component in the case of electrical waves. The various amplitudes may be either positive, negative or zero, and so it is possible for either all the cosine or sine terms to be absent. If all the cosine terms are absent the function is said to be *odd* and the origin may be chosen so that $f(t)=-f(-t)$ for all values of t; the wave-form on opposite sides of this origin will then be reversed (see Fig. A7.1*a*). On the other hand, if no sine terms appear the function is an *even* one and it is possible to choose the origin so that $f(t)=+f(-t)$ for all values of t. The wave-form in this case is symmetrical about the ordinate at the origin (Fig. A7.1*b*).

The method of evaluating the amplitude coefficients mathematically is indicated below, but it should be pointed out that except for

comparatively simple wave-forms the process is a difficult one, and mechanical devices known as "harmonic analysers" are employed.

The choice of a sine wave, say, instead of a saw-tooth wave, as the basic component of any wave-form is due to a number of considerations, two of which may be mentioned here, viz. (*a*) that the sine function always occurs in the mathematical solution of the differential equations representing the vibratory motion of mechanical systems, and (*b*) that it is the simplest periodic function which fulfils the condition that itself and all its derivatives should be continuous.

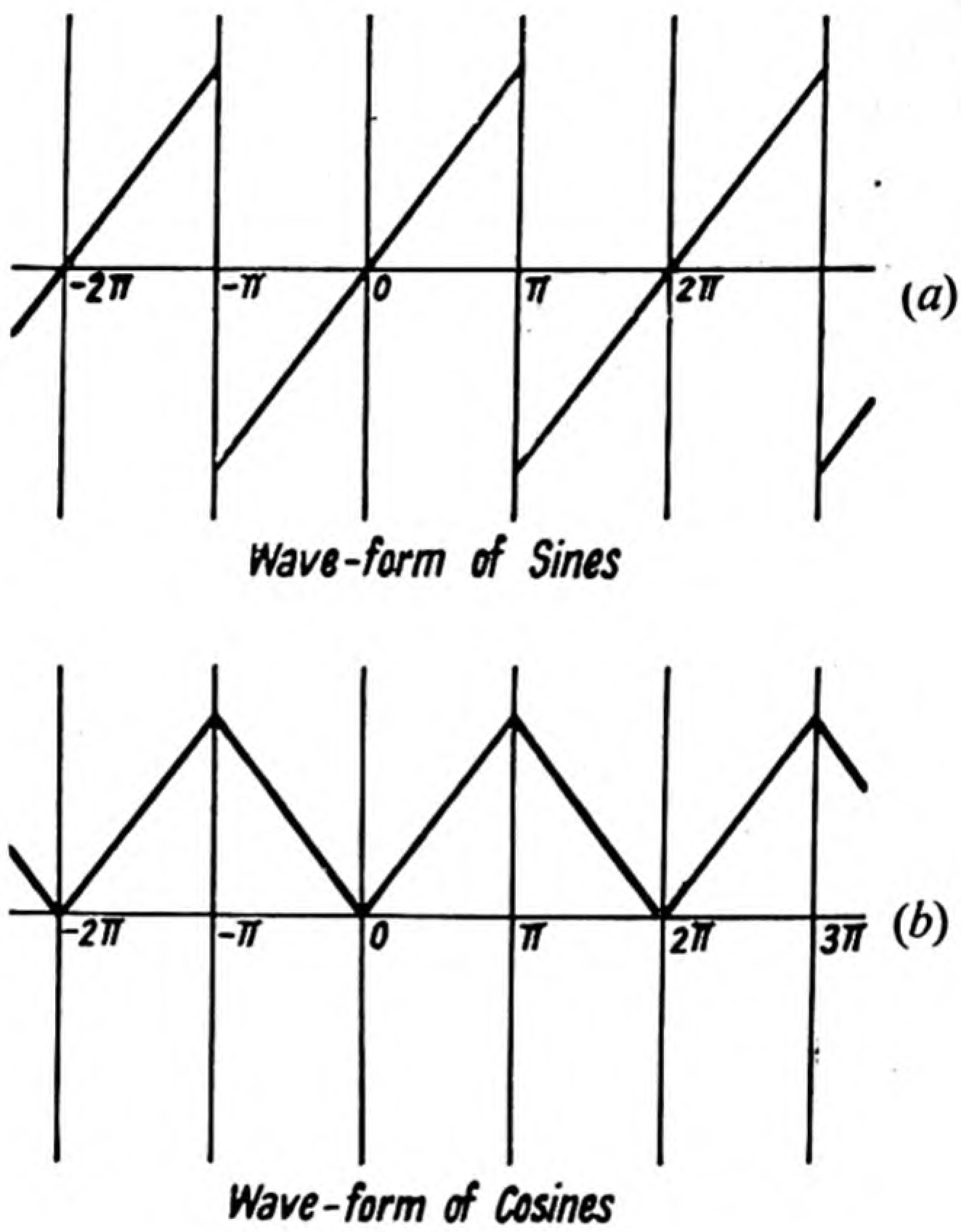


Fig. A7.1.

A given curve is only capable of Fourier analysis if it conforms with the two following restrictions:

(*a*) That the ordinates of the curve are finite, *i.e.* do not assume an infinite value as at *A* in Fig. A7.2*a*, and

(*b*) That the curve is single-valued, *i.e.* it progresses always in the same direction and therefore does not exhibit any re-entrant portion as at *B* in Fig. A7.2*b*. Interpreted in terms of sound waves these restrictions mean that a particle cannot undergo an infinite displacement, and secondly, that it cannot suffer two different displacements at the same time.

Evaluation of amplitude coefficients

If the two sides of equation (A7.2) are supposed to be identical for all values of t, then they must remain so when each side is subjected

to the same operation, unless this process should render the series divergent.

Integrate both sides of (A7.2) with respect to t over a *complete* vibration of *period* $T=\frac{2\pi}{\omega}$. It follows that all the terms on the right-hand side are zero except A_0, hence $\int_0^T f(t)dt=\int_0^T A_0 dt=A_0 T$,

i.e.
$$A_0=\frac{1}{T}\int_0^T f(t)dt \quad . \quad . \quad . \quad . \quad . \quad (A7.3)$$

It is evident that A_0 is the average value of $f(t)$ over one cycle and represents the "mean height" of the curve during this period; if A_0 is zero the curve must be as much below as above the zero line during a cycle.

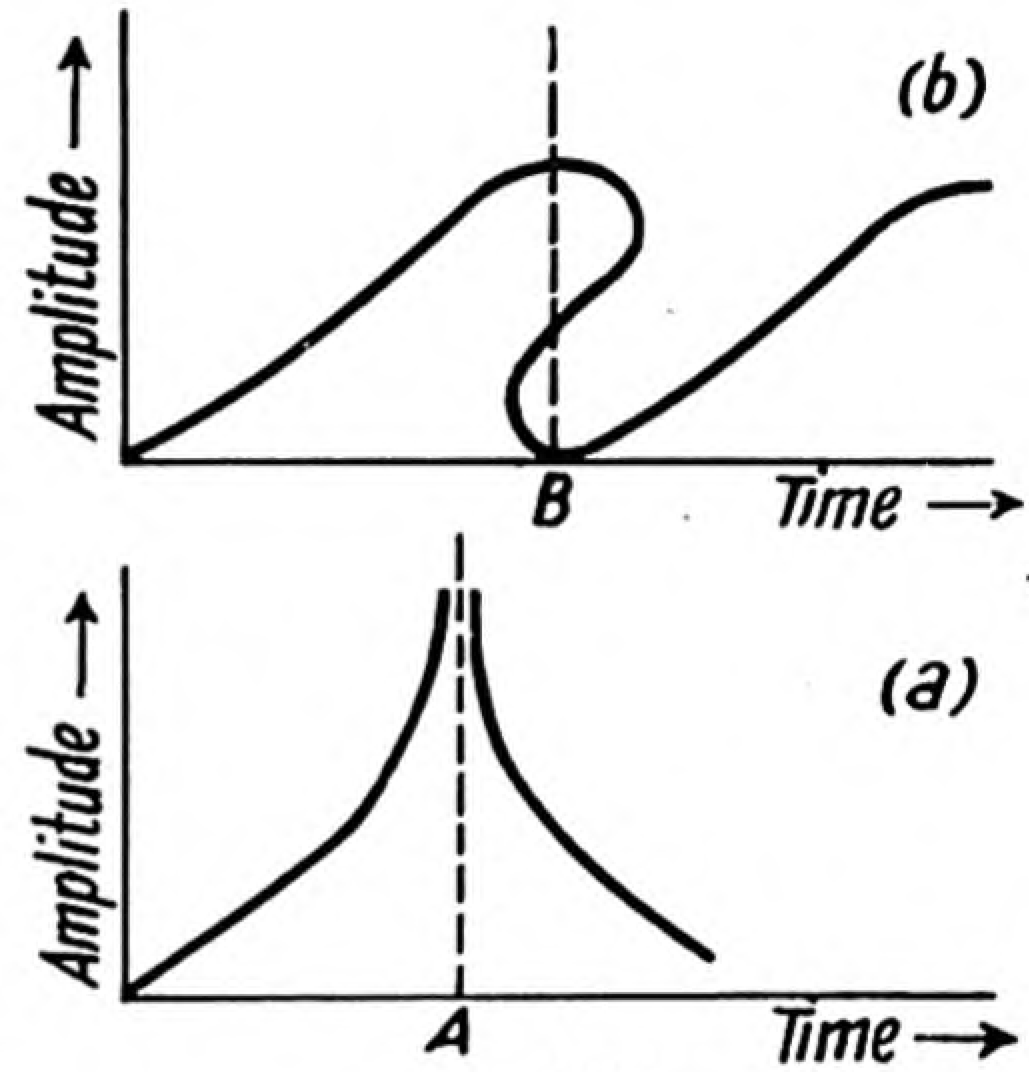


Fig. A7.2.

In order to evaluate the amplitude coefficients use is made of the so-called orthogonality of sinusoids, which is the collective term for sines and cosines. If the product of any two sinusoids of *different* frequencies is integrated over a complete cycle, then, according to this property, the integral will be zero. Expressed algebraically:—

$$\left.\begin{aligned}
&\int_0^T \sin m(\omega t) \sin n(\omega t).dt=0 \quad {\scriptstyle m\neq n}\\
&\int_0^T \cos m(\omega t) \cos n(\omega t).dt=0 \quad {\scriptstyle m\neq n}\\
\text{and } &\int_0^T \sin m(\omega t) \cos n(\omega t).dt=0 \quad {\scriptstyle m=n \text{ or } m\neq n}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (A7.4)$$

where m and n are integers.

But
$$\int_0^T \sin^2 m(\omega t).dt=\int_0^T \cos^2 m(\omega t).dt=\frac{T}{2} \quad . \quad . \quad (A7.5)$$

Hence if both sides of (A7.2) are multiplied by $\cos r.(\omega t)$ and integrated for a complete cycle (0 to T), all terms on the right-hand side are zero except that containing A_r. It follows therefore that

$$\int_0^T f(t) \cos r(\omega t).dt=\frac{A_r T}{2} \text{ or } A_r=\frac{2}{T}\int_0^T f(t) \cos r(\omega t).dt \quad (A7.6)$$

Similarly, by multiplying all terms by sin $r(\omega t)$ and integrating over a complete cycle the coefficients of the sine series may be determined, thus

$$B_r = \frac{2}{T}\int_0^T f(t) \sin r(\omega t).dt \quad . \quad . \quad . \quad . \quad \text{(A7.7)}$$

Example of Fourier analysis

By way of applying the above analysis the case of a periodic wave of rectangular shape will be chosen, the importance of this form of wave

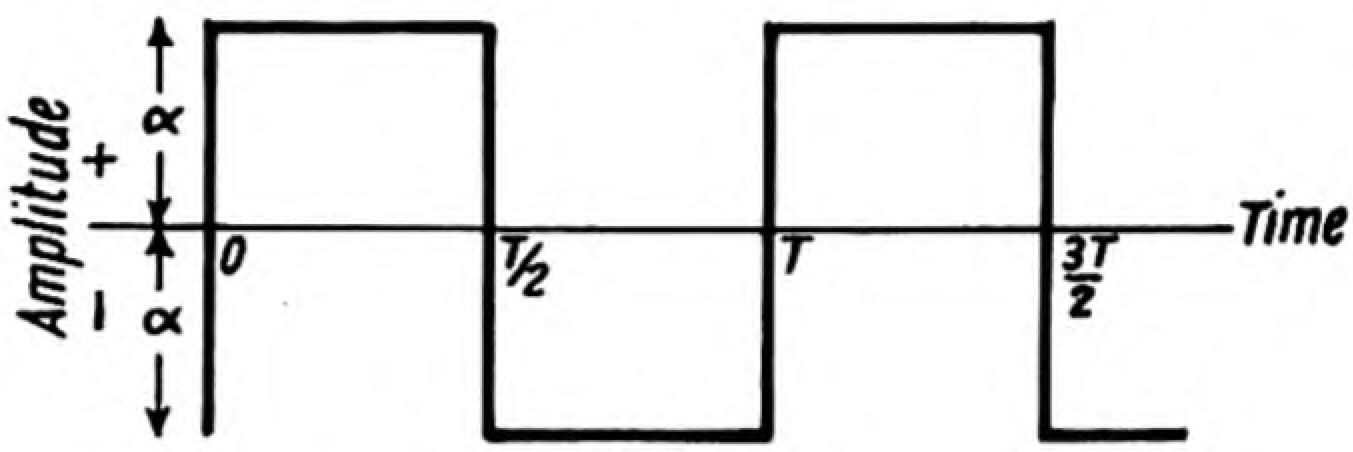


Fig. A7.3.

being its use, amongst others, for testing the frequency response of electrical apparatus. Then it is evident from the square wave shown in Fig. A7.3 that

$$y = f(t) = +a \quad \text{from } t=0 \text{ to } t=\frac{T}{2}$$

and

$$y = f(t) = -a \quad \text{from } t=\frac{T}{2} \text{ to } t=T.$$

Hence

$$A_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_0^{\frac{T}{2}} a dt + \frac{1}{T}\int_{\frac{T}{2}}^T -a dt = 0 \quad . \quad . \quad \text{(A7.8)}$$

$$A_r = \frac{2}{T}\int_0^{\frac{T}{2}} a \cos r(\omega t)dt - \frac{2}{T}\int_{\frac{T}{2}}^T a \cos r(\omega t)dt$$

$$= \frac{2a}{T}\left[\int_0^{\frac{T}{2}} \cos\left(\frac{2\pi r}{T}.t\right)dt - \int_{\frac{T}{2}}^T \cos\left(\frac{2\pi r}{T}.t\right)dt\right]$$

$$= \frac{a}{\pi r}\left[2 \sin \pi r - \sin 2\pi r\right]$$

$$= 0 \quad \text{for all integral values of } r \quad . \quad . \quad . \quad \text{(A7.9)}$$

and

$$B_r = \frac{2a}{T}\left[\int_0^{\frac{T}{2}} \sin\left(\frac{2\pi r}{T}.t\right)dt - \int_{\frac{T}{2}}^T \sin\left(\frac{2\pi r}{T}.t\right)dt\right]$$

$$= \frac{a}{\pi r}[1 - 2 \cos \pi r + \cos 2\pi r]$$

$$= \frac{2a}{\pi r}[1 - \cos \pi r] \quad \text{for all integral values of } r \quad \text{(A7.10)}$$

$$\left.\begin{aligned} r \text{ even}, B_r &= 0 \\ r \text{ odd}, B_r &= \frac{4a}{r\pi} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (A7.11)$$

The complete Fourier series is therefore

$$y = f(t) = \frac{4a}{\pi}(\sin \omega t + \tfrac{1}{3}\sin 3\omega t + \tfrac{1}{5}\sin 5\omega t + \ldots) \quad (A7.12)$$

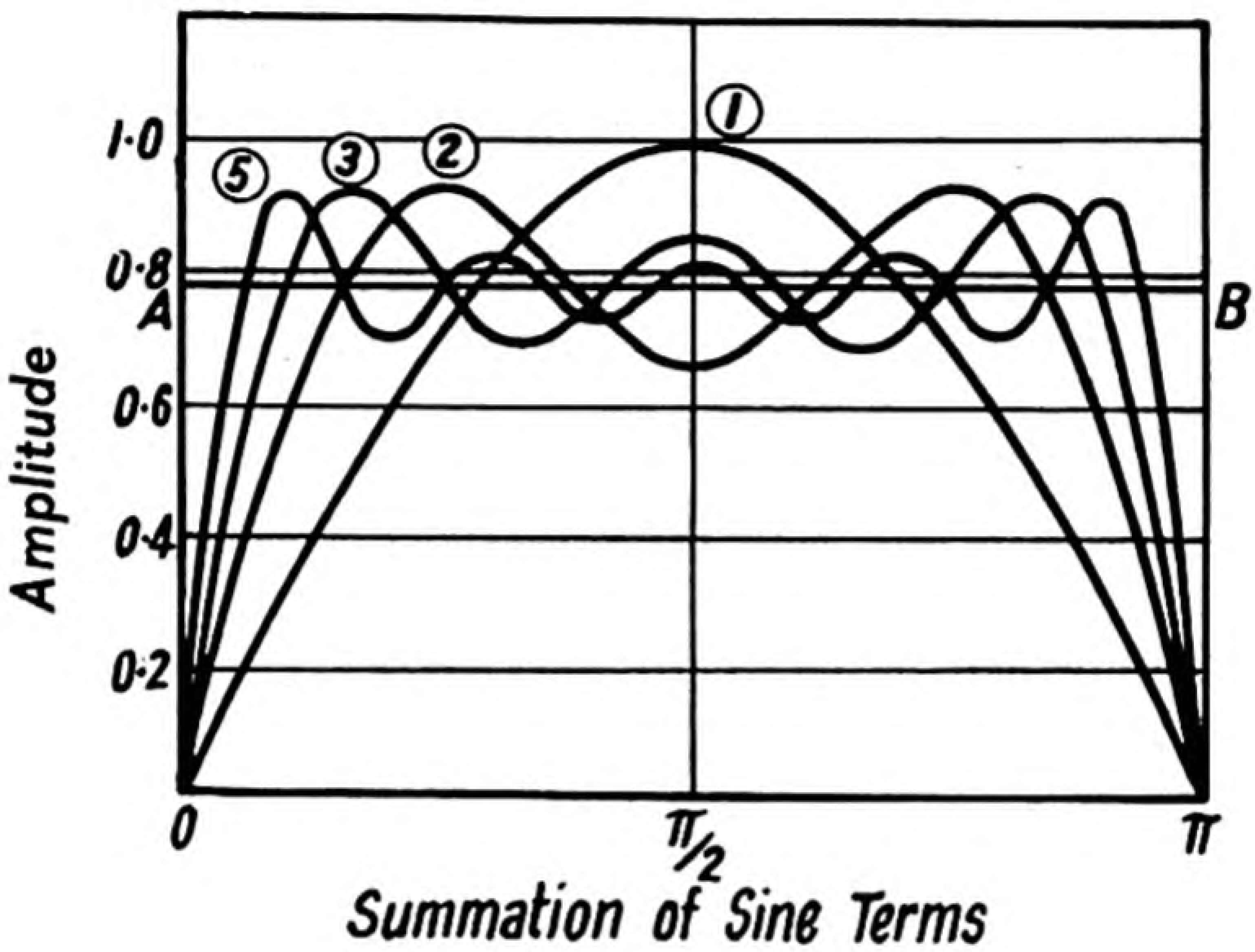


Fig. A7.4.

Curve 5 in Fig. A7.4 is obtained by combining the first five terms of this expression (A7.12) and suggests that quite a large number of terms are required to obtain any high degree of approximation to the given wave-form $OAB\pi$ when it contains sharp discontinuities as in the case of the square wave.

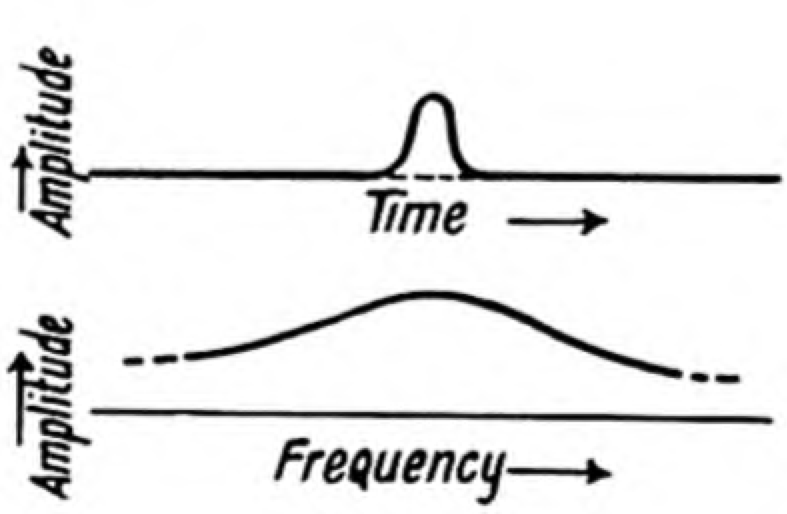


Fig. A7.5.

The above analysis has been concerned with a periodic wave-form, but in practice the generation of such waves is usually limited to a small train, and the limiting case is when this train is reduced to a single wave or *pulse*, as it is termed. Such a pulse is shown in the upper part of Fig. A7.5 which might represent, for example, the sound wave of a pistol shot. Since this wave is non-periodic it cannot be

represented by the ordinary Fourier series, yet it may be expressed in terms of what are known as Fourier's integrals in which the component frequencies differ by infinitesimal amounts. The lower curve of Fig. A7.5 indicates the *continuous* form of the frequency spectrum obtained by such an analysis of the pulse.

It should be noted that Fourier analysis is not capable of direct application to light waves since their shape cannot be observed directly. However, the lower curve of Fig. A7.5 is very similar to the intensity-frequency curve of the continuous spectrum from an incandescent solid, and it consequently suggests that white light might be considered as a succession of random pulses which are Fourier analysed by the prism.

Further it should be noted that to form a pulse having a finite time of duration, waves of different frequencies are necessary. Furthermore, these different waves must annul one another at all times except for the duration of the pulse, and the shorter this time interval the greater will be the number of different frequencies required. If a curve is drawn showing frequency against pressure amplitude in the pulse, $\Delta\nu$ is taken to represent the difference in frequency of the two points on the curve having amplitudes one-half of the maximum; the corresponding time interval is denoted by Δt. It may be shown, even for those cases where the starting and stopping of the pulse is not abrupt, that the following relation is approximately true, viz. $\Delta t\, \Delta\nu \simeq 1$. During speech the duration Δt of a particular consonant may be of the order of one-fifth of that for music, hence the corresponding $\Delta\nu$ is large and the sensation of an associated definite pitch is lost.

APPENDIX 8

Non-linear Systems

Distortion of a signal in its passage through a practical communication channel may arise either from the frequency selectivity or from the non-linearity of the component systems. Non-linearity implies that the level of the output signal is not strictly proportional to that of the input, and when the applied signal is of a complex nature then frequencies, additional to those of the signal, are introduced. The thermionic triode valve is an example of a non-linear device for grid input signals of appreciable magnitude, although for very small voltages they may be regarded as having a linear response. The current-potential characteristic for crystal detectors, metal-oxide rectifiers and thermionic triode valves may be expressed in the form $I=kV^n$, where for a triode valve n usually lies between 1·5 and 2·5, depending on its geometrical construction. Hence, if V_s is the standing voltage due to the steady grid and anode potentials the anode current $I=k(V_s+\Sigma E_1 \sin \omega_1 t)^n$, where $E_1 \sin \omega_1 t$, $E_2 \sin \omega_2 t$, etc., refer to the various E.M.F.s applied simultaneously (or alternatively to the components of a complex wave-form).

Assuming $n=2$ as a fair approximation and considering only two applied E.M.F.s it follows on expanding the above expression that

$$\begin{aligned}
I=&k(V_s^2+E_1^2\sin^2\omega_1 t+E_2^2\sin^2\omega_2 t\\
&+2V_sE_1\sin\omega_1 t+2V_sE_2\sin\omega_2 t\\
&+2E_1E_2\sin\omega_1 t\sin\omega_2 t)\\
=&k\Big[V_s^2+\frac{E_1^2}{2}+\frac{E_2^2}{2}-\frac{E_1^2\cos 2\omega_1 t}{2}-\frac{E_2^2\cos 2\omega_1 t}{2}\\
&+2V_sE_1\sin\omega_1 t+2V_sE_2\sin\omega_2 t\\
&+E_1E_2\cos(\omega_1-\omega_2)t-E_1E_2\cos(\omega_1+\omega_2)t\Big] \quad . \quad (A8.1)
\end{aligned}$$

It is evident from the above analysis that the two impressed voltages have given rise to a current comprising six different alternating components. Two have the same frequencies as the input voltages, two of double these frequencies and the other two components have respectively the summation $\frac{(\omega_1+\omega_2)}{2\pi}$ and difference $\frac{(\omega_1-\omega_2)}{2\pi}$ frequencies.

In the modulation of broadcast radio waves $\frac{\omega_1}{2\pi}$ could refer to the carrier frequency and $\frac{\omega_2}{2\pi}$ would be a pure tone *sound* frequency modulating the carrier wave, and as a consequence of the non-linear action of the transmitting valve there will exist the two so-called *side-band* frequencies $\frac{(\omega_1-\omega_2)}{2\pi}$ and $\frac{(\omega_1+\omega_2)}{2\pi}$ as explained by the above equation. Sometimes, in order to economise power, one side-band is suppressed by a rejector filter, while in commercial radio-telephony only one side-band is transmitted, but such a procedure requires in reception a local oscillator to supply the suppressed carrier frequency. Other non-linear devices which may be mentioned are transformers with cores of variable permeability as used for frequency multiplication and in voltage regulators, low-pressure gas discharge tubes used for "triggering" purposes, etc., and in acoustics the ear is a notable example of a non-linear device as made evident by the aural detection of sum and difference frequencies and harmonics when two pure tones of suitable intensity are impressed upon the ear. In such mechanical systems the displacement η of the non-linear element may be expressed generally as $\eta=af+bf^2$, where f is the actuating force, and if two or more harmonic forces are simultaneously impressed upon the element, the resultant motion will include the various frequencies as given in equation (A8.1).

Alternatively it should be noted that the restoring force F for a given displacement x of a non-linear mechanical system may be expressed generally as $F=F_0+Ax+Bx^2$, where F_0, A and B are constants. It follows that the *restoring force* changes in *magnitude* if the *displacement* changes only in *sign*. In these **asymmetrical** vibrations, as compared with those of a corresponding linear system, the fundamental frequency becomes slightly reduced and the mean position of vibration is displaced.

APPENDIX 9

Complex Quantities

A complex number is a quantity z of the form $z=x+jy$, where x and y are real numbers, and $j=\sqrt{-1}$ is an operator (see below) which acting upon a vector advances its phase by $\frac{\pi}{2}$, *i.e.* turns it through $90°$ in an anti-clockwise direction; jy (or often merely y) is termed the imaginary part of the complex number. These numbers obey the ordinary laws of algebraic addition, etc., and also $j^2=-1$. It follows that two complex numbers z_1 and z_2 are only equal when their real and imaginary parts are *separately* equal. The geometrical representation of this complex notation is given in Fig. A9.1, where the position of the point P (representing z) with respect to the origin O is attained by a displacement ON along the real axis X combined with a displacement NP along the imaginary axis Y, *perpendicular* to OX. If the complex number is expressed in polar form (see Fig. A9.1), it is evident that $r=\sqrt{x^2+y^2}$ where $r\cos\theta=x$, $r\sin\theta=y$ and θ is the angle the radius vector r makes with the x-axis, so that $z=r(\cos\theta+j\sin\theta)$. This figure is known as an *Argand* diagram, named after its inventor, and in it r (always taken as positive) expresses the *magnitude* of z. This numerical value of z is known as its *modulus* and is written $|z|$ or mod. z. The angle θ is termed the *argument* or *phase* of z, and whereas the modulus is unique the argument of a complex number is not, since the value of z is unaffected by the substitution $(\theta+2\pi n)$ for θ, where n is an integer. These complex quantities are usefully employed in dealing with harmonic waves or vibrations, which involve sine or cosine functions, for the trigonometrical forms may be transformed into imaginary exponentials and the subsequent computation is considerably simplified. From the identities $e^{j\omega t}=\cos\omega t+j\sin\omega t$ and $e^{-j\omega t}=\cos\omega t-j\sin\omega t$, it follows that the general displacement expression for a progressive wave, viz. $\eta=a\cos\{(kx-\omega t)+\epsilon\}$ may be *regarded* as the *real* part of $\eta=ae^{\pm j\{(kx-\omega t)+\epsilon\}}$. It is found more convenient to employ the negative sign and rewrite the equation as $\eta=Be^{j(\omega t-kx)}$, where $B=ae^{-j\epsilon}$ will be in general complex. Since only linear equations are involved and the complex function is a solution, it, and not merely the real part, may be used in the analysis of the problem. Care must be taken, however, in the use of complex quantities in non-linear expansions, as for instance in the calculation of the instantaneous power developed in an electrical circuit. In this case, if the current and voltage are expressed in complex notation by $i=i_m e^{j(\omega t-\phi_1)}$ and $v=v_m e^{j(\omega t-\phi_2)}$ respectively, then it must be noted that the instantaneous power is *not* given by $W_i=$real part

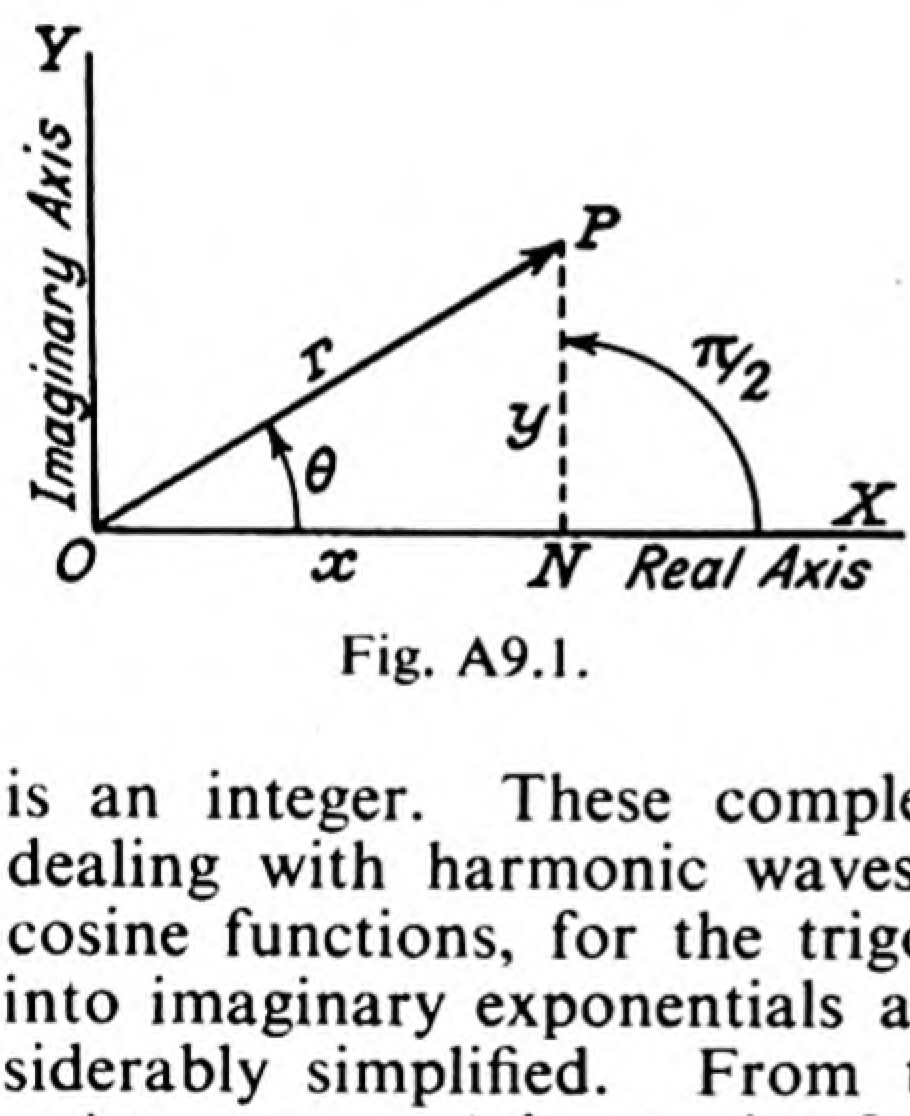


Fig. A9.1.

$(i_m v_m e^{j(2\omega t-\phi_1-\phi_2)})$, but by W_i=real part$(i_m e^{j(\omega t-\phi_1)})\times$real part$(v_m e^{j(\omega t-\phi_2)})$.

As an example of their application to a vibrational problem the case of a forced vibration will be investigated by imaginary exponentials. Equation $M\ddot{x}+R\dot{x}+Sx=F\cos ft$ may be written as $M\ddot{z}+R\dot{z}+Sz=Fe^{jft}$ and $z=Ae^{jft}$ is an obvious solution, which on substitution gives $A=\frac{F}{(-Mf^2+jRf+S)}$

so that
$$z=\frac{Fe^{jft}}{[jRf-Mf^2+S]}$$

and
$$\dot{z}=jfz.$$

Hence
$$\dot{z}=\frac{Fe^{jft}}{\left\{R+j\left[Mf-\frac{S}{f}\right]\right\}}$$
$$=\frac{Fe^{jft}}{|Z|e^{j\phi}}$$
$$=\frac{F}{|Z|}e^{j(ft-\phi)}$$

where
$$|Z|=\sqrt{R^2+\left(Mf-\frac{S}{f}\right)^2}$$

and
$$\tan\phi=\frac{Mf-\frac{S}{f}}{R}$$

Hence (cf. eqn. (20), Chap. 13.)
$$\dot{x}=\text{real part of }\dot{z}$$
$$=\frac{F}{Z}\cos(ft-\phi),$$

where, as is common practice, the modulus sign of Z has been omitted.

APPENDIX 10

Fermat's Principle of Least Time

This principle, due to Fermat (1667), although usually applied to the propagation of light waves, is equally true for waves of any kind. It is applicable to a medium with a varying index of refraction, or to

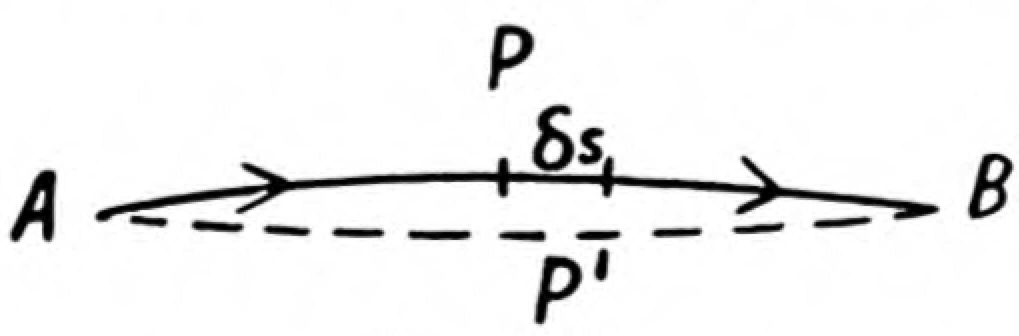


Fig. A10.1.

the case of a ray crossing the boundary separating two media of different refractive indices. Expressed mathematically the principle states that the true ray path APB (Fig. A10.1) between any two points

A and B will be such that the time for the disturbance to traverse the path will be a minimum, or, more strictly, an extremum since in certain cases it may be a maximum. A simple illustration due to Sir W. R. Hamilton (1833) is worthy of mention. "If an eye is placed at the interior but not at the centre of a reflecting hollow sphere it may itself be reflected in two opposite points, of which one indeed is nearest to it but the other on the contrary is the furthest, so that of the two different paths of light corresponding to these two opposite points one indeed is the shortest but the other is quite the longest of any."

Let v be the velocity of propagation of the disturbance (*e.g.* light or sound) at a point P in the path, then the total time for the disturbance to travel from A to B is

$$\tau=\int_A^B \frac{ds}{v} \qquad \text{(A10.1)}$$

In the case of a light wave $v=\dfrac{c}{n}$, where c is the velocity of light *in vacuo* and n is the appropriate value of the refractive index at P for the particular wave-length of light considered. It follows from (A10.1) that $\tau=\dfrac{1}{c}\displaystyle\int_A^B nds$, the *line integral* $\displaystyle\int_A^B nds$ is termed the *optical length* of the path between the two given points.

If a neighbouring path $AP'B$ is considered, very close to the natural path APB, and the time taken for the new path time is $\tau+d\tau$, it follows from Fermat's principle of least time that

$$\delta\tau=\frac{1}{c}\,\delta\left(\int_A^B nds\right)=0 \qquad \text{(A10.2)}$$

In order to apply Fermat's principle to find the correct path of a disturbance in a particular problem, the procedure is to calculate the times needed for the disturbance to pass between the two given points along any two paths very close together. In the expression for the difference of these times the second order or higher terms are neglected, and the natural or correct path is defined by equating the remaining terms to zero.

As a particular application Fermat's principle will be used to derive Snell's law of refraction for a light (or sound) ray at the boundary s_1s_2 (Fig. A10.2) between two media of refractive indices n_1 and n_2, and corresponding velocities of propagation v_1 and v_2. It is evident from the diagram that the difference in times of the two paths AOB and $AO'B$ is given by $\delta\tau=\dfrac{O'P_1}{v_1}-\dfrac{OP_2}{v_2}=OO'\left(\dfrac{\sin i}{v_1}-\dfrac{\sin r}{v_2}\right)$ approximately, neglecting higher powers of OO'.

By Fermat's principle $\delta\tau=0$, *i.e.* $\dfrac{\sin i}{v_1}-\dfrac{\sin r}{v_2}=0$, or $\dfrac{\sin i}{\sin r}=\dfrac{v_1}{v_2}=\dfrac{n_2}{n_1}$, which is Snell's law.

Equation (A10.2) is of the same form as the mechanical law which determines the path taken by a mass particle in moving from one point to another under the action of conservative forces. In this example

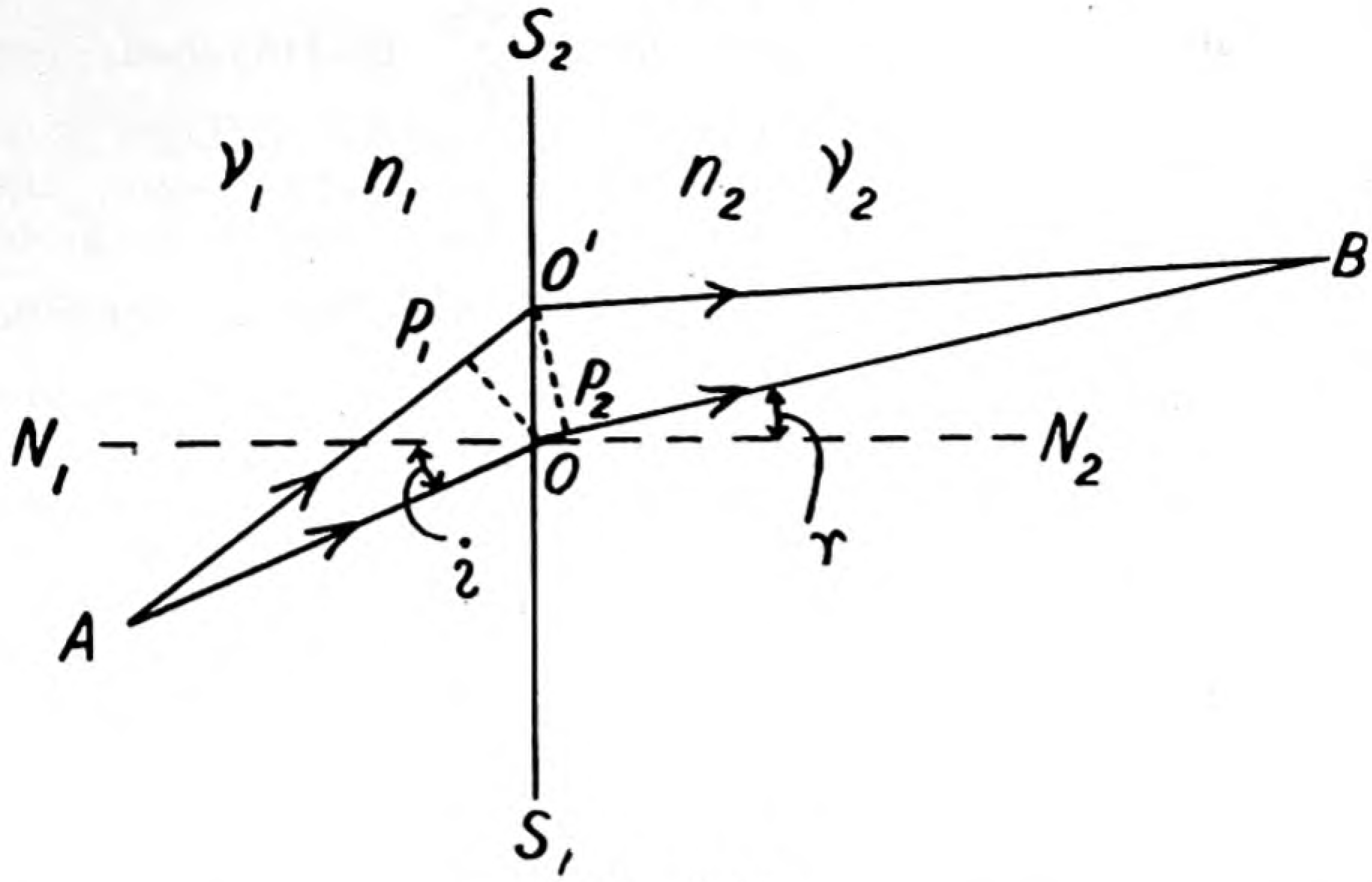


Fig. A10.2.

the true or natural path is defined by the line integral of the *momentum* over the path, and the *principle of least action* as it is termed, states that $\delta\left(\int_A^B mvds\right)=0$, where m is the mass and v the velocity of the particle.

APPENDIX 11

Note on Diffraction—Cornu's Spiral and Fresnel's Integrals

Diffraction by a straight edge. Let C represent a long, narrow slit (Fig. A11.1) which is the source of cylindrical waves, and suppose TM represents an opaque plate with a straight edge at T, parallel to the slit and situated in the path of the diverging waves. C is joined by

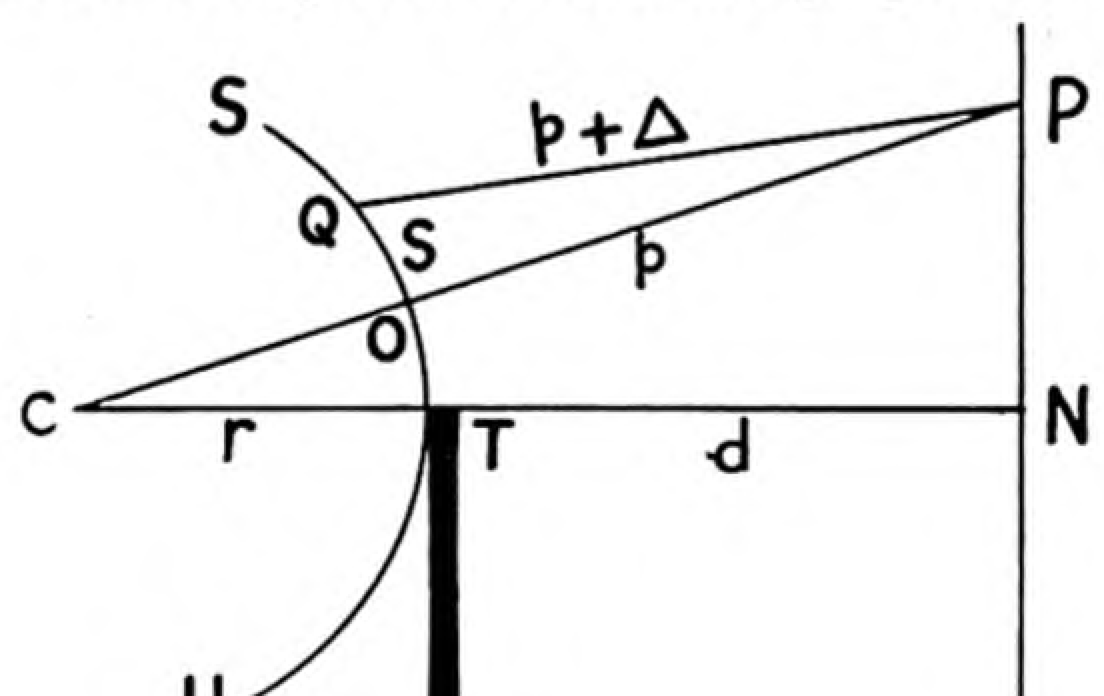


Fig. A11.1.

a straight line COP to any point P on the screen PN, O being on the cylindrical wave-front touching the straight edge at T. Q is a point chosen on this wave-front at a *short* distance S from O and a straight line is drawn connecting Q and P. From the geometry of the figure

it follows that to a first approximation $\Delta = \frac{S^2}{2p}$. By Huyghens' method of wave construction, each point on the surface *UTOQS* is to be regarded as a secondary source of waves, and furthermore, suppose that the vibration of the particles on this surface is given by $y = A \sin \frac{2\pi t}{T}$. For a strip of width δs at Q the amplitude contribution to the effect of the wave at P will be given by $\delta y = B . \delta s \sin 2\pi\left(\frac{t}{T} - \frac{p+\Delta}{\lambda}\right)$, where B includes A and assuming that the amplitude is proportional to δs and the phase is retarded by $(p+\Delta)/\lambda$. It is also tacitly assumed that the obliquity factor and the decrease of amplitude with distance from the source (pp. 127, 129) may be neglected, assumptions which are justified by results.

Hence the resultant amplitude of the vibration at P due to the whole wave-front is given by

$$y = B\int_{-S_T}^{S_S} \sin 2\pi\left(\frac{t}{T} - \frac{p+\Delta}{\lambda}\right) ds \quad . \quad . \quad . \quad . \quad \text{(A11.1)}$$

where $OS = +S_S$ and $OT = -S_T$.

Since Δ is different for each point on the wave-front it is convenient to rewrite equation (A11.1) as follows:—

$$y = B\int_{-S_T}^{S_S} \sin 2\pi\left\{\left(\frac{t}{T} - \frac{p}{\lambda}\right) - \frac{\Delta}{\lambda}\right\} ds$$

$$= B \sin \left\{2\pi\left(\frac{t}{T} - \frac{p}{\lambda}\right)\right\} \int_{-S_T}^{S_S} \cos \left(\frac{2\pi\Delta}{\lambda}\right) ds$$

$$- B \cos \left\{2\pi\left(\frac{t}{T} - \frac{p}{\lambda}\right)\right\} \int_{-S_T}^{S_S} \sin \left(\frac{2\pi\Delta}{\lambda}\right) ds \, . \quad \text{(A11.2)}$$

This equation may be rewritten as

$$y = R \sin\left\{2\pi\left(\frac{t}{T} - \frac{p}{\lambda}\right) - \theta\right\} \quad . \quad . \quad . \quad . \quad \text{(A11.3)}$$

where
$$R \cos \theta = B\int_{-S_T}^{S_S} \cos \left(\frac{2\pi\Delta}{\lambda}\right) ds$$

and
$$R \sin \theta = B\int_{-S_T}^{S_S} \sin \left(\frac{2\pi\Delta}{\lambda}\right) ds \quad . \quad . \quad \text{(A11.4)}$$

Equation (A11.3) indicates that the resultant vibration at P will have the same period as the incident wave but a different phase constant and amplitude. The intensity of the vibration at P being given by (amplitude)^2 will be [from (A11.4)],

$$R^2 = B^2\left[\left\{\int_{-S_T}^{S_S} \cos \left(\frac{2\pi\Delta}{\lambda}\right) ds\right\}^2 + \left\{\int_{-S_T}^{S_S} \sin \left(\frac{2\pi\Delta}{\lambda}\right) ds\right\}^2\right] \quad \text{(A11.5)}$$

But $\Delta = S^2/2p$ and so

$$R^2 = B^2\left[\left\{\int_{-S_T}^{S_S} \cos\left(\frac{\pi s^2}{p\lambda}\right)ds\right\}^2 + \left\{\int_{-S_T}^{S_S} \sin\left(\frac{\pi s^2}{p\lambda}\right)ds\right\}^2\right]$$

or $$R^2 = C^2\left[\left\{\int_{-V_T}^{V_S} \cos\left(\frac{\pi v^2}{2}\right)dv\right\}^2 + \left\{\int_{-V_T}^{V_S} \sin\left(\frac{\pi v^2}{2}\right)dv\right\}^2\right] \quad \text{(A11.6)}$$

where v is a new variable chosen so that $s = v\sqrt{\frac{p\lambda}{2}}$. It follows that $\frac{2\pi\Delta}{\lambda} = \frac{\pi v^2}{2}$ and $ds = \sqrt{\frac{p\lambda}{2}}.dv$. Now λ is small and so if $s = S_s$ it follows that $v_s = \frac{S_s}{\sqrt{\lambda}}\sqrt{\frac{2}{p}}$ will be very large and may be taken as ∞ without serious error.

Rewriting equation (A11.6) as $R^2 = C^2(x^2 + y^2)$, then

$$x = \int_{-V_T}^{\infty} \cos\left(\frac{\pi v^2}{2}\right)dv \quad \text{and} \quad y = \int_{-V_T}^{\infty} \sin\left(\frac{\pi v^2}{2}\right)dv \quad \text{(A11.7)}$$

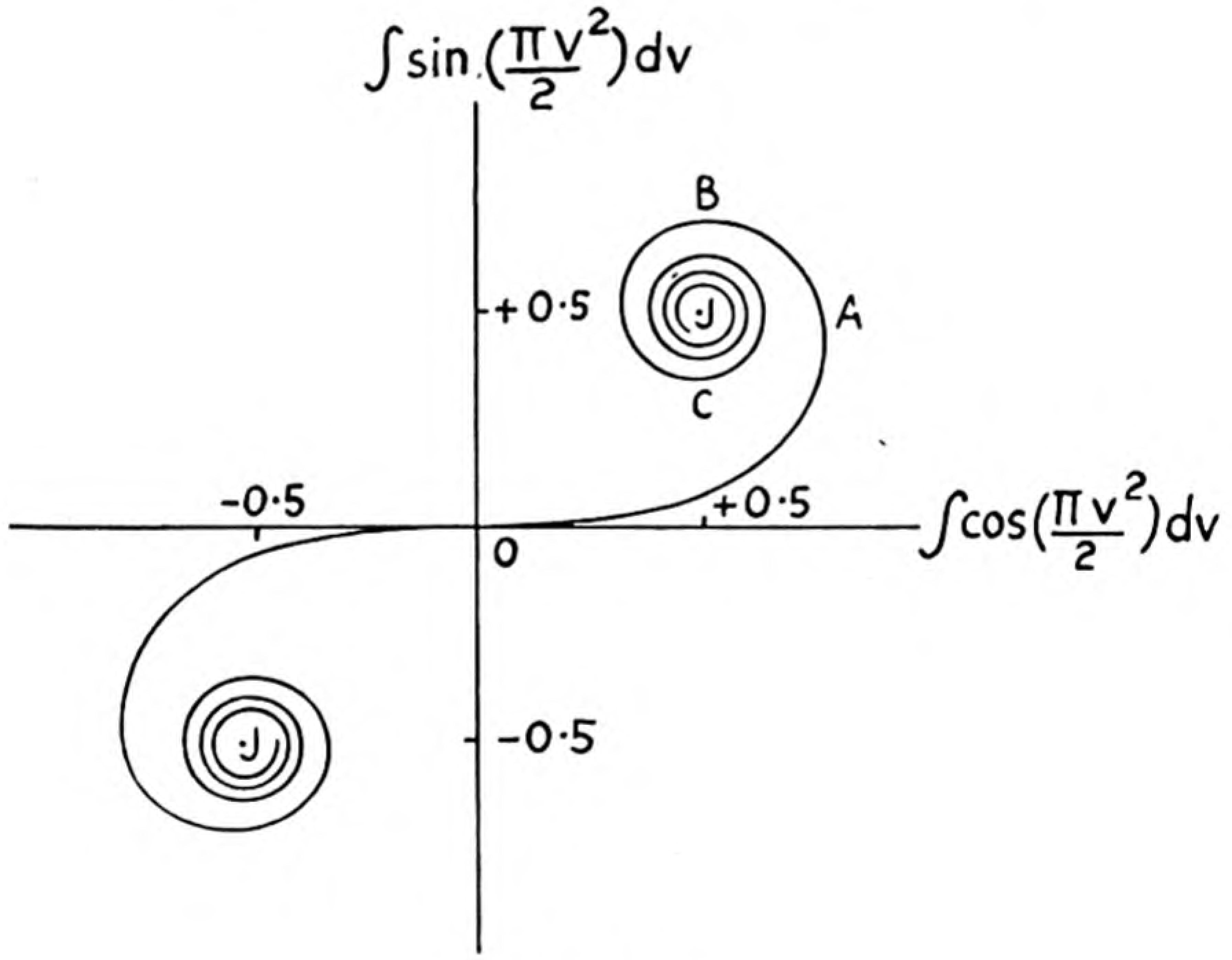


Fig. A11.2.

where x and y are known as Fresnel's integrals. These integrals have been evaluated in the form of infinite series and so the value of R^2 for different positions of P may be calculated with the help of the tables. If y as ordinate is plotted against x as abscissa a curve (see Fig. A11.2) known as Cornu's spiral is obtained, and this provides a fascinating geometrical method of solving diffraction problems.

Some properties of this curve will be briefly mentioned. The slope of the curve is given by $\frac{dy}{dx}=\frac{\sin\frac{\pi v^2}{2}}{\cos\frac{\pi v^2}{2}}=\tan\frac{\pi}{2}v^2$, and so when v^2 increases by 4 the tangent of the curve comes back to its initial value. Each point on the spiral curve corresponds to a particular value of v. Again, if l denotes distances measured along the curve,

$$dl=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx=\left[1+\tan^2\left(\frac{\pi v^2}{2}\right)\right]^{\frac{1}{2}}\cos\left(\frac{\pi v^2}{2}\right)dv=dv, \text{ i.e. } l=v,$$

and so if θ is the angle made by the tangent to the x-axis, $\tan\theta=\frac{dy}{dx}=\tan\frac{\pi v^2}{2}=\tan\frac{\pi l^2}{2}$, or $\theta=\frac{\pi l^2}{2}$ represents the intrinsic equation of the curve. The curvature of the curve is defined by $\frac{1}{\rho}=\frac{d\theta}{dl}=\pi l$, and so changes sign at the origin where it has a zero value. It also follows from the above that the difference of v between any two points on the curve is equal to the length (l) of the curve between these points. Again the coordinates of the end points J of the spiral are respectively $(+\frac{1}{2}, +\frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$ so that $OJ=\frac{1}{\sqrt{2}}$ represents the contribution to the amplitude of one-half of the wave-front.

APPENDIX 12

Energy of a Simple Harmonic Oscillator

It has been shown (p. 42) that for a particle executing S.H.M.:—

Total energy of motion = Maximum value of kinetic energy of oscillator
= Maximum value of potential energy of oscillator
$=\frac{1}{2}ma^2\omega^2$.

Hence for a displacement x the potential energy is given by $\frac{1}{2}mx^2\omega^2$, and the kinetic energy of the oscillator in passing through this displacement is given by $\frac{1}{2}ma^2\omega^2-\frac{1}{2}mx^2\omega^2$. Since the potential energy for a given frequency is proportional to x^2, then the curve representing potential energy against displacement will be a parabola. These facts are exhibited in Fig. A12.1.

This graphical representation of the energy of a simple harmonic oscillator is of significance in atomic theory, when considering the relative probabilities of finding an electron at various displacements, when vibrating in a particular quantum state. On the above classical theory the maximum displacement of an electron oscillator will *definitely* occur whenever the potential energy equals its total energy. This definiteness is lost, however, in wave mechanics, and in the lower quantum states there is a fairly large probability that the electron will *exceed* the classical amplitude.

Energy of a vibrating string. The displacement η of a vibrating string in which a standing-wave system exists is given on the usual notation by

$$\eta=2a\sin\left(2\pi\frac{x}{\lambda}\right)\sin 2\pi nt \quad . \quad . \quad . \quad . \quad \text{(A12.1)}$$

the amplitude factor being $2a\sin\left(2\pi\frac{x}{\lambda}\right)$, where n is the frequency and λ is the wave-length.

If m is the mass per unit length of the string, then, for vibrations of small amplitude, the kinetic energy dE of an element of length dx is given by

$$dE=m.dx\frac{\dot{\eta}^2}{2}=2a^2m(2\pi n)^2\sin^2\left(2\pi n\frac{x}{\lambda}\right).\cos^2 2\pi nt.dx.$$

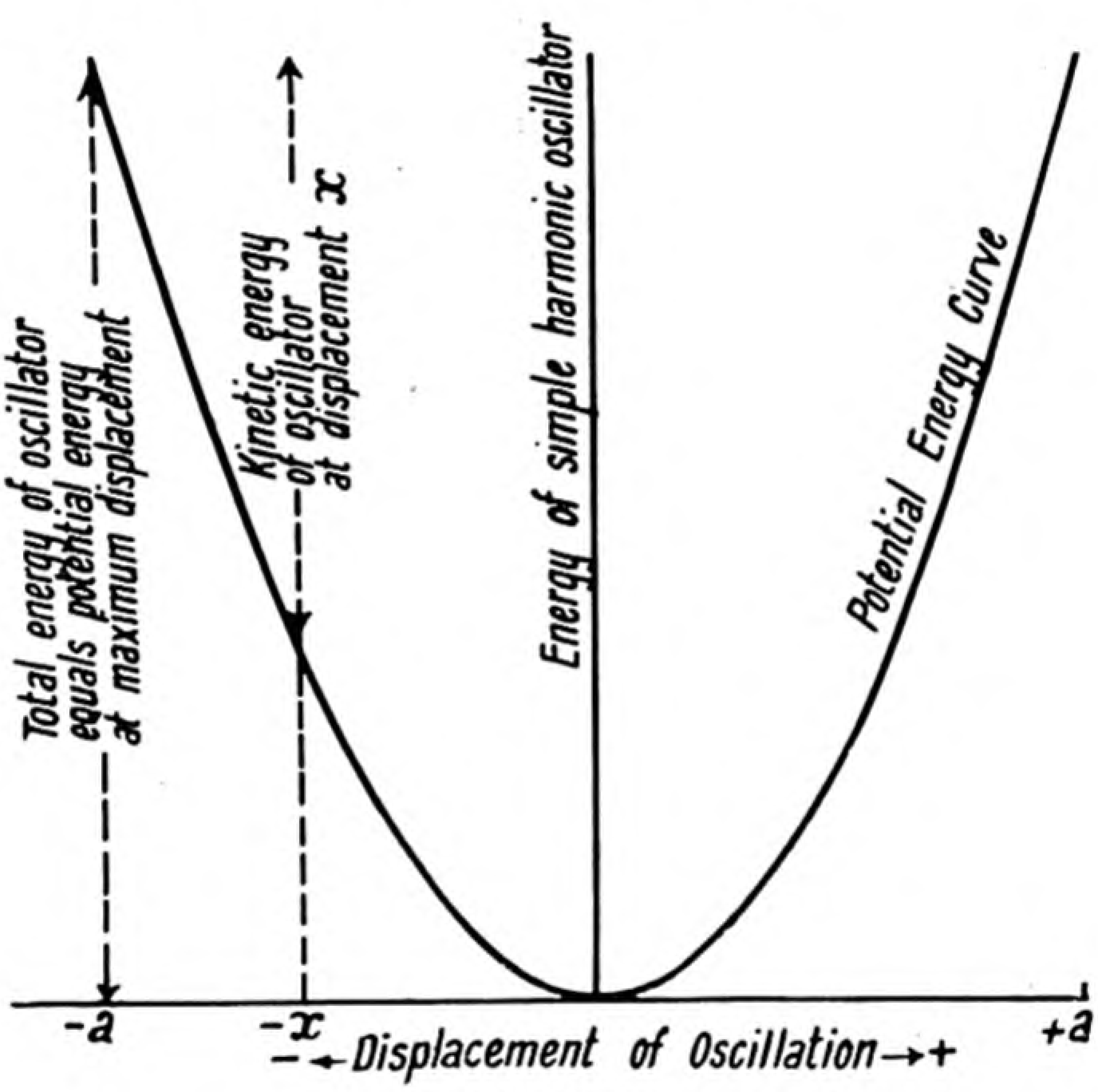


Fig. A12.1.

Hence the K.E. of one vibrating loop is given by

$$8\pi^2n^2a^2m\int_0^{\frac{\lambda}{2}}\sin^2\left(2\pi\frac{x}{\lambda}\right).\cos^2 2\pi nt.dx=4\pi^2n^2a^2m\lambda.\cos^2 2\pi nt, \quad \text{(A12.2)}$$

at any instant t.

Now at values of $t=0, \frac{T}{2}, T$, etc., in which T is the period, the displacement is zero, and the P.E.$=0$; so that the energy is wholly kinetic; *i.e.* the total energy, *i.e.* maximum kinetic energy of the loop, is obtained by putting $t=0$ in the cosine term, which becomes $4\pi^2n^2a^2m\lambda$, and for N loops, $4\pi^2n^2a^2mN\lambda$. The length of the string in vibration is $N\lambda/2$, hence the energy of vibration is $8\pi^2n^2a^2M$, $M=mN\lambda/2$ being the total mass of the vibrating string.

APPENDIX 13

Correction of "Standing-wave" Formula for a Phase Change on Reflection

Referring to Fig. 6.19 on p. 107, let δ be the phase change due to reflection at P, then the resultant displacement at a point x along the tube will be given by

$$\eta = a \sin(\omega t - kx) + fa \sin\{\omega t + k(x+\delta)\} \quad . \quad \text{(A13.1)}$$

Suppose now that new zeros of time and position are chosen so that $x_1 = x + \frac{\delta}{2}$ and $t_1 = t + \frac{k\delta}{2\omega}$. On substitution for x and t in (A13.1) it is found that

$$\eta = a \sin(\omega t_1 - kx_1) + fa \sin(\omega t_1 + kx_1) \quad . \quad \text{(A13.2)}$$

This expression is of the same form as (1) on p. 106, and it is seen that the effect of the phase change $k\delta$ is merely to shift the whole of the standing-wave system a distance $\frac{\delta}{2}$ away from the origin. Hence, if the position of the standing-wave system with respect to the specimen is obtained and compared with that when the specimen is replaced by one of infinite impedance, it follows that the magnitude (from the standing-wave ratio) and phase of the reflection coefficient become known. From these data the acoustic impedance of the specimen may be evaluated (see Appendix 30).

APPENDIX 14

Vibrational Modes of Some Mechanical Systems

A consideration of the vibrational modes of a system is of great importance in other branches of physics besides acoustics; for example, in the theory of the specific heats of a solid the thermal vibrations are supposed to result from many simultaneous modes, whose phases have a random distribution to one another.

(*a*) **One-dimensional system.** Standing waves are produced when a string fixed at each end is set into vibration and, moreover, it may vibrate in two or more modes simultaneously, the criterion for the existence of any particular mode being that the total length l of the string should contain a whole number n of half-wave-lengths ($\lambda/2$), i.e. $l = n\frac{\lambda}{2}$ or $n = \frac{2l}{\lambda}$. This expression will also be valid for an organ pipe open at both ends. In this latter case the vibrations are *longitudinal* and it follows by differentiation of the above expression that the number dn of vibrational modes in the wave-length range λ to $\lambda + d\lambda$ is numerically $dn = 2l.\frac{d\lambda}{\lambda^2}$. The corresponding expression for the *transverse* vibrations of a string is doubled, viz. $dn = 4l.\frac{d\lambda}{\lambda^2}$, since the vibration of a

string in any direction intermediate to two directions at right angles to its length may be resolved vectorially into the sum of two perpendicular vibrations. Each of these vibrations is counted as a separate mode for it is independent of the other both in phase and in amplitude.

(*b*) **Two-dimensional system.** Let $OCDE$ (Fig. A14.1) represent a membrane of rectangular form which is rigidly fixed at its periphery and in which a wave is travelling with velocity v in the direction ON. D_x and D_y are the lengths of sides of the membrane, and AB and $A'B'$ represent adjacent positions of a wave-crest at any time t. The direction of travel of the wave makes angles α and β with the x and y axes respectively. If the membrane is subjected to equal tension in

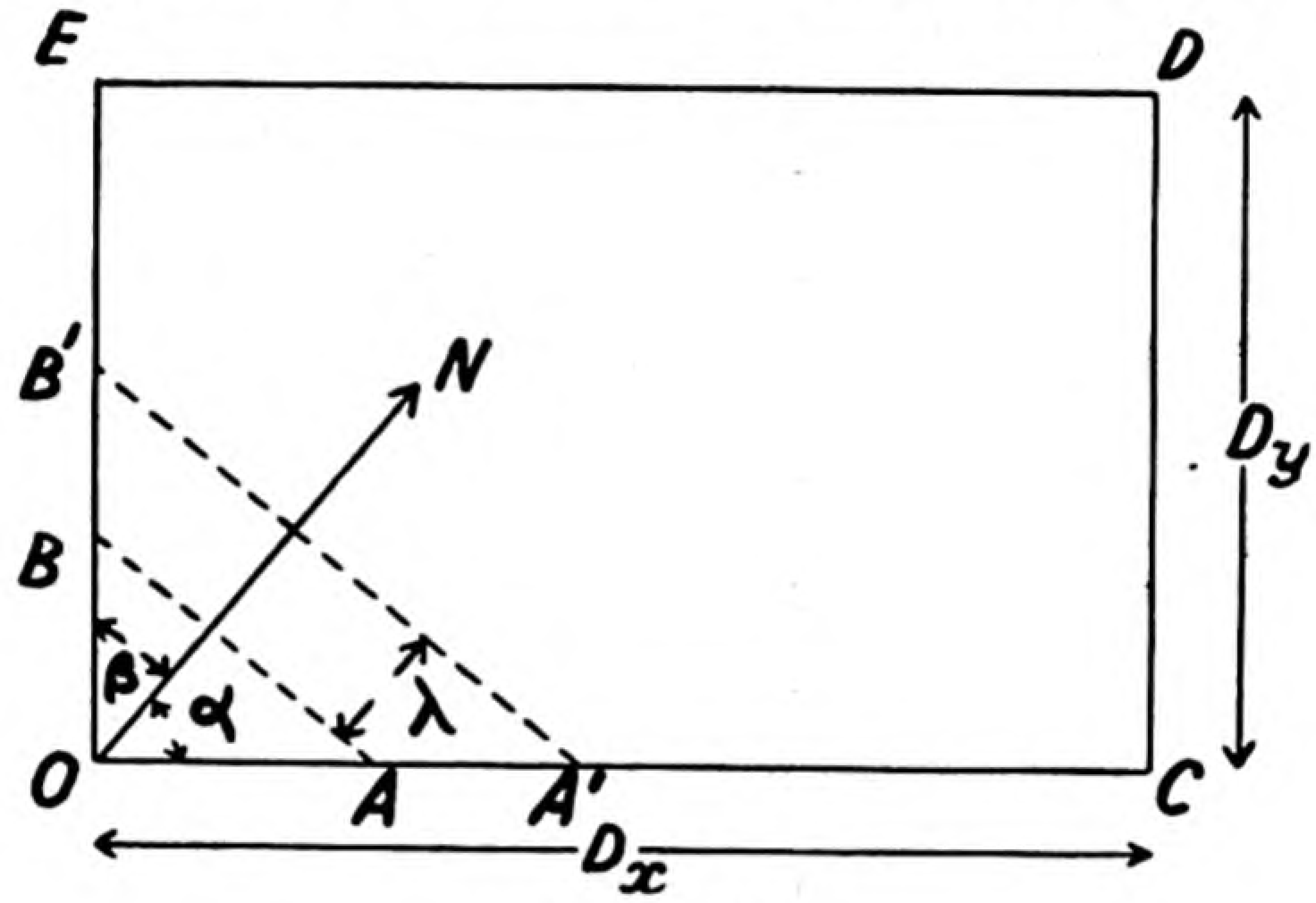


Fig. A14.1.

all directions, then the velocity of wave propagation v may be taken as constant in any direction of travel. The conditions for a standing-wave system are evidently $\dfrac{D_x}{\frac{AA'}{2}}=n_p$ and $\dfrac{D_y}{\frac{BB'}{2}}=n_q$, where n_p and n_q are whole numbers. Hence $\dfrac{D_x}{\frac{\lambda}{2\cos\alpha}}=n_p$ and $\dfrac{D_y}{\frac{\lambda}{2\cos\beta}}=n_q$, and since $\cos^2\alpha+\cos^2\beta=1$, it follows that

$$\sqrt{\frac{n_p^{\,2}}{D_x^{\,2}}+\frac{n_q^{\,2}}{D_y^{\,2}}}=\frac{2}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad (A14.1)$$

The membrane is now said to be vibrating in its $(n_p.n_q)th$ mode, indicating that it is in its n_pth mode along the x-axis and its n_qth mode along the y-axis.

(*c*) **Three-dimensional system.** Consider now a box of sides D_x, D_y, and D_z in the direction of the perpendicular axes x, y, z respectively. Then, assuming rigid walls, the conditions for a standing-wave system are easily seen to be $\dfrac{D_x}{\frac{\lambda}{2\cos\alpha}}=n_p$, $\dfrac{D_y}{\frac{\lambda}{2\cos\beta}}=n_q$, and $\dfrac{D_z}{\frac{\lambda}{2\cos\gamma}}=$

where γ is the angle between the direction of the wave and the z-axis, n_r is an integer and the other symbols have their previous significance. Since $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ then it follows that

$$\sqrt{\frac{n_p^2}{D_x^2}+\frac{n_q^2}{D_y^2}+\frac{n_r^2}{D_z^2}}=\frac{2}{\lambda} \quad . \quad . \quad . \quad . \quad (A14.2)$$

If ν denotes the frequency corresponding to the wave-length λ then $\nu\lambda = c$ and (A14.2) may be written

$$\nu=\frac{c}{2}\sqrt{\frac{n_p^2}{D_x^2}+\frac{n_q^2}{D_y^2}+\frac{n_r^2}{D_z^2}} \quad . \quad . \quad . \quad . \quad (A14.3)$$

It is interesting to note that the system of standing-waves given by this expression may be divided into three groups, viz.

(i) *Oblique* waves defined by $n_x \neq 0$, $n_y \neq 0$, and $n_z \neq 0$, and these are formed from eight travelling waves which suffer reflection from all the six walls.

(ii) *Tangential* waves which are built up of four travelling waves, and these suffer reflection from four walls and are directed parallel to the other two. *e.g.* if $n_z = 0$ but n_x and n_y are not zero, then the tangential waves will move parallel to the xy walls.

(iii) *Axial waves* consist of two travelling waves propagated parallel to one axis and only striking two walls, *e.g.* if $n_x = 0 = n_y$ then a z-axial wave is indicated.

An inspection of equation (A14.3) suggests that each characteristic frequency in a rectangular box or room behaves as a vector having components $\frac{cn_x}{2D_x}$, $\frac{cn_y}{2D_y}$, and $\frac{cn_z}{2D_z}$, and hence may be represented by one point in a rectangular lattice in frequency space. The lattice spacings in the x, y, z directions would be $\frac{c}{2D_x}$, $\frac{c}{2D_y}$, and $\frac{c}{2D_z}$ respectively. The "volume occupied in frequency space" by one point is consequently a rectangular parallelepiped of volume $\frac{c^3}{8D_xD_yD_z}$ with the point located at the centre. The direction of the vector joining the point to the origin thus gives the direction of the standing wave and its length is a measure of the wave frequency.

This conception of a "frequency lattice" has found application in the problem of finding the number and type of normal modes having frequencies in a given frequency range. For consider the volume of an octant of a sphere of radius ν, it will be given by $\frac{1}{8}\cdot\frac{4}{3}\pi\nu^3 = \frac{\pi\nu^3}{6}$. It follows that the number of frequency points within this volume, *i.e.* the number of modes having frequencies less than or equal to ν, will be $\frac{\pi\nu^3}{6} \div \frac{c^3}{8D_xD_yD_z}$ or $\frac{4\pi\nu^3}{3c^3}(D_xD_yD_z)$. This approximate result was obtained by Lord Rayleigh and others when deriving the laws governing electromagnetic radiation and was expressed in the form that the number of frequencies less than ν was

$$n(\nu)=\frac{4\pi(D_xD_yD_z)\nu^3}{3c^3} \quad . \quad . \quad . \quad . \quad (A14.4)$$

The *number* of vibrational modes with frequencies between ν and $\nu+d\nu$ is obtained by differentiating the above expression whence

$$dn(\nu)=\frac{4\pi(D_xD_yD_z)\nu^2}{c^3}d\nu, \text{ i.e. } dn(\nu)=\frac{4\pi(D_xD_yD_z)d\lambda}{\lambda^4} \quad \text{(A14.5)}$$

since $c=\nu\lambda$ and $d\nu=\frac{-c}{\lambda^2}d\lambda$.

As an example of the use of the foregoing analysis in room acoustics consider a rectangular enclosure $10\times20\times40$ metres and suppose $\nu=300$ c.p.s. and $d\nu=10$ c.p.s. It follows from substitution in

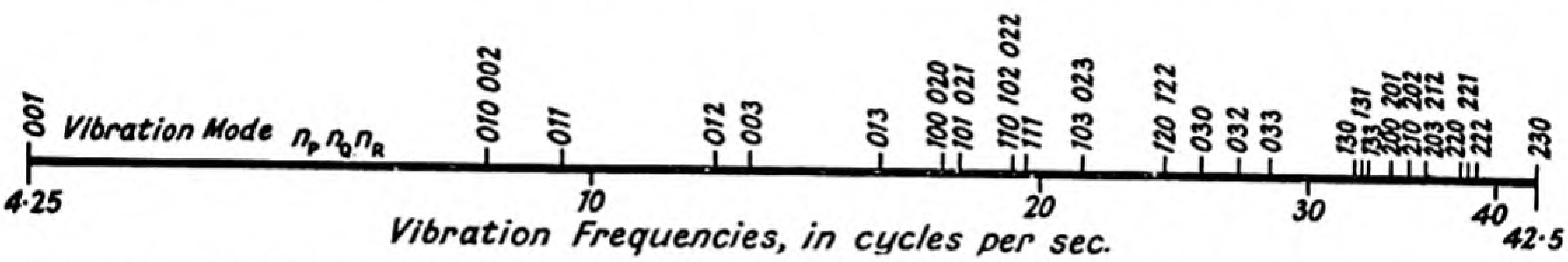


Fig. A14.2. Spectrum of resonant frequencies from 4·25 c.p.s. (fundamental) to 42·5 c.p.s. in the rectangular auditorium.

(A14.5) that the number of vibrational modes between 300 c.p.s. and 310 c.p.s. for the above enclosure is approximately equal to 2550.

The more exact expression for $n(\nu)$ may be shown to be given by

$$n(\nu)=\frac{4\pi(D_xD_yD_z)\nu^3}{3c^3}+\frac{\pi A}{4c^2}\nu^2+\frac{D}{8c}\nu+O(\nu) \quad . \quad \text{(A14.6)}$$

where A is the area of the wall surfaces and D is the total length of the edges of the room. The first three terms are mainly due to the oblique, the tangential and the axial waves respectively, while $O(\nu)$ is an irregular step-function whose effect is to give a "stepped" nature to the curve connecting the number of eigen-tones against the corresponding frequency.

APPENDIX 15

Mean Free Path of Sound in a Rectangular Room*

Instead of mirror images of the source of sound in the room, images of the room are used in conjunction with the original source S, as in Fig. A15.1 *a* and *b*, in which the dimensions are shown, the former being a plan; actually the image rooms occur in three dimensions.

Consider any "ray" of sound SC projected in an upward direction at an angle θ to the vertical and ϕ to the x-axis which coincides with the dimensions a at floor level.

After the source has emitted sound energy for t sec. the ray will be vt in length, v being the velocity in units of length per second. The number of encounters *per second* with any pair of opposite walls will be the component of v perpendicular to those walls divided by the distance between, and is actually the number of impacts per second within the room on those walls.

* Bate and Pillow, *Proc. Phys. Soc.*, **59**, 535, 1947.

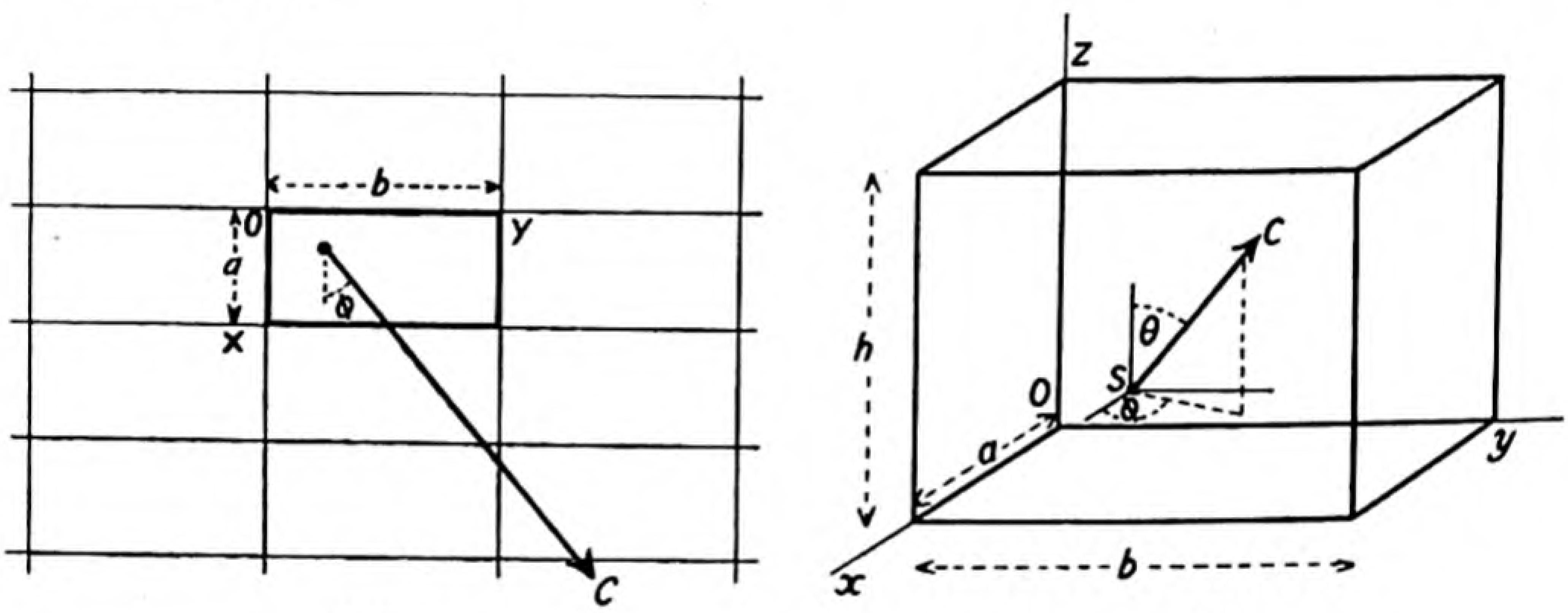
The number of impacts with the walls will, therefore, be $\left(\frac{v}{a}\right)\sin\theta\cos\phi$, $\left(\frac{v}{b}\right)\sin\theta\sin\phi$, and $\left(\frac{v}{h}\right)\cos\theta$ on the walls parallel to the yz, xz, and xy planes respectively.

Assume $4\pi n$ units of sound energy to be emitted from S in 1 sec., then the number projected in the (θ, ϕ) direction in the element of solid angle $\sin\theta.d\theta.d\phi$ will be $n\sin\theta.d\theta.d\phi$ per sec., so the total number of impacts will be

$$8n\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left\{\frac{v}{a}\sin\theta\cos\phi+\frac{v}{b}\sin\theta\sin\phi+\frac{v}{h}\cos\theta\right\}\sin\theta.d\theta.d\phi \quad \text{(A15.1)}$$

$$=2n\pi v\frac{(bh+ah+ab)}{abh}$$

$=\pi nv\dfrac{A}{V}$, A being the total area of the walls and V the volume of the room.



(a) (b)

Fig. A15.1.

The total distance traversed by all the units in 1 sec. $=4\pi n.v$, so that the mean distance traversed between impacts

$$=\frac{4\pi nv}{\left(\frac{\pi nvA}{V}\right)}=\frac{4V}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A15.2)}$$

This result is independent of the position of S, and is therefore the mean free path.

The same result is obtained in the case of spherical and cylindrical enclosures; for their calculation the reader should consult the original paper.

APPENDIX 16

Spherical Waves

A spherical wave is one in which any of the characteristics displacement η, excess pressure δP, condensation s, etc., of a sound wave is a function of the radius r, which will be the distance measured to the wave-front from the source of origin.

Fig. A16.1 shows a spherical wave system consisting of alternate spherical regions of compression (C) and rarefaction (R) emanating from the source S. In the ideal case this generator would consist of a small elastic sphere $ABCD$ (Fig. A16.2) which is alternately compressed and dilated at a definite frequency so that the excess pressure

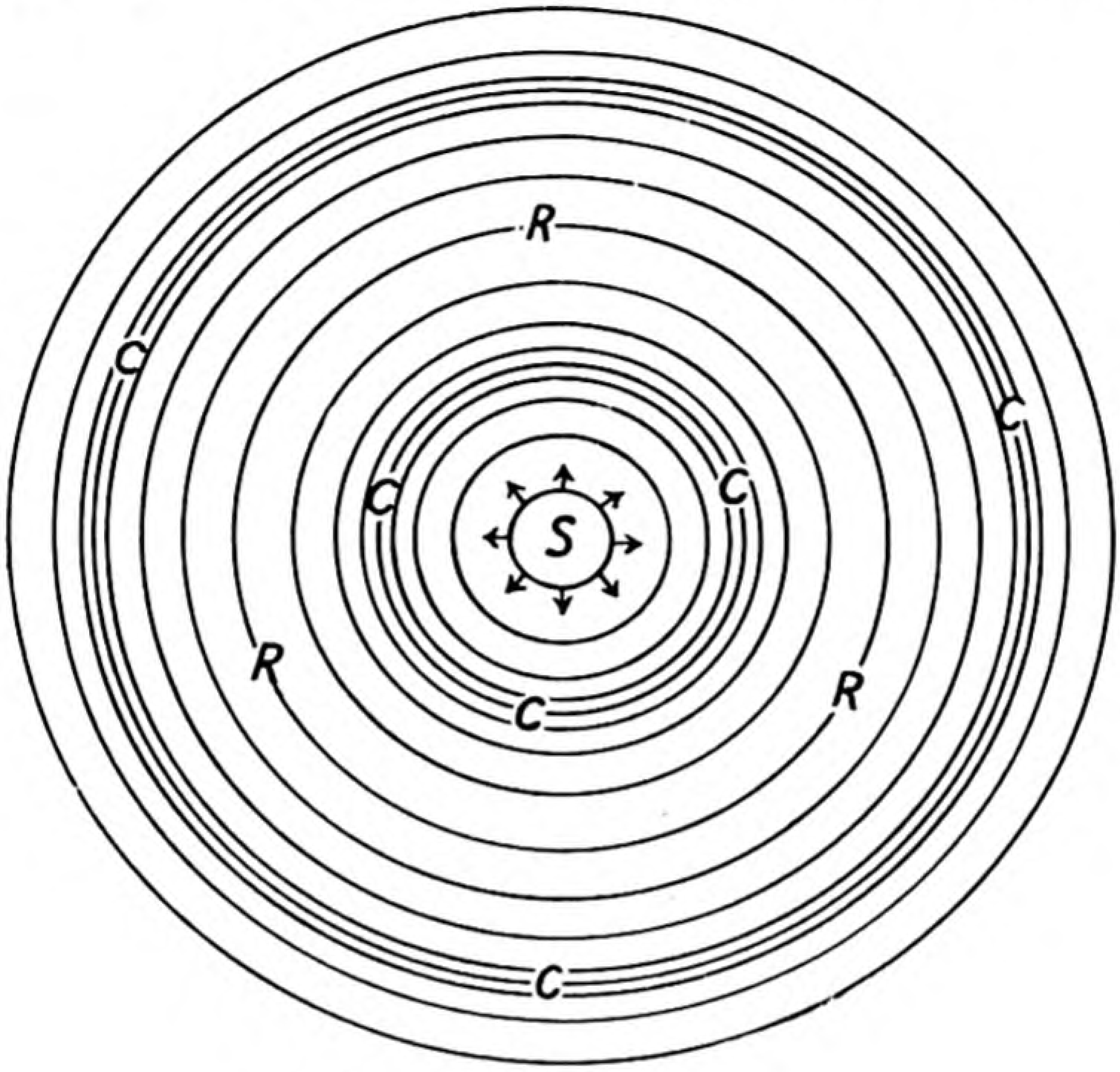


Fig. A16.1.

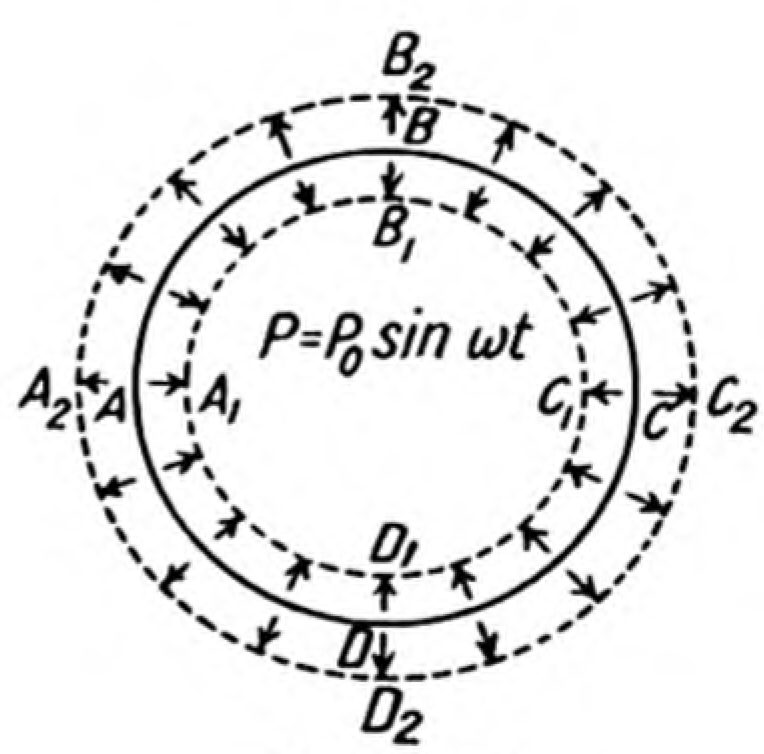


Fig. A16.2.

(p) at any instant is given by $p = p_0 \sin \omega t$, where p_0 is the maximum excess pressure and $\frac{\omega}{2\pi}$ is the frequency.

In a plane progressive wave the particle-velocity and the excess pressure are everywhere and at all times in phase with each other, but in the case of a standing-wave system there is a phase difference of $\frac{\pi}{2}$, and in consequence no transfer of acoustic energy takes place.

Analogies which immediately come to the mind of the radio engineer are the corresponding cases of a correctly terminated (or infinite) transmission line in which no energy is reflected back, and one which is of a suitable length to obtain electromagnetic standing-waves. In the case of a diverging spherical sound wave the particle-velocity u will be of the form (see p. 413)

$$u=\frac{Ak}{r}\sin k(r-ct)+\frac{A}{r^2}\cos k(r-ct) \quad . \quad . \quad . \quad \text{(A16.1)}$$

whereas the excess pressure

$$p=\rho\frac{Akc}{r}\sin k(r-ct) \quad . \quad . \quad . \quad . \quad . \quad \text{(A16.2)}$$

Now equation (A16.1) may be rewritten as

$$u=A'\sin[k(r-ct)+\theta] \quad . \quad . \quad . \quad . \quad \text{(A16.3)}$$

where $\quad A'=\frac{A}{r^2}\sqrt{1+k^2r^2},\ \theta=\tan^{-1}\frac{1}{kr},$ and $k=\frac{2\pi}{\lambda}.$

It follows that near to the source the particle-velocity is $\frac{\pi}{2}$ ahead of the excess pressure, but as r increases they come more nearly into phase, a condition which is to be expected at very large distances when the wave-front becomes planar.

From (A16.1) and (A16.3) it follows that

$$p_{\max}=\frac{\rho c . u_{\max}}{\sqrt{\left\{1+\frac{1}{4\pi^2}\left(\frac{\lambda}{r}\right)^2\right\}}} \quad . \quad . \quad . \quad . \quad \text{(A16.4)}$$

which approximates to the expression for plane waves, $p=\rho c u_{\max}$, at large distances from the source. It follows from equation (A16.4) that the amplitude of the particle-velocity, $u_{\max}$, will increase, at a given point for a wave of given amplitude, as the frequency $\left(\propto\frac{1}{\lambda}\right)$ of the source *decreases*; this fact is of importance in the design of velocity microphones.

Returning to the example of the pulsating sphere it is evident that since the motion of any point on the sphere is strictly radial, then the sound intensity at a point P_R will be unaffected by the insertion of a baffle board BB (Fig. A16.3) surrounding, but not touching, the sphere in a diametral plane. Similarly at a point P_L to the left of BB the intensity will remain unaltered by the presence of the baffle. Furthermore, if the motion of the right-hand half of the sphere be *reversed* (Fig. A16.4) no essential difference will be noted at P_L, and it is evident that a close approximation has been obtained to the case of an oscillating flat disc, of diameter small compared with λ, set in a baffle board (Fig. A16.5). This problem was investigated by Lord Rayleigh and applied to the practical sound generator, viz. a loud-speaker membrane in a large baffle board. By using the latter device the output of the source is approximately doubled, since by virtue of being situated at a small distance (compared with λ) away from the plane of the baffle it gives rise to an image source (see p. 98) of the same

frequency and phase as itself. The reason for the doubling, of course, arises from the fact that the radiation is limited by reflection to one hemisphere only.

If the baffle board is removed then it is evident that a movement of the diaphragm in one direction will simultaneously give rise to a compression in front of it, and a rarefaction in its wake. In other words, a pressure difference will exist in the air at the two surfaces, which tends to become equalised by a flow of air round the disc. Hence some of the energy of the membrane will be employed in imparting kinetic energy to these air streams and less will be radiated as sound waves. This fact is well illustrated by the case of an oscillating sphere (Fig. A16.6*a*) on which all points have the same mutually parallel velocity, H say; the two extreme positions of the vibrating sphere are shown dotted. Now the velocity H of each point on the sphere is the resultant of a tangential (T) and a radial (R) component

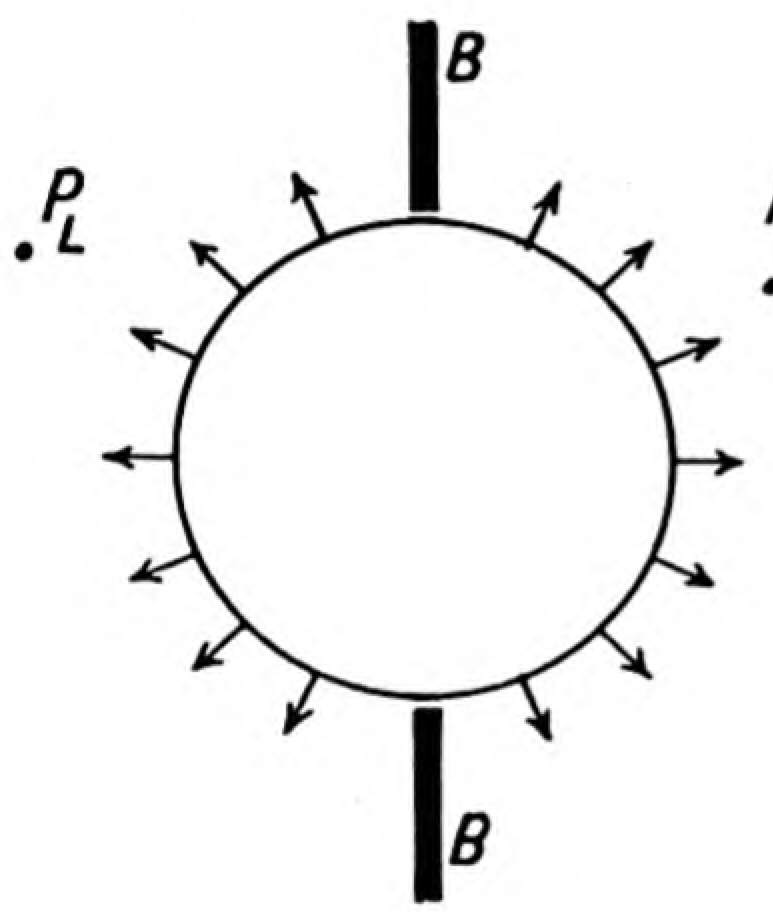


Fig. A16.3.

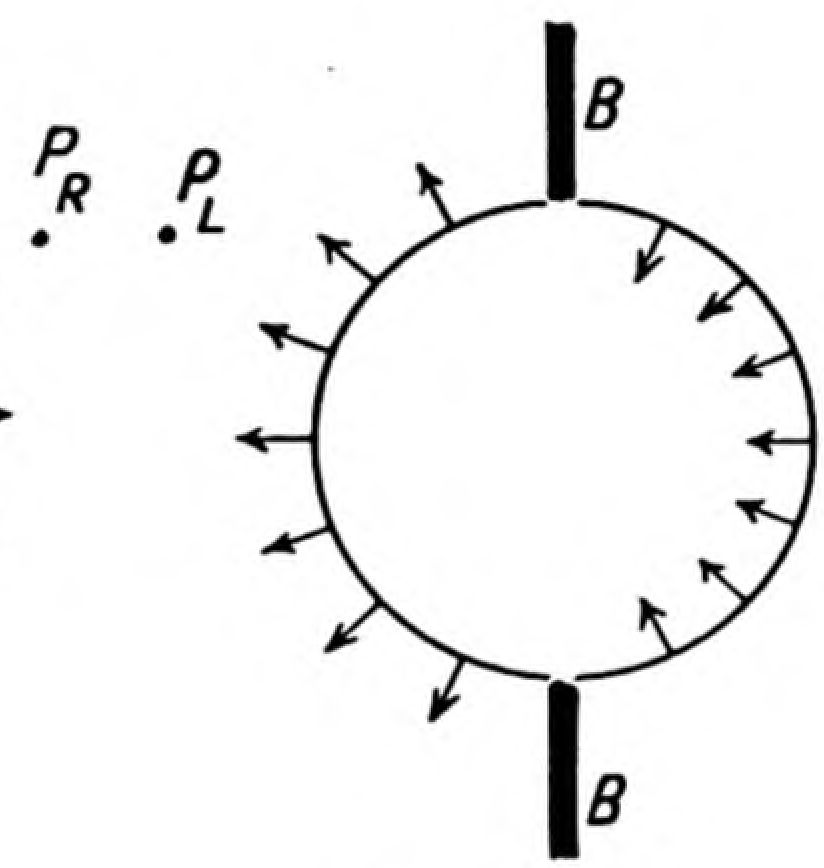


Fig. A16.4.

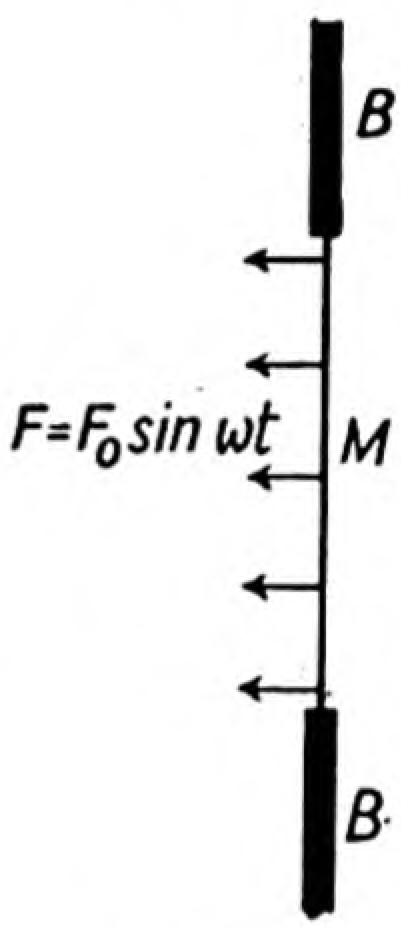


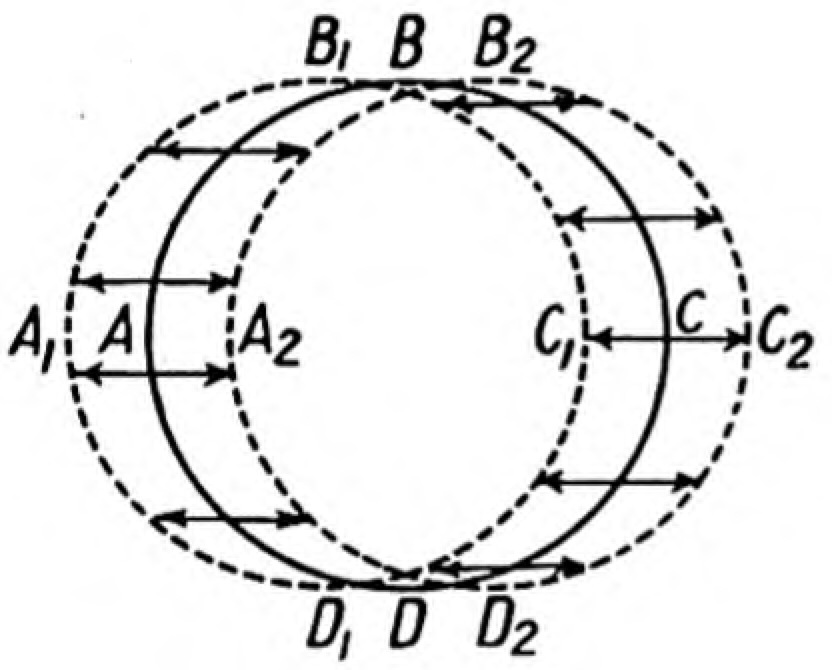
Fig. A16.5.

as indicated in Fig. A16.6*b*. The radial components may be associated with a non-uniform *pulsating* sphere while the tangential components will give rise to an oscillation of air (Fig. A16.6*c*) round the spherical surface, which van Urk and Vermeulen have indicated is analogous to that in an organ pipe closed at both ends and of length $\pi \times$ radius of sphere.

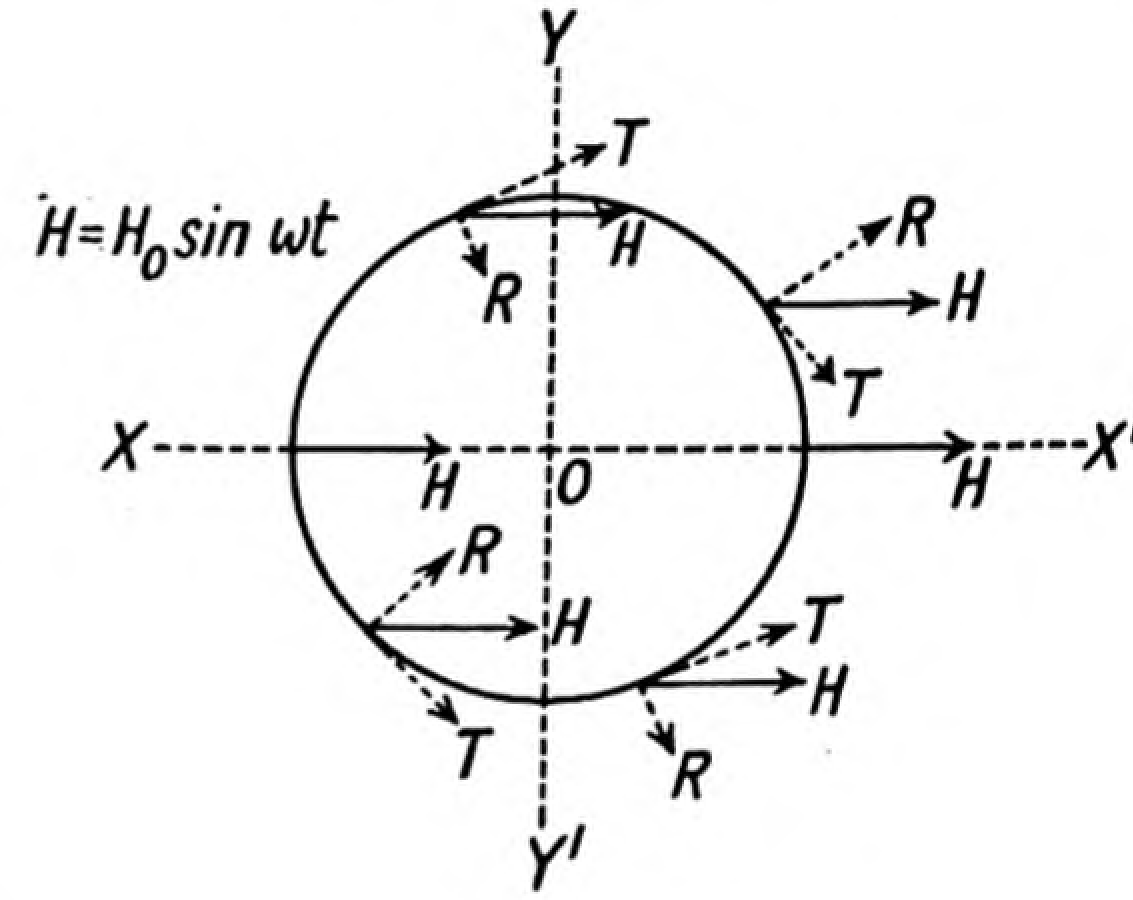
Besides vibrating membranes exposed on both sides to the propagating medium, tuning-forks and vibrating wires are other common examples of double sources (or doublets), which are the acoustic analogy to magnetic and electrostatic dipoles (p. 412). Fig. 7.5 shows the approximate sound field due to a vibrating tuning-fork, and it is evident how the placing of a card in the positions indicated will help to reduce the interference from the out of phase sources of the doublets.

The velocity potential (p. 411) at a point P distant r from the centre of a doublet is proportional to $\frac{\cos\theta}{r^2}$, where θ is the angle between the line joining the two component sources and the line from P to the mid point. Furthermore, if S is the strength (see Appendix 21)

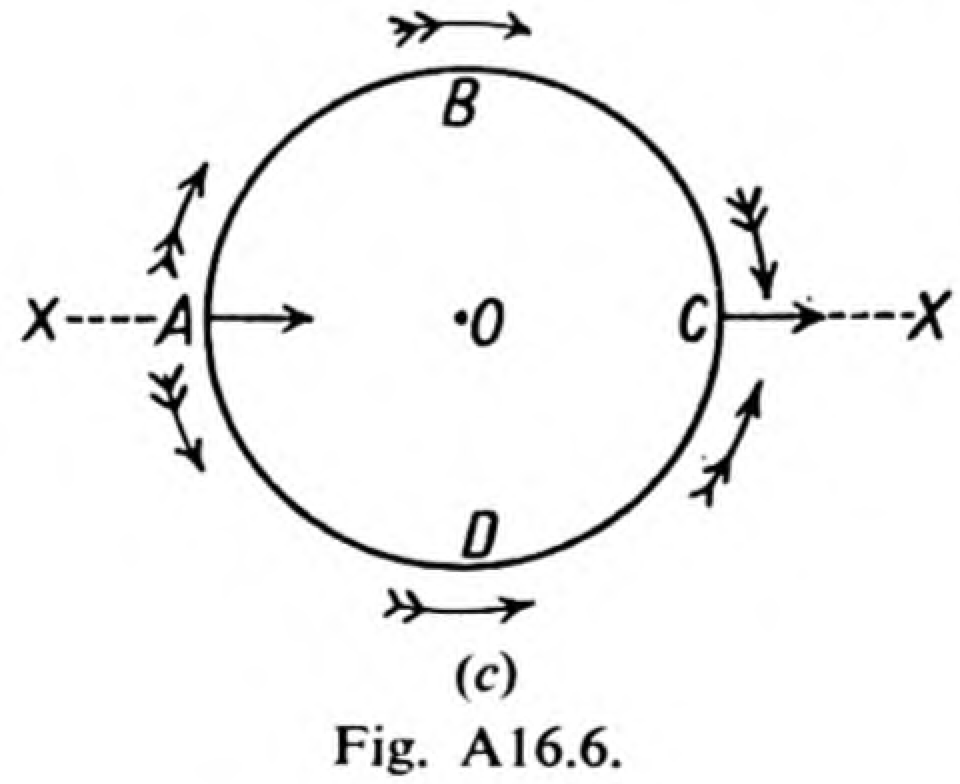
of the double source then the rate of emission of energy is $\frac{\rho c k^4 S^2}{24\pi}$ where $k=\frac{2\pi}{\lambda}$, and for a single source the rate is $\frac{\rho c k^2 S^2}{8\pi}$. Hence the power radiated is in both cases proportional to ρc, but whereas for a double source it is $\propto \frac{1}{\lambda^4}$ in the simple source it is $\propto \frac{1}{\lambda^2}$. Hence the doublet becomes more efficient as a radiator than the single source as the frequency $\left(\propto \frac{1}{\lambda}\right)$ increases. In both cases, however, it is evident that it is only with fairly rapid *changes* of motion of a body that appreciable acoustical energy is produced in the form of sound waves, just as in the analogous electrical problem of the radiation of electromagnetic waves.



(a)



(b)



(c)

Fig. A16.6.

APPENDIX 17

Transmission of Energy by Longitudinal Waves in Fluid Media

There is obviously a limit to the maximum energy which may be transmitted by longitudinal waves through a liquid, and it is imposed by the condition that a pressure *amplitude* p greater than the static equilibrium pressure P_0 will result in a negative pressure, *i.e.* a tension, in the region of the minimum of the alternating cycle. This condition

will be given by $P_0 \leqq p$ but $W = \frac{1}{2}\frac{p^2}{R}$, where W is the mean power transmitted across unit area and R is the acoustical impedance of the medium.

Hence $W_{max} = \frac{P_0{}^2}{2R}$, an expression which is independent of the frequency.

For air at 20° C., $W_{max} = \frac{10^{12}}{2 \times 41 \cdot 2} = 1 \cdot 2 \times 10^{10}$ erg sec. per cm.^{2}.

$= 1 \cdot 2 KW$ per cm.^{2},

while for water (near surface) at 20° C.,

$$W_{max} = \frac{10^{12}}{2 \times 1 \cdot 5 \times 10^5} = 3 \cdot 3 \times 10^{-4} KW \text{ per cm.}^2.$$

The amplitude a corresponding to the breakdown excess pressure P is given by $\omega a = 2\pi fa = v = \frac{P}{R} = \frac{P_0}{R}$ or $a = \frac{1}{2\pi f} \cdot \frac{P_0}{R}$,

where f is the frequency. Calculation shows that for ordinary frequencies the amplitudes required for breakdown are quite large, as, for example, at 256 c.p.s. the critical amplitude at the surface of sea-water is 0·004 cm., but at 25,600 c.p.s. it would be 0·00004 cm. The smaller critical amplitude at higher frequencies accounts for the streams of bubbles observed to rise from the pressure anti-nodes in a standing system of ultrasonic waves in a liquid, due to the escape of dissolved gases at the times of low pressure. Even in the absence of these gases the energy of the high frequency vibrations is sometimes sufficient to vaporise the liquid so that vapour bubbles are seen to appear.

APPENDIX 18

Acoustic Impedance of Pipes

The following theoretical treatment is after the method suggested by Irons, and it is assumed that

(i) the cross-sectional *area* S of the pipe is uniform and $<< \lambda^2$, λ being the wave-length of the sound waves;

(ii) the fundamental acoustic equations are applicable, and there is no particle displacement in directions perpendicular to the axis of the tube, *i.e.* the walls of the tube are rigid;

(iii) both the particle displacement η and the excess pressure ΔP are *everywhere* analytical functions of time and space, so that we can interchange the x and t derivatives;

(iv) the particle-velocity $\dot{\eta}$ and condensation s are both small; and

(v) dissipation of energy due to viscosity is neglected.

The equation for propagation of plane waves in the direction of the x-axis is

$$\frac{\partial^2 \eta}{\partial t^2} = c^2 \frac{\partial^2 \eta}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (A18.1)$$

where c is the velocity of propagation.

The waves are assumed to be simple harmonic in type of frequency $\frac{\omega}{2\pi}$ and hence $$\eta = f e^{j\omega t} \quad \ldots \ldots \quad (A18.2)$$
where f is a function of x to be determined.

From (1) and (2) it follows that

$$\frac{\partial^2 \eta}{\partial x^2} - \frac{1}{c^2} \cdot \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial^2 \eta}{\partial x^2} + \frac{\omega^2}{c^2}\eta = 0 \quad \ldots \quad (A18.3)$$

The solution of this equation is

$$\eta = A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}$$

so that the complete solution of (A18.1) is

$$\eta = \left(A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}\right) e^{j\omega t} \quad \ldots \quad (A18.4)$$

Now the particle-velocity at time t at any point P (Fig. A18.1a) is given by $\dot{\eta} = \frac{\partial \eta}{\partial t} = j\omega\eta$, where η is the particle displacement at P, and hence the corresponding volume current

$$S\dot{\eta} = Sj\omega\eta = Sj\omega\left[A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}\right] e^{j\omega t} \quad . \quad (A18.5)$$

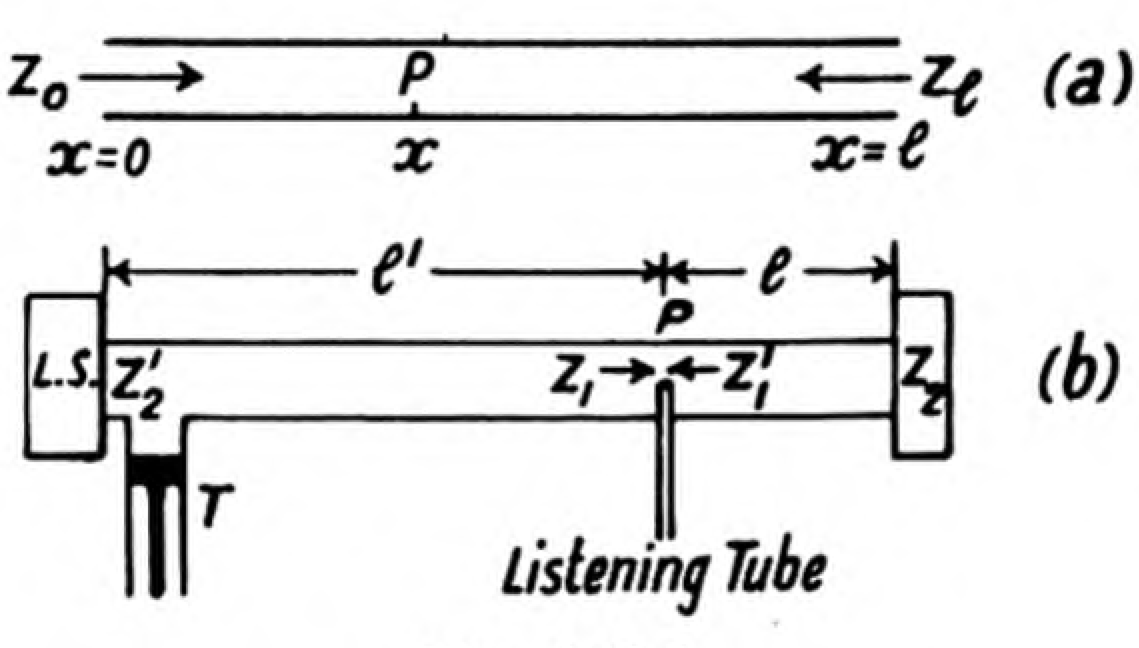


Fig. A18.1.

The excess pressure at P at time t is expressed by

$$\Delta P = Ks = -c^2 \rho \frac{\partial \eta}{\partial x} = -c\rho\omega\left[A \cos \frac{\omega x}{c} - B \sin \frac{\omega x}{c}\right] e^{j\omega t} \quad (A18.6)$$

where K is the bulk modulus of elasticity, s is the condensation and ρ the density of the medium.

The impedance Z of the pipe at P to the passage of sound waves, is therefore, by definition,

$$Z = \frac{\Delta P}{\text{Volume current}} = \frac{\Delta P}{S\dot{\eta}} = \frac{j\rho c}{S}\left[\frac{A \cos \frac{\omega x}{c} - B \sin \frac{\omega x}{c}}{A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}}\right] \quad (A18.7)$$

Now if Z_0 and Z_l are the terminating impedances at $x=0$, $x=l$ respectively, and these values are substituted in equation (A18.7), the following relations are obtained:

$$\left.\begin{aligned} Z_0&=\frac{j\rho c}{S}\left[\frac{A}{B}\right], \text{ and} \\ Z_l&=\frac{j\rho c}{S}\left[\frac{A\cos\left(\frac{\omega l}{c}\right)-B\sin\left(\frac{\omega l}{c}\right)}{A\sin\left(\frac{\omega l}{c}\right)+B\cos\left(\frac{\omega l}{c}\right)}\right]\end{aligned}\right\} \quad . \quad \text{(A18.8)}$$

Eliminating A and B from these equations

$$Z_l=\frac{j\rho c}{S}\left[\frac{Z_0\cos\frac{\omega l}{c}-\frac{j\rho c}{S}\sin\frac{\omega l}{c}}{Z_0\sin\frac{\omega l}{c}+\frac{j\rho c}{S}\cos\frac{\omega l}{c}}\right] \quad . \quad . \quad \text{(A18.9}a\text{)}$$

Alternatively

$$Z_0=\frac{jZ_l\cos\frac{\omega l}{c}-\frac{\rho c}{S}\sin\frac{\omega l}{c}}{-\frac{S}{\rho c}Z_l\sin\frac{\omega l}{c}+j\cos\frac{\omega l}{c}} \quad . \quad . \quad . \quad . \quad \text{(A18.9}b\text{)}$$

If the pipe is stopped at $x=l$, *i.e.* it is a closed end, then $Z_l\to\infty$, since the volume displacement is negligible, however large the excess pressure. It follows, therefore, from (A18.9a) that

$$Z_0\sin\frac{\omega l}{c}+\frac{j\rho c}{S}\cos\frac{\omega l}{c}=0$$

i.e.

$$Z_0=-\frac{j\rho c}{S}\cot\frac{\omega l}{c} \quad . \quad . \quad . \quad . \quad . \quad \text{(A18.10)}$$

If the other end $x=0$ is open, then $Z_0=0$, since a small excess pressure gives rise to a large volume displacement.

Hence from (A18.10) $\cot\frac{\omega l}{c}=0$, so that the possible stationary modes of vibration in the open-closed pipe are given by $\frac{\omega_m l}{c}=(2m-1)\frac{\pi}{2}$, where m is any integer. Now $\frac{\omega_m}{2\pi}=f_m=\frac{c}{\lambda_m}$, f_m and λ_m being the frequency and wave-length respectively of the mth mode of vibration, and so it follows that

$$\lambda_m=\frac{4l}{2m-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A18.11)}$$

In the case of the open-open pipe $Z_l=0$, and this condition implies, from equation (A18.9a), that

$$Z_0\cos\frac{\omega l}{c}-\frac{j\rho c}{S}\sin\frac{\omega l}{c}=0,$$

i.e.

$$Z_0=\frac{j\rho c}{S}\tan\frac{\omega l}{c} \quad . \quad . \quad . \quad . \quad . \quad \text{(A18.12)}$$

But $Z_0=0$ also and therefore $\tan\frac{\omega l}{c}=0$, and the possible stationary modes of vibration are given by, m being any integer,

$$\frac{\omega_m l}{c}=m\pi \text{ or } \lambda_m=\frac{c}{f_m}=\frac{2l}{m} \quad . \quad . \quad . \quad . \quad . \quad \text{(A18.13)}$$

Measurement of the "conductivities" of orifices. The method to be described is that used by Robinson and it is dependent upon the theory developed above.

Consider a tube (Fig. A18.1*b*) terminating at the left-hand end in a loud-speaker unit and at the right-hand end in an impedance Z_2. A listening-tube is inserted at the point P so that the main tube may be considered to be divided into two parts l' and l as shown. Let Z_1 and Z_1' be the impedances presented at P when looking into the right- and left-hand portions respectively.

It follows from the previous analysis that

$$Z_1=\frac{jZ_2\cos\frac{\omega l}{c}-\frac{\rho c}{S}\sin\frac{\omega l}{c}}{-\frac{S}{\rho c}Z_2\sin\frac{\omega l}{c}+j\cos\frac{\omega l}{c}} \quad . \quad . \quad . \quad \text{(A18.14)}$$

and

$$Z_1'=\frac{jZ_2'\cos\frac{\omega l'}{c}-\frac{\rho c}{S}\sin\frac{\omega l'}{c}}{-\frac{S}{\rho c}Z_2'\sin\frac{\omega l'}{c}+j\cos\frac{\omega l'}{c}} \quad . \quad . \quad . \quad \text{(A18.15)}$$

The impedances Z_1 and Z_1' are obviously determined by the values of the lengths l_1 and l_1' and the end impedances Z_2 and Z_2'. Under certain circumstances the loud-speaker will give rise to a standing-wave system and this state may be attained by suitable adjustment of the positions and magnitudes of Z_2 and Z_2'.

Suppose that initially the right-hand end of the tube is closed by a *rigid* stop and that l is varied until the point P is judged to be a position of *minimum* pressure, then l must equal $\frac{n\lambda}{4}$ where n is an *odd* integer.

But looking towards the right at P, it follows from equation (A18.9*b*) that

$$Z_1=\frac{jZ_2\cos\frac{\omega l}{c}-\frac{\rho c}{S}\sin\frac{\omega l}{c}}{-\frac{S}{\rho c}Z_2\sin\frac{\omega l}{c}+j\cos\frac{\omega l}{c}}$$

$$=\frac{j\cos\frac{\omega l}{c}-\frac{\rho c}{SZ_2}\sin\frac{\omega l}{c}}{-\frac{S}{\rho c}\sin\frac{\omega l}{c}+\frac{j}{Z_2}\cos\frac{\omega l}{c}}$$

$$=-\frac{j\rho c}{S}\cot\frac{\omega l}{c} \text{ since } Z_2\to\infty. \quad . \quad . \quad . \quad \text{(A18.16)}$$

Now $l=\frac{n\lambda}{4}$ by adjustment, and hence on substitution

$$Z_1=-\frac{j\rho c}{S}\cot\frac{2\pi}{\lambda}\cdot\frac{n\lambda}{4}=-\frac{j\rho c}{S}\cot\frac{n\pi}{2}=0.$$

In order to obtain complete silence in the microphone at P it is necessary to have means of varying the end impedance Z_2', and this is effected by adjusting the position of a plunger in a side tube T near the loud-speaker. When this adjustment is made it implies that $Z_1'\to 0$.

The unknown impedance Z_x is now substituted for the rigid end Z_2 and, with the left-hand side setting unchanged, the length l is adjusted until silence is obtained at P. When this condition has been attained it follows that $Z_1=0$ and so from equation (A18.9*b*)

$$jZ_x\cos\frac{\omega l}{c}-\frac{\rho c}{S}\sin\frac{\omega l}{c}=0,$$

i.e.

$$Z_x=\frac{\rho c}{jS}\tan\frac{\omega l}{c}=-\frac{j\rho c}{S}\tan\frac{\omega l}{c} \quad . \quad . \quad . \quad \text{(A18.17)}$$

Fig. A18.2*a* shows a particular form of impedance consisting of two pipes of different diameters, S_1 and S_2 respectively, joined together

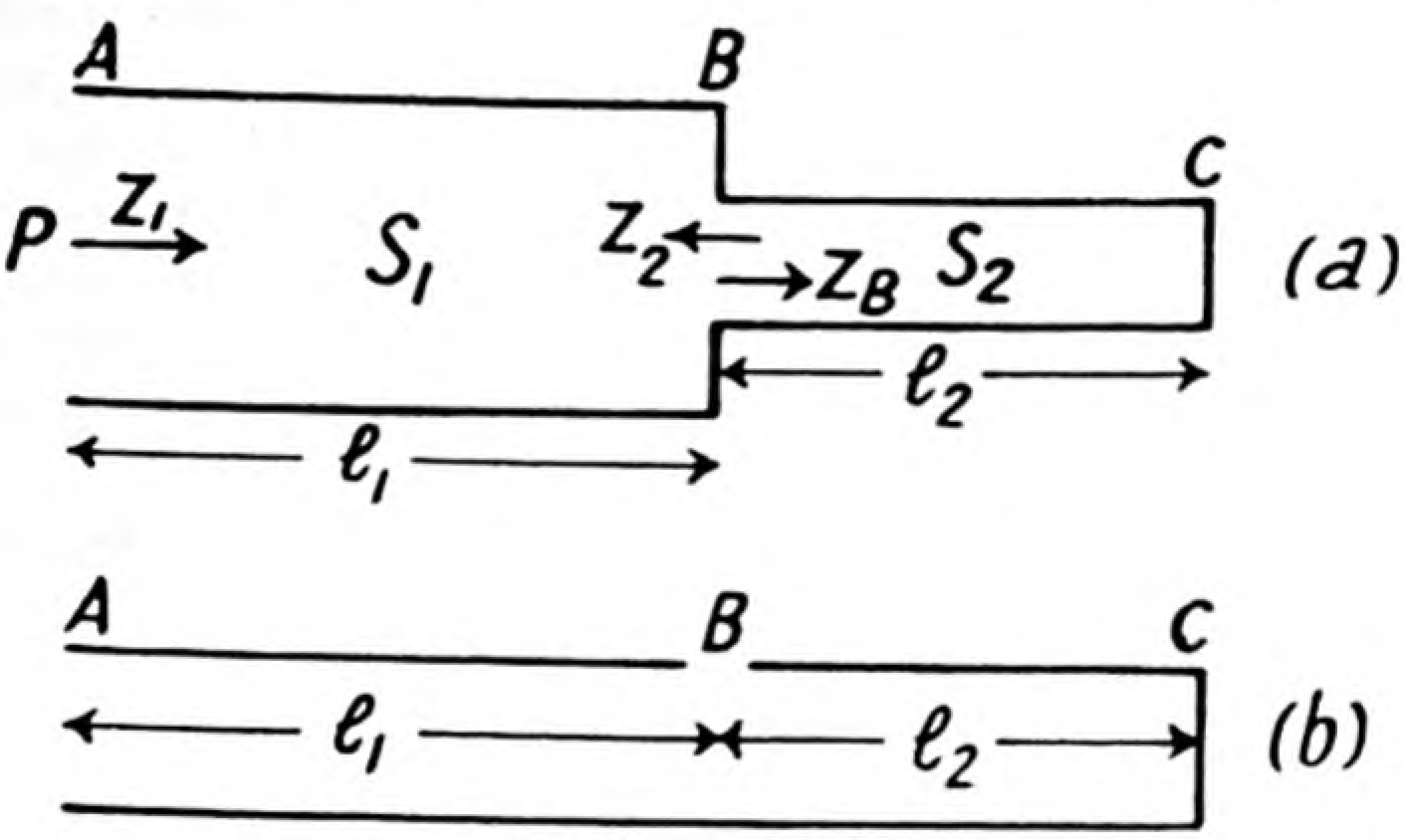


Fig. A18.2.

at B. Considering the section AB, whose end impedance at B is denoted by Z_2, it follows that since l_1 is adjusted for silence, then $Z_1=0$. Hence from equation (A18.9*b*)

$$jZ_2\cos\frac{\omega l_1}{c}-\frac{\rho c}{S_1}\sin\frac{\omega l_1}{c}=0$$

or

$$Z_2=-\frac{j\rho c}{S_1}\tan\frac{\omega l_1}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A18.18)}$$

Now considering the length BC, its end impedance $Z_C\to\infty$, since C is a rigid end.

Hence from equation (A18.9*a*)

$$Z_B\sin\frac{\omega l_2}{c}+\frac{j\rho c}{S_2}\cos\frac{\omega l_2}{c}=0 \text{ or } Z_B=-\frac{j\rho c}{S_2}\cot\frac{\omega l_2}{c}. \quad \text{(A18.19)}$$

But $Z_2=Z_B+Z_{\text{orifice}}$ since the impedances are in series. The impedance of an orifice, however, $=\dfrac{j\rho\omega}{K}$ where K is the conductivity of the orifice (see Helmholtz resonator), so it follows that

$$\frac{j\rho\omega}{K}=Z_{\text{orifice}}=Z_2-Z_B=-\frac{j\rho c}{S_1}\tan\frac{\omega l_1}{c}+\frac{j\rho c}{S_2}\cot\frac{\omega l_2}{c},$$

i.e.

$$\frac{\omega}{Kc}=\frac{1}{S_2}\cot\frac{\omega l_2}{c}-\frac{1}{S_1}\tan\frac{\omega l_1}{c} \quad . \quad . \quad . \quad . \quad \text{(A18.20)}$$

In Fig. A18.2*b* the impedance of the orifice at B is in parallel with the tube impedance of BC and in this case

$$\frac{1}{Z_2}=\frac{1}{Z_B}+\frac{1}{Z_{\text{orifice}}}.$$

APPENDIX 19

Equation of Continuity

A special case of this equation has already been encountered on p. 67, viz. equation (8), which was derived on the assumption that the total mass of fluid between two planes parallel to the wave-front of the propagated plane wave was constant at all times and for any part of the medium, provided it does not contain any source or sink. The equation is of general application to other transport phenomena, *e.g.* transfer of heat or electricity, and infers that the quantity concerned is uncreatable and indestructible.

If the fluid is confined within a tube whose cross-sectional area A is a function of x then equation (9) on p. 67 becomes

$$s=-\frac{1}{A}\cdot\frac{\partial}{\partial x}(A\eta) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A19.1)}$$

This result follows from consideration of the fluid contained between two planes, parallel to the wave-front, *i.e.* perpendicular to the tube axis, at x and $x+\delta x$. At a particular instant due to the passage of a sound wave these planes are displaced to $(x+\eta)$ and $\left(x+\delta x+\eta+\dfrac{\partial\eta}{\partial x}\delta x\right)$ respectively, and suppose V_u and V_d are the values of the enclosed volume in the corresponding undisplaced and displaced positions of the planes. Then it follows, if A_x represents the cross-sectional area at x, etc., that $V_u=\left[A_x+\dfrac{1}{2}\left(\dfrac{\partial A_x}{\partial x}\right)\delta x\right]\delta x$ and

$$V_d=\left[A_x+\left(\frac{\partial A_x}{\partial x}\right)\eta+\frac{1}{2}\left(\frac{\partial A_x}{\partial x}\right)\left(\delta x+\frac{\partial\eta}{\partial x}.\,\delta x\right)\right]\left(\delta x+\frac{\partial\eta}{\partial x}\delta x\right),$$

whence

$$V_d-V_u=\left[A_x.\frac{\partial\eta}{\partial x}+\left(\frac{\partial A_x}{\partial x}\right)\eta\right]\delta x=\frac{\partial}{\partial x}(A_x\eta)\delta x,$$

neglecting second order terms. Hence to required degree of approximation the condensation

$$s=\frac{V_u-V_d}{V_u}=\frac{-\frac{\partial}{\partial x}(A_x\eta)\delta x}{A_x\delta x}=-\frac{1}{A_x}\frac{\partial}{\partial x}(A_x\eta).$$

It is proposed now to obtain a more general expression for the equation of continuity and for this purpose the elementary box, Fig. A19.1, with centre Q, coordinates x, y, z, is supposed stationary while the fluid medium flows through it. Let the average components of velocity flow at P be u, v and w in the direction of the coordinate axes x, y and z respectively. Then if ρ is the density of the fluid at Q at *time* t, it follows that the rate of *increase* of the fluid mass within the cube due to the x component of velocity is given by

$$\left\{\left(\rho u-\frac{\partial(\rho u)}{\partial x}\cdot\frac{\delta x}{2}\right)-\left(\rho u+\frac{\partial(\rho u)}{\partial x}\cdot\frac{\delta x}{2}\right)\right\}\delta y\delta z$$

i.e.
$$-\frac{\partial(\rho u)}{\partial x}\cdot\delta x\delta y\delta z.$$

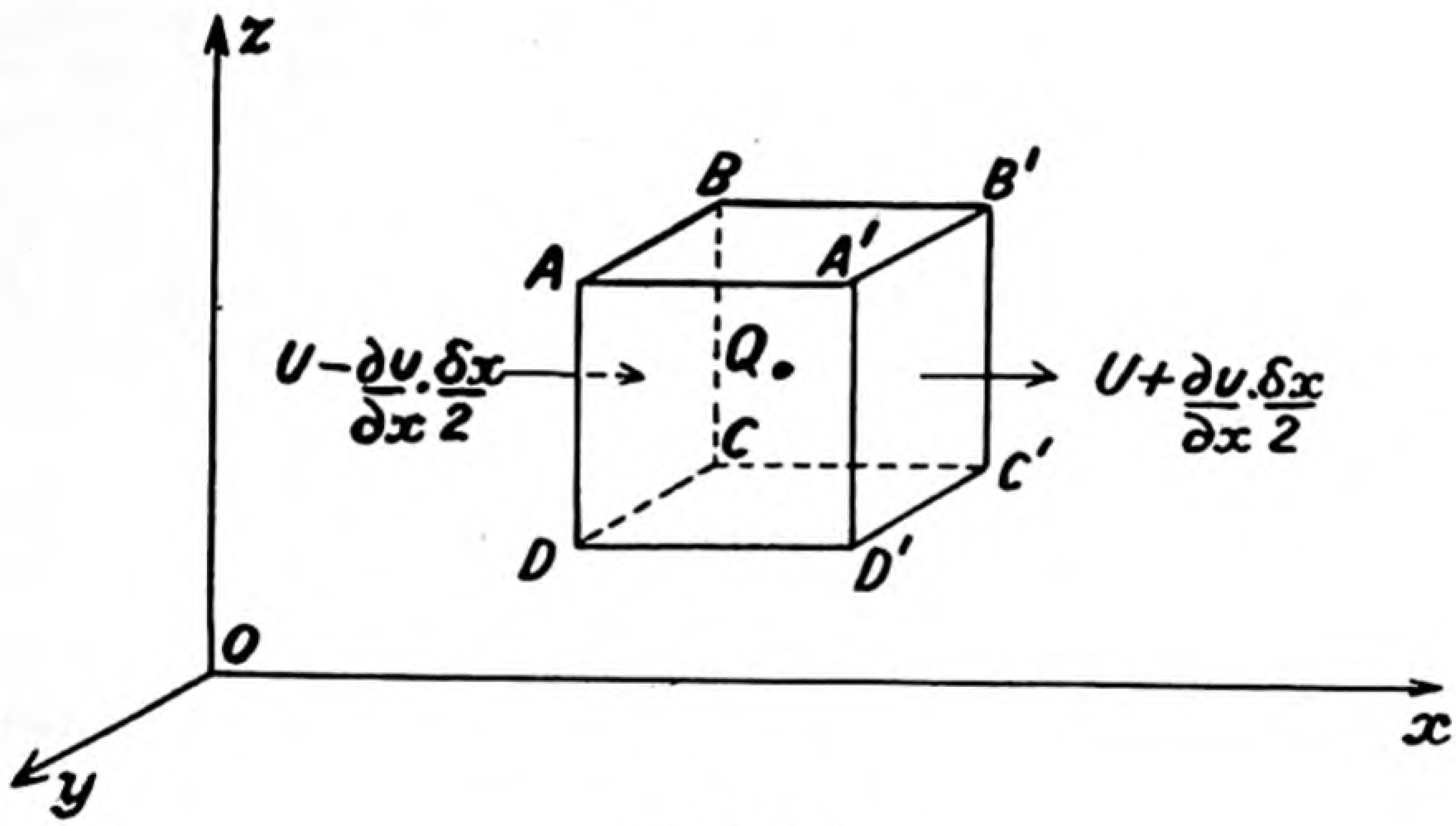


Fig. A19.1.

∴ the total rate of increase of mass within the cube due to components u, v and w is

$$-\left(\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right)\delta x\delta y\delta z \quad . \quad . \quad (A19.2)$$

But the rate of increase will also be given by

$$\frac{\partial\rho}{\partial t}\cdot\delta x\delta y\delta z \quad . \quad . \quad . \quad . \quad . \quad (A19.3)$$

Equating these expressions

$$\frac{\partial\rho}{\partial t}+\left\{\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right\}=0 \quad . \quad . \quad (A19.4)$$

which holds for *any* deformable continuous medium. It should be noted that for an incompressible fluid $\frac{\partial\rho}{\partial t}=0$, and furthermore, if this

incompressible fluid is homogeneous everywhere ρ is constant, and equation (A19.4) reduces to the form $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$.

Now $\frac{\partial(\rho u)}{\partial x}=\rho\frac{\partial u}{\partial x}+u\frac{\partial \rho}{\partial x}$ which simplifies to $\rho\frac{\partial u}{\partial x}$ if $u\frac{\partial \rho}{\partial x}<<\rho\frac{\partial u}{\partial x}$, *i.e.* if $\frac{1}{\rho}\cdot\frac{\partial \rho}{\partial x}<<\frac{1}{u}\cdot\frac{\partial u}{\partial x}$.

Since, for the most intense waves the ear can tolerate, the fluctuations in ρ about ρ_0 are only about 1 in 10^3, it follows (see Fig. A19.2) that the above condition is fulfilled for sound waves in air.

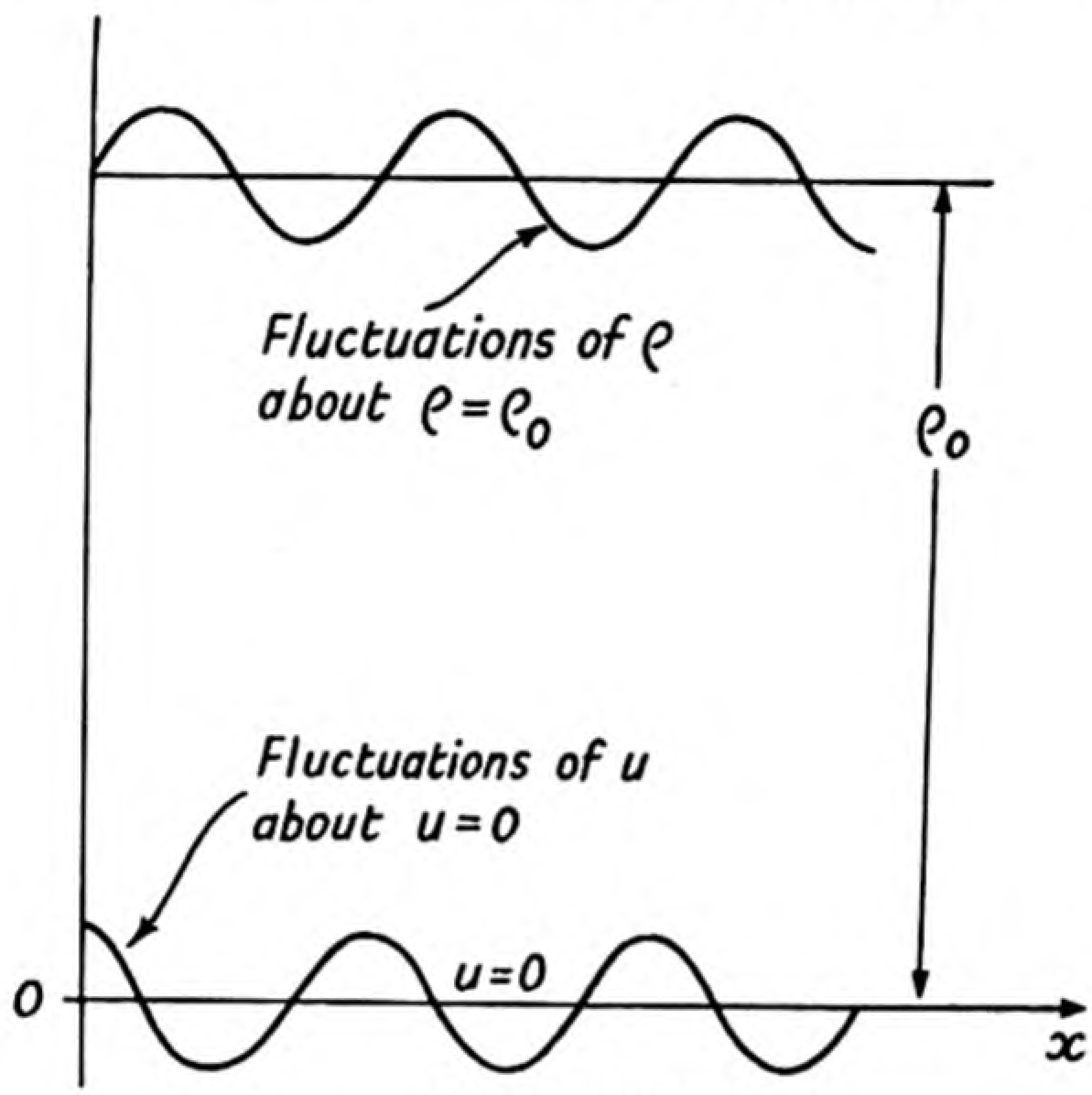


Fig. A19.2.

Hence equation (A19.4) may now be written as

$$\frac{\partial \rho}{\partial t}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0 \qquad \text{(A19.5)}$$

Alternatively, since

$\rho=\rho_0(1+s)$ and $\frac{\partial \rho}{\partial t}=\rho_0\frac{\partial s}{\partial t}$ and therefore $\rho_0\frac{ds}{\partial t}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$,

or to a sufficient approximation as $\rho_0\to\rho$ to within 1 in 1000,

$$\frac{ds}{\partial t}+\frac{\partial u}{\partial x}+\frac{dv}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad \text{(A19.6)}$$

In the case of an *incompressible* fluid this equation becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

The general equation of heat conduction, etc.

The mass flow of equation (A19.6) is now replaced by q_x, q_y and q_z which are respectively the heat flows per unit area per second in the x, y and z directions, and the amended equation is

$$\frac{\partial(q_x)}{\partial x}+\frac{\partial(q_y)}{\partial y}+\frac{\partial(q_z)}{\partial z}+\rho c\frac{\partial \theta}{\partial t}=0$$

where θ is temperature, ρ is density and c the specific heat of the medium, but $q_x=-K\frac{\partial\theta}{\partial x}$, $q_y=-K\frac{\partial\theta}{\partial y}$, $q_z=-K\frac{\partial\theta}{\partial z}$, where K is the coefficient of thermal conductivity;

$$\therefore \frac{K}{\rho c}\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right)=\frac{\partial\theta}{\partial t} \quad . \quad . \quad . \quad . \quad \text{(A19.7)}$$

$\frac{K}{\rho c}$ is generally written as h^2, where h is known as the coefficient of thermal diffusivity.

When steady heat flow conditions have been established, $\frac{\partial\theta}{\partial t}=0$ and equation becomes $\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right)=0$, or, as it is generally written, $\nabla^2\theta=0$.

The above equation is attributed to Laplace and is probably the most important partial differential equation of applied mathematics, and the function θ may represent other quantities than that of the temperature of bodies in thermal equilibrium. For instance, θ may stand for the electrostatic potential in free space, the gravitational potential in regions outside that occupied by attracting matter, or the *velocity potential* in the irrotational motion of a homogeneous liquid. Hence the problem of determining the distribution of electric intensity within a region under defined boundary conditions becomes synonymous mathematically with the task of ascertaining the temperature in a solid whose boundary surface is maintained at a constant temperature. The reader is directed to mathematical treatises for the solution of Laplace's equation.*

Again attention should also be directed to the universal nature of equation (A19.7), viz. $\nabla^2\theta=\frac{1}{h^2}\cdot\frac{\partial\theta}{\partial t}$, which deals with heat flow in the variable state.

Analogous problems are, (*a*) the diffusion of a substance in solution which is governed by the equation $\nabla^2 n=\frac{1}{K}\frac{\partial n}{\partial t}$ where K is the coefficient of diffusion (in cm.^{2} per sec.) of a homogeneous medium and n represents the concentration of the solute at any point (in gms. per c.c.), and (*b*) the distribution of potential (V) (in volts) along an insulated electrical transmission line or cable. In the latter case the equation simplifies to $\frac{\partial^2 V}{\partial x^2}=RC\frac{\partial V}{\partial t}$, where R is the resistance in ohms per unit length and C is the capacitance in farads per unit length. This particular analogy between heat flow and electrical currents is being utilised to solve heat conduction problems by means of electric-circuit theory, the the heat resistivity corresponding to the inverse of the thermal conductivity and the heat capacitance to the thermal capacity (ρc, where c is the specific heat of the material). It may also be mentioned that the

* *Modern Analysis*, Whittaker and Watson (Cambridge University Press); *Vector Methods*, Rutherford (Oliver and Boyd); *The Mathematics of Physics and Chemistry*, Margenau and Murphy (Van Nostrand), etc.

temperatures at different depths within the body of a semi-infinite solid whose plane face is subjected to a periodic fluctuating temperature show amplitude values and phase lags identical with those of the electric currents within the body of a conductor carrying an alternating current. This latter effect is the well known "skin-effect" phenomenon of electrodynamics and is governed by the equation, for the case of semi-infinite solid, of $\nabla^2 i=\mu\sigma\frac{\partial i}{\partial t}$ where μ and σ are respectively the permeability and electrical conductivity of the medium.

When the dimensions of constrictions, valves, etc., of a tube in which a sound wave is being propagated are longer than the wave-length of the sound, then the system becomes analogous to the electrical transmission line, otherwise the resemblance is to an electrical circuit of "lumped" constants.

Equations of motion. These are deduced by equating the resultant force acting on a small element of fluid to the product of the mass of the element and its acceleration in the direction of the force.

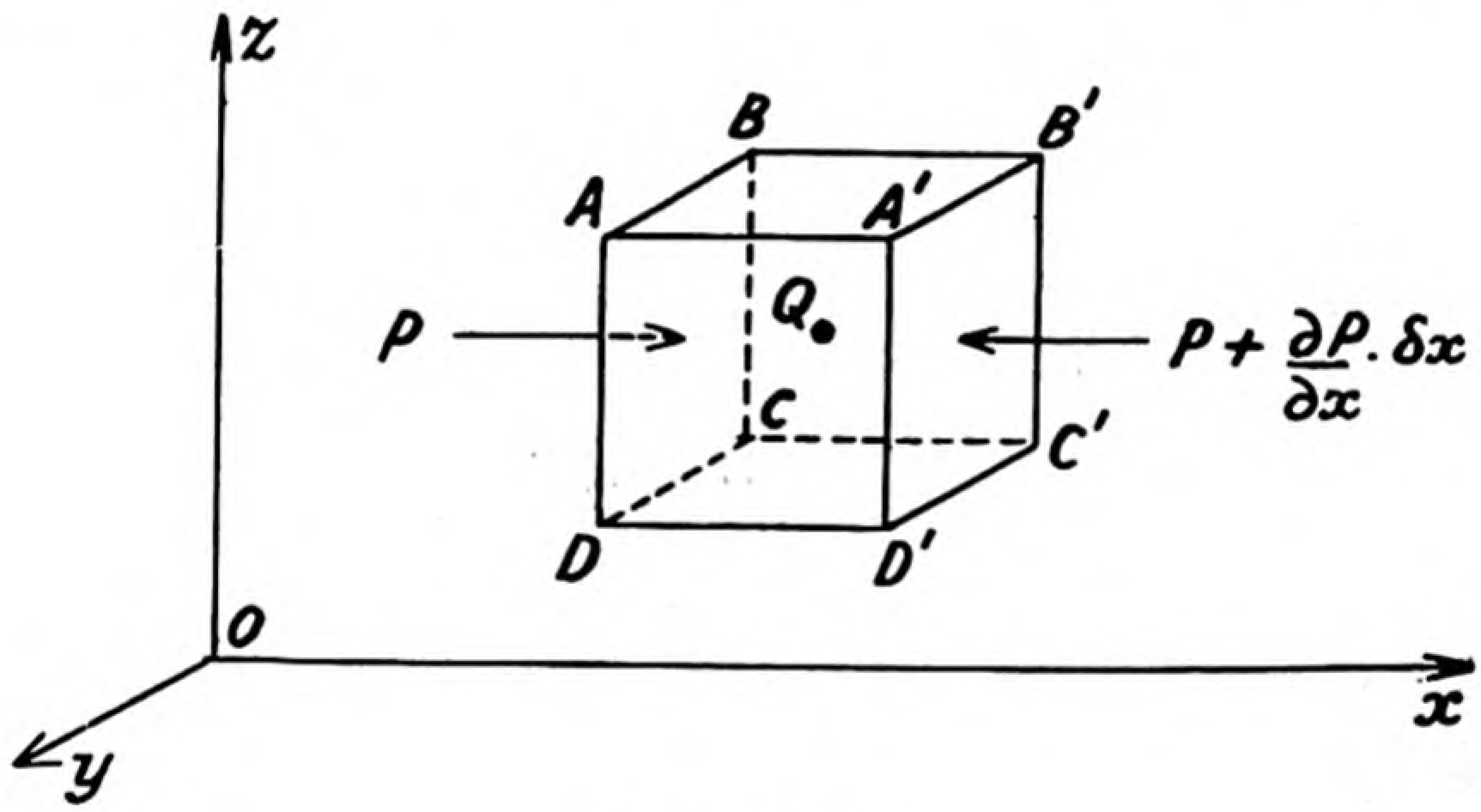


Fig. A19.3.

Consider the medium confined within the cube (Fig. A19.3), at the centres of the yz faces of which mean excess pressures p and $p+\frac{\partial p}{\partial x}.\delta x$ are exerted.

The net instantaneous force acting on the element at Q in the x direction will be

$$\left[p-\left(p+\frac{\partial p}{\partial x}.\delta x\right)\right]\delta y\delta z=-\frac{\partial p}{\partial x}.\delta x\delta y\delta z.$$

It follows that the equation of motion is therefore given by

$$-\frac{\partial p}{\partial x}.\delta x\delta y\delta z=\rho.\delta x\delta y\delta z.\frac{du}{dt} \quad . \quad . \quad . \quad . \quad \text{(A19.8)}$$

The acceleration $\frac{du}{dt}$ is expressed as the *total* derivative in consequence of the lapse of time and of change of position of the fluid element, but

$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$, and these last three terms are negligible if the particle-velocity is small compared with the wave-velocity.

Hence equation (A19.8) may be written

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\cdot\frac{\partial p}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A19.9)}$$

Now equation (10), p. 67, gives $Ks=\Delta p$, where Δp is the *excess* pressure, K is the bulk modulus, and s the condensation. It follows that here $\frac{\partial p}{\partial x}=K\frac{\partial s}{\partial x}$ and so equation (A19.9) becomes

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}=-\frac{K}{\rho}\cdot\frac{\partial s}{\partial x}\\ \text{Similarly}\quad &\frac{\partial v}{\partial t}=-\frac{K}{\rho}\cdot\frac{\partial s}{\partial y}\\ \text{and}\quad &\frac{\partial w}{\partial t}=-\frac{K}{\rho}\cdot\frac{\partial s}{\partial z}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A19.10)}$$

Differentiating the equations of motion with respect to x, y and z respectively and adding

$$\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 v}{\partial y\partial t}+\frac{\partial^2 w}{\partial z\partial t}=-\frac{K}{\rho}\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}+\frac{\partial^2 s}{\partial z^2}\right) \qquad \text{(A19.11)}$$

By differentiating the equation of continuity with respect to t

$$\frac{\partial^2 s}{\partial t^2}+\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 v}{\partial y\partial t}+\frac{\partial^2 w}{\partial z\partial t}=0 \quad . \quad . \quad . \quad \text{(A19.12)}$$

Combining equations (A19.11) and (A19.12)

$$\frac{\partial^2 s}{\partial t^2}=\frac{K}{\rho}\left(\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}+\frac{\partial^2 s}{\partial z^2}\right) \quad . \quad . \quad . \quad \text{(A19.13)}$$

which is the wave equation expressed in terms of the condensation and it gives the velocity of propagation $c=\sqrt{\frac{K}{\rho}}$.

In the case of a fluid in which so-called *body forces* cannot be neglected, the above equations (A19.10) of motion require modification. An example of such a force is that due to the gravitational attraction of the earth. In the general case the body force is assumed to have components X, Y, Z respectively in the directions of the coordinate axes. Since the magnitude of the forces exerted will depend on the *volume or mass* of the element, the amended equations will be

$$\left.\begin{aligned} \rho\frac{\partial u}{\partial t}&=X\rho-K\frac{\partial s}{\partial x}\\ \rho\frac{\partial v}{\partial t}&=Y\rho-K\frac{\partial s}{\partial y}\\ \rho\frac{\partial w}{\partial t}&=Z\rho-K\frac{\partial s}{\partial z}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A19.14)}$$

The above equations are usually known as Euler's hydrodynamical equations.

APPENDIX 20

Elasticity and Viscosity

Bulk-modulus of elasticity of a solid. Consider a unit cube of an isotropic solid subjected to a uniform pressure P on each of its faces, and let the contraction of each side be denoted by a. It follows that the *decrease* in volume is given by $1^3-(1-a)^3=3a$ approximately, and hence the bulk-modulus $k=\frac{P}{3a}$.

Shear-modulus of elasticity of a solid—*the equivalence of a shear to a combined extension and compression.*

Reference to Fig. 5.6 on p. 61 suggests that the application of a shearing force to the upper face ab of a solid cube, the lower face cd being rigidly fixed to a plane, results in one diagonal of the face $abcd$ (or a parallel plane) increasing and the other decreasing in length. In Fig. A20.1 the angle of shear, θ, is shown greatly exaggerated and

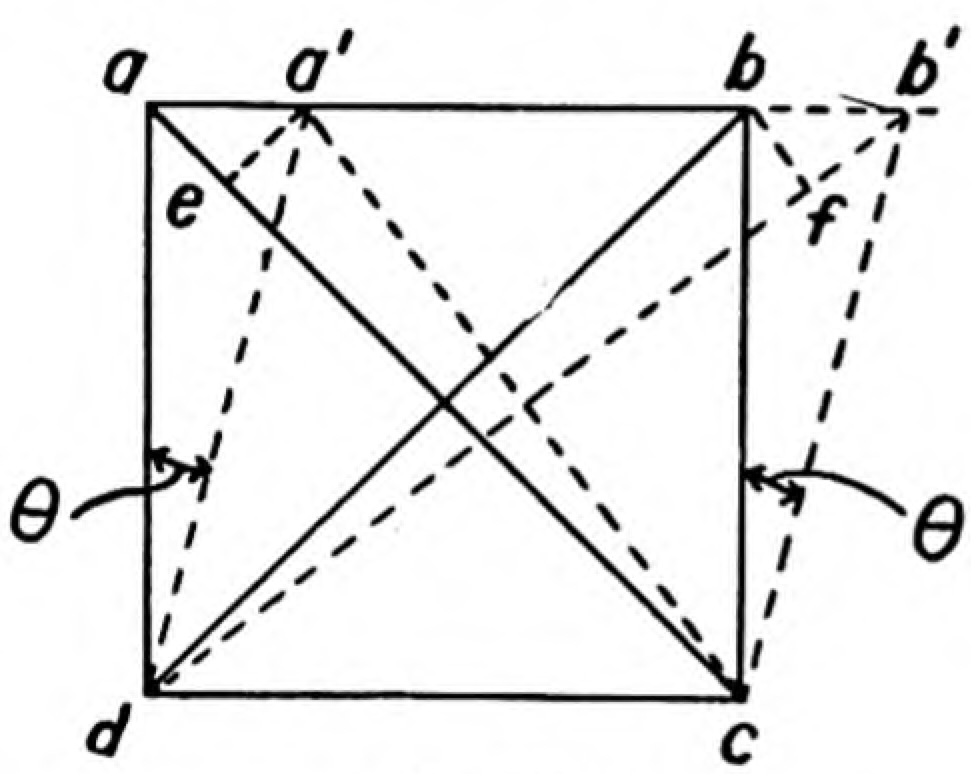


Fig. A20.1.

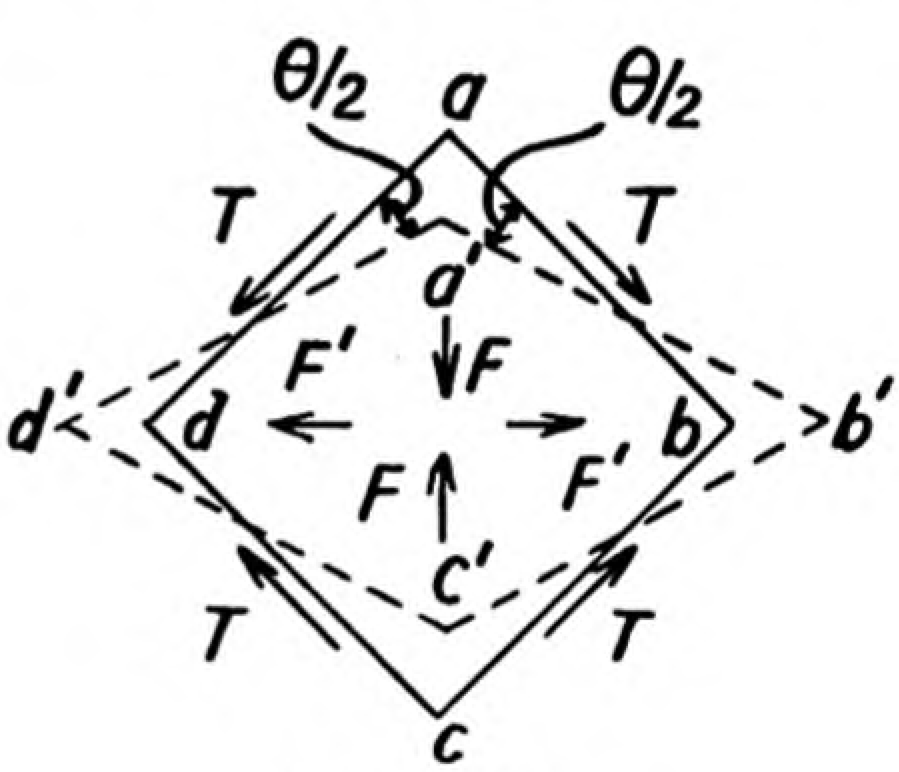


Fig. A20.2.

effectively the angles aea' and bfb' may be each taken as 90°, the remaining angles of triangles $aa'e$ and $bb'f$ being 45° each. It follows from the geometry of the figure that $bb'=\sqrt{2}fb'$ and $db=df=\sqrt{2}$, since the cube has sides of unit length. Hence the *extension* strain along the diagonal $db=\frac{fb'}{db}=\frac{bb'}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}=\frac{\theta}{2}$, for bc is of unit length.

In a similar manner the *compression* strain along the diagonal $ac=\frac{\theta}{2}$, *i.e.* a simple shear of magnitude θ is equivalent to an extension and a compression at right angles to each other, each being of magnitude $\frac{\theta}{2}$. The converse statement is also true, viz. that a simultaneous extension and compression, at right angles to each other and of equal magnitude, are equivalent to a simple shear. The effect of applying tensile and compressive *forces*, F' and F respectively, along the direction of the diagonals so that $aa'=bb'$, is seen in Fig. A20.2 which on inspection is seen to conform with Fig. A20.1 if $d'c'$ is rotated through an angle $\frac{\theta}{2}$

and d' and c' made to coincide with d and c respectively. The magnitude of F may be obtained by resolving the stresses T and T' of Fig. 5.6*a* along the diagonal direction *ac*. Since *abcd* is a cube it follows, by taking moments about any corner *a*, say, that $T=T'$, and hence $F=2T\cos 45°=\sqrt{2}T$. This compressive *force* F acts over the area *db*, i.e. $\sqrt{2}$ units, and hence the compressive *stress*$=\frac{\sqrt{2}T}{\sqrt{2}}=T$. Similarly the tensile *stress* corresponding to the force F' is also equal to T. Thus the normal tensile and compressive stresses which require to be applied to produce the same shear have precisely the values of the tangential stresses they replace.

Reference Books on Elasticity

Prescott, J.: *Applied Elasticity*. Longmans Green, 1924.

Love, A.: *Mathematical Theory of Elasticity*. Cambridge Univ. Press.

Newman, F. H., and Searle, V. H. L.: *Properties of Matter*. Arnold, 1948.

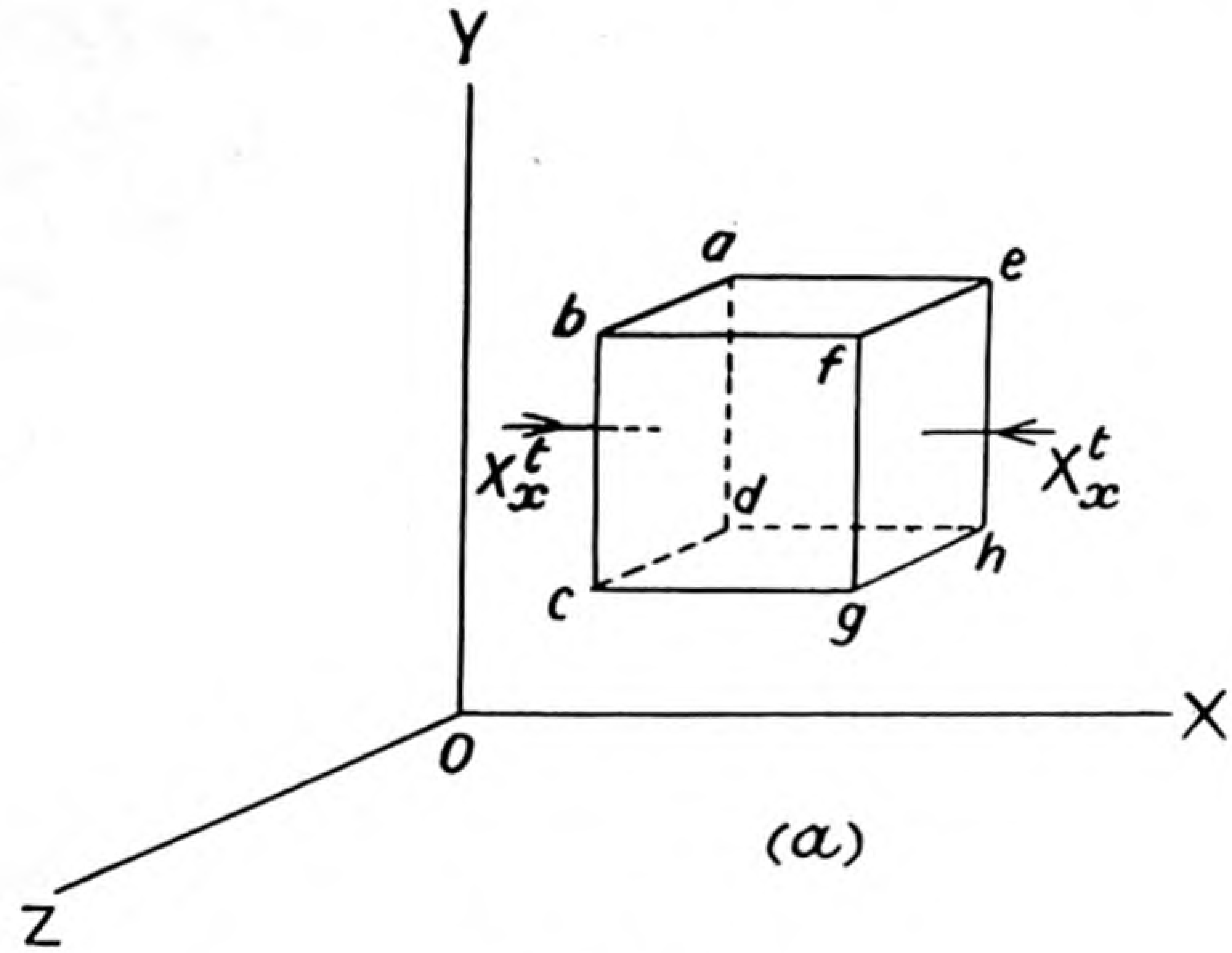


Fig. A20.3*a*.

Propagation of a longitudinal plane wave in an isotropic solid medium of infinite extent

Consider a portion of the infinite medium in the form of a unit cube of the material (Fig. A20.3*a*) whose faces *abcd* and *efgh* are acted upon by total applied normal stresses X_x^t to produce relative length changes in the direction of the three coordinate axes of $e_x=e_x^t$, $e_y=0$, and $e_z=0$ respectively. Such a state of affairs would result from the passage of a plane wave in the direction of the x-axis, the wave-front being parallel to the plane YOZ. Let G be the effective modulus of elasticity for the type of deformation considered, and let it be further assumed that G may be expressed in terms of the bulk modulus K and the shear modulus (*i.e.* rigidity modulus) n.

It will be shown now that the state of strain envisaged in the problem may be regarded as being attained by the compounding of three separate stages as follows:

(1) A *uniform compression* (Fig. A20.3*b*) in which $X_x = Y_y = Z_z$, the suffixes indicating that the stresses act normally to the respective faces, such that the corresponding strains are equal, viz. $e_x = e_y = e_z = \frac{e'_x}{3}$, where e'_x is the total contraction per unit length in the x direction.

Hence by definition

$$3\left(K.\frac{e'_x}{3}\right) = Ke'_x = X_x = Y_y = Z_z \quad . \quad . \quad . \quad . \quad (A20.1)$$

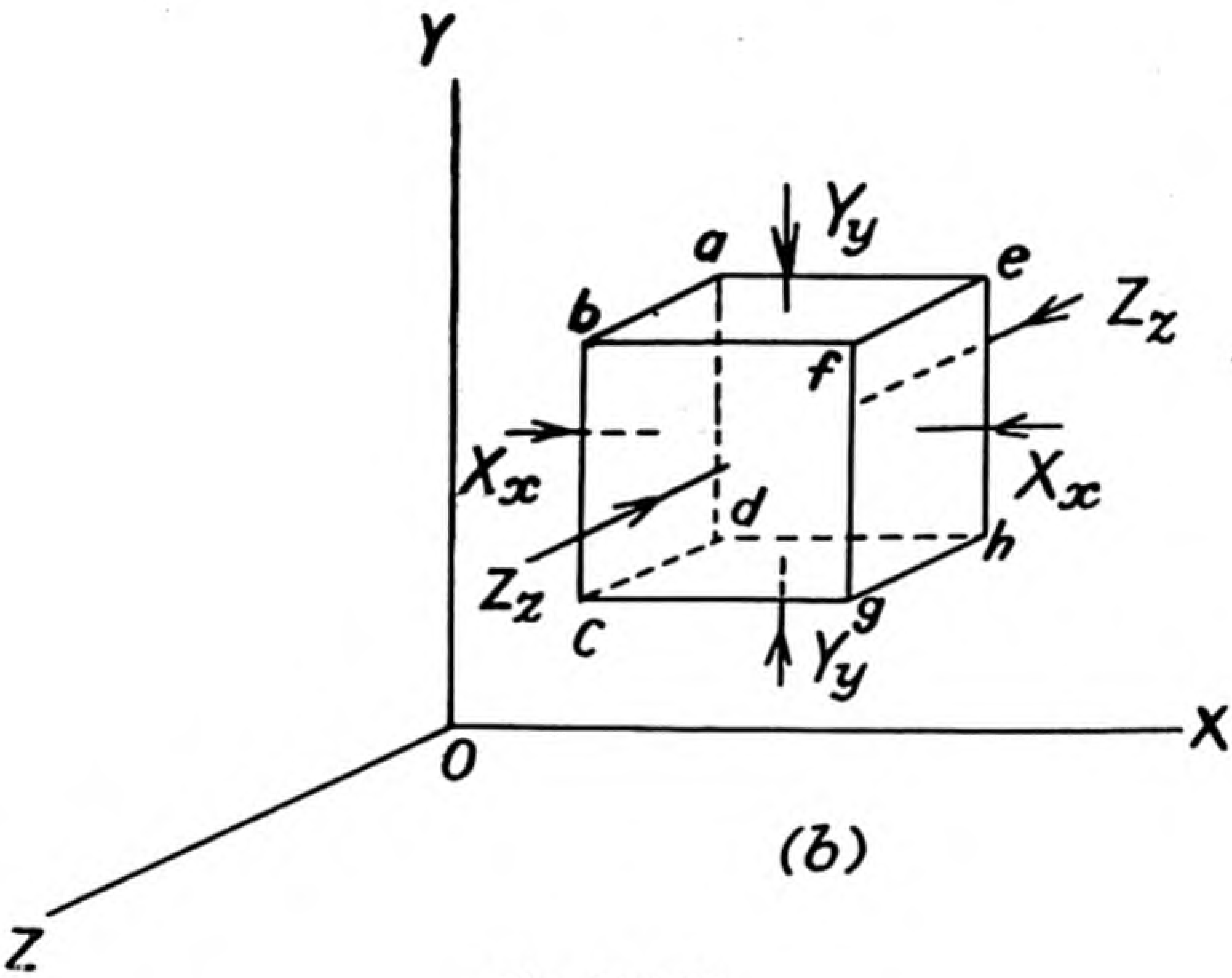


Fig. A20.3*b*.

(2) A compressive stress X_x applied in the x direction and a tensile stress Y_y in the y direction such that $X_x = -Y_y$, and correspondingly $e_y = -e_x = -\frac{e'_x}{3}$. Such a combination of stresses has been shown to be equivalent to a simple shear of magnitude $\frac{2e'_x}{3}$.

Hence from definition of rigidity modulus,

$$X_x = -Y_y = n.\tfrac{2}{3}e'_x \quad . \quad . \quad . \quad . \quad . \quad (A20.2)$$

(3) Similarly a compressive stress X_x and a tensile stress Z_z combine to give a simple shear of magnitude $\frac{2}{3}e'_x$ in the xz or a parallel plane. Evidently $X_x = -Z_z$, $e_z = -e_x = -\frac{e'_x}{3}$, and it follows that

$$X_x = -Z_z = n.\tfrac{2}{3}e'_x \quad . \quad . \quad . \quad . \quad . \quad (A20.3)$$

From (A20.1), (A20.2), (A20.3) the total applied stress is

$$X'_x = 3X_x = e'_x(K + \tfrac{4}{3}n) \quad . \quad . \quad . \quad . \quad (A20.4)$$

Consequently the effective elastic modulus (G) for the plane wave type of deformation considered is

$$G=\frac{X_x^t}{e_x^t}=\left(K+\frac{4}{3}n\right).$$

The corresponding wave equation is therefore

$$\left(K+\frac{4}{3}n\right)\frac{\partial^2\xi}{\partial x^2}=\rho\frac{\partial^2\xi}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad (A20.5)$$

where ξ is the displacement in the x direction and ρ is the density of the medium. The appropriate velocity of propagation $c=\sqrt{\left(\frac{K+\frac{4}{3}n}{\rho}\right)}$.

In a fluid medium it is the *time* rate of change of strain which is fundamental, and the classical coefficient of viscosity (μ) corresponding to the rigidity coefficient (n) is defined by considering the relative motion of two parallel planes in the fluid. If these planes are distant δy apart and move with relative velocity δu then the relevant tangential stress (X_y) in the x direction is given by $X_y=-\mu\frac{du}{dy}$, the negative sign being taken since the force exerted *by* the medium is always opposite to that of the velocity of the surface with respect to the medium. X_y signifies that the force acts in the x direction on unit area, having its normal in the y direction.

By considering the forces on an elementary cube it is shown in treatises on hydrodynamics* that

$$\frac{\partial u}{\partial t}-\frac{4}{3}\frac{\mu}{\rho}\cdot\frac{\partial^2 u}{\partial x^2}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \quad . \quad . \quad . \quad . \quad (A20.6)$$

which is an extension of the equation developed on p. 406 and is a simplification of the generalised Navier-Stokes equations. The attenuation of a plane progressive wave through the fluid medium may be deduced from the above equation as $\frac{8\mu\pi^2 f^2}{3\rho c^3}$ nepers per cm., *i.e.* it is proportional to the square of the frequency (f).

This predicted frequency variation of attenuation is verified by experiment, but in the case of gases and liquids of low shear viscosity the discrepancy between the calculated and *absolute* values is large. In order to explain this discrepancy it is suggested that there are two viscosity coefficients, so that by analogy with equation (A20.5) the *effective* viscosity coefficient is taken as $(\chi+\frac{4}{3}\mu)$ where χ is termed the *compressional* (or volume) *viscosity coefficient*. The energy dissipation due to this coefficient arises from what is known as thermal relaxation, *i.e.* the incomplete establishment of thermal equilibrium in a system. The absorption of energy may often be a hundred times or more than that due to the effect of heat conduction or of the ordinary viscosity coefficient. For polyatomic gases under standard conditions the *time lag* in the energy transfer between the *internal* vibrational degrees of freedom and the *external* translational degrees of freedom

* e.g. *Fundamentals of Hydro- and Aero-Mechanics*, Prandtl, L., and Tietjens, O. G. (McGraw-Hill, 1934).

may be of the order of 10^{-5} sec., *i.e.* anomalous energy absorption will be evident at frequencies in the neighbourhood of 100 Kc. per sec. The observed velocity dispersion arises from the fact that the ratio of the principal specific heats of a gas is a measure of the completeness of the excitation of the internal degrees of freedom.

APPENDIX 21

Velocity Potential and Spherical Wave Theory

Velocity potential

The theory of plane sound waves has so far been developed in terms of the displacement (η) of the particles of the transmitting medium [or of the *excess* pressure (p) or condensation (s) of a gas], but in many acoustic problems it is most convenient to make use of a function known as the *velocity potential*. As its name suggests, the velocity of the medium at any point is deducible from the function in the same manner that the force in a conservative dynamical system is obtained from the potential energy, but it must be emphasised that the velocity potential itself is *not* a potential energy. In employing the function, denoted by ϕ, it is assumed that the motion of the fluid is *irrotational*, *i.e.* that the flow is streamline so that eddies are absent and the *curl* of the velocity, viz. $\int u\,ds$ round any closed contour, is zero.

The fluid-velocity at any point of the medium will be normal to that member of the system of surfaces, ϕ=constant, which passes through the point, *i.e.* the *stream lines* will cut orthogonally the surfaces ϕ=constant. If u, v and w are the rectangular components of the fluid-velocity, then it follows from the definition of ϕ that $u=-\dfrac{\partial\phi}{\partial x}$, $v=-\dfrac{\partial\phi}{\partial y}$ and $w=-\dfrac{\partial\phi}{\partial z}$. The use of the negative sign in these expressions emphasises the analogy with gravitational and electrostatic potentials, but it should be noted that some writers, *e.g.* Rayleigh, Stewart and Lindsay, use the positive sign.

A *simple source* is *conceived* to be a *point* in the medium from which is emitted a volume of $4\pi m$ per sec., where m in hydrodynamical analysis is usually defined as the *strength* of the source (cf. p. 418). This volume is often correspondingly termed the output but in acoustics it is the quantity which writers usually call the strength. Correspondingly a *sink* refers to a point where *inward* radial *flow* takes place continuously, *i.e.* it is a *negative source.*

If a source and a sink of equal strength m are situated at an infinitesimal distance δl apart the combination is termed a *doublet*, or double source. Let δl tend to zero when m tends to infinity in such a manner that the product $m\delta l$, which defines the *strength* μ of doublet, still remains finite. The positive direction of the axis of the doublet is given by that of the line δl drawn in the sense from $-S$ to $+S$, referring to the sink and source respectively. This idea of a doublet is analogous to the magnetic or electrostatic dipole, and it may be applied

to the solution of such problems as that of a vibrating membrane open to transmitting media on both sides. Although modern practice is for sound generators and detecting devices to be "one-sided," yet the conception of the doublet is extremely useful in such problems as finding the magnitude of the scattering effect of obstacles in the path of sound waves.

Calculation of velocity potential (ϕ). *At a point due to a simple source of strength m.* Dealing with the *two-dimensional* case the flow of fluid per sec. across a closed curve surrounding the source is given by $2\pi m$. But at radius r the radial fluid-velocity is $-\frac{\partial\phi}{\partial r}$, hence the rate of flow across a circle of circumference r is

$$-2\pi r\frac{\partial\phi}{\partial r}=2\pi m \quad . \quad . \quad . \quad . \quad . \quad . \quad (A21.1)$$

The integration of this equation gives

$$\phi=\text{constant}-m \log r \quad . \quad . \quad . \quad . \quad (A21.2)$$

The curves of equal ϕ are therefore concentric circles.

In the three-dimensional case it is evident that equation (A21.1) becomes

$$-4\pi r^2\frac{\partial\phi}{\partial r}=4\pi m \quad . \quad (A21.3)$$

whence on integration (assuming $\phi=0$ at infinity)

$$\phi=\frac{m}{r} \quad . \quad . \quad (A21.4)$$

The velocity potential distant r from a *sink* of strength m would be

$$\phi=-\frac{m}{r} \quad . \quad . \quad (A21.5)$$

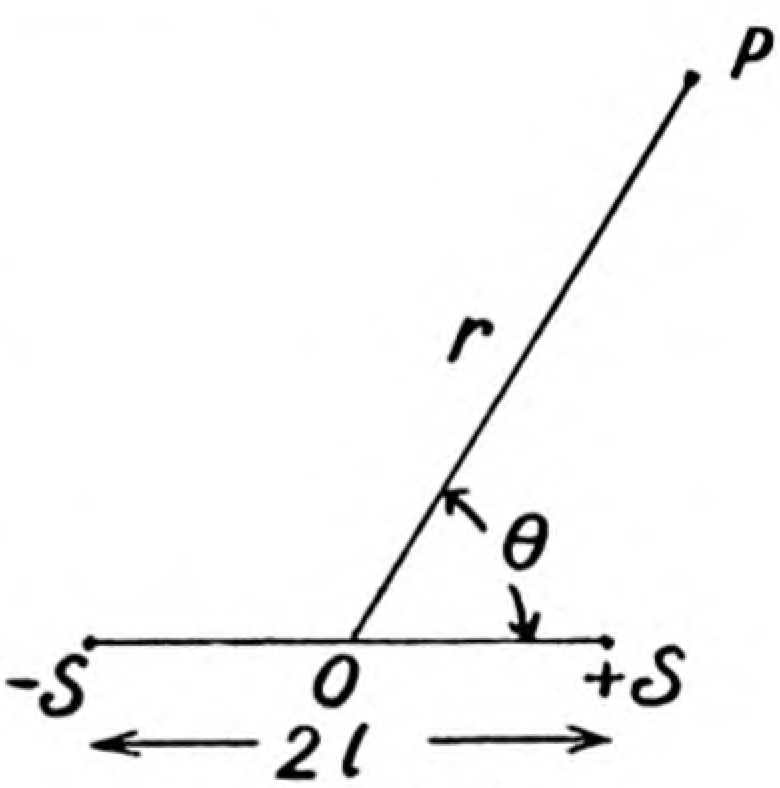


Fig. A21.1.

Potential at a point due to a doublet, i.e. a so-called double source.

In Fig. A21.1 the axis of the doublet of strength m is in the sense $-S$ to $+S$ and its length is assumed to be $2l$.

Let $-SP=r_2$ and $+SP=r_1$, then

$$\phi_p=\phi_{+s}+\phi_{-s}=\frac{m}{r_1}-\frac{m}{r_2}=m\frac{(r_2-r_1)}{r_1r_2}$$

$$=\frac{2ml\cos\theta}{r^2}=\frac{\mu\cos\theta}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (A21.6)$$

since l is very small and $r_1r_2\rightarrow r^2$. μ is known as the strength of the doublet. An alternative expression for ϕ_p is easily seen to be

$$\phi_P=\frac{-m}{r}+\frac{m}{r}+\frac{\partial}{\partial l}\left(\frac{m}{r}\right)\delta l=\mu\frac{\partial}{\partial l}\left(\frac{1}{r}\right) \quad . \quad . \quad . \quad (A21.7)$$

where δl is infinitesimal distance between $-S$ and $+S$.

The velocity component of the fluid at P is easily deduced, along the radius vector as

$$-\frac{\partial\phi}{\partial r}=\frac{2\mu\cos\theta}{r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (A21.8)$$

and perpendicular to the radius vector (in the sense of θ increasing) as

$$-\frac{1}{r}\frac{\partial\phi}{\partial\theta}=\frac{\mu\sin\theta}{r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A21.9)$$

The close similarity of the above with corresponding formulae in electrostatics and magnetism should be noted.

In the case of sound generators it is evident that on the average a single radiator is better than a doublet of equivalent strength, for in the latter case the radiation is zero when $\theta=\frac{\pi}{2}$.

The wave equation expressed in terms of the velocity potential. The importance of the velocity potential in acoustics is that when this function has been determined all the other important characteristics of a wave are easily deduced from it. Since $\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$, etc., and by definition the particle-velocity $u=-\frac{\partial\phi}{\partial x}$, it should be easily verified that

$$s=\frac{1}{c^2}\frac{\partial\phi}{\partial t},\quad \frac{\partial\eta}{\partial t}=u=-\frac{\partial\phi}{\partial x} \quad \text{and} \quad p=Ks=\rho\frac{\partial\phi}{\partial t}.$$

Consider now the case of a plane wave travelling in the direction of the x-axis. Then employing the usual notation it follows that the continuity equation $\frac{\partial s}{\partial t}+\frac{\partial u}{\partial x}=0$ becomes

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}=0 \quad \text{or} \quad \frac{\partial^2\phi}{\partial t^2}=c^2\frac{\partial^2\phi}{\partial x^2} \quad . \quad . \quad (A21.10)$$

The most general form of this d'Alembertian wave equation will evidently be

$$\frac{\partial^2\phi}{\partial t^2}=c^2\nabla^2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (A21.11)$$

Spherical Waves

In the case of spherical waves diverging from a point then $r^2=x^2+y^2+z^2$, where r is the distance from the central source expressed in terms of the x, y, z coordinates of any point on the wave-front. It follows that

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial r}\cdot\frac{\partial r}{\partial x}=\frac{x}{r}\frac{\partial\phi}{\partial r}, \text{ etc.} \quad . \quad . \quad . \quad . \quad (A21.12)$$

Hence
$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)$$
$$=\frac{1}{r}\frac{\partial\phi}{\partial r}+x\frac{\partial}{\partial r}\left[\frac{1}{r}\cdot\frac{\partial\phi}{\partial r}\right]\frac{\partial r}{\partial x}$$
$$=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{x^2}{r}\left[\frac{1}{r}\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r^2}\frac{\partial\phi}{\partial r}\right] \quad . \quad . \quad . \quad (A21.13)$$

and similarly for $\frac{\partial^2\phi}{\partial y^2}$ and $\frac{\partial^2\phi}{\partial z^2}$.

Therefore
$$\nabla^2\phi=\frac{3}{r}\frac{\partial\phi}{\partial r}+\frac{(x^2+y^2+z^2)}{r}\left[\frac{1}{r}\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r^2}\frac{\partial\phi}{\partial r}\right]$$
$$=\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial r^2}$$
$$=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) \qquad \text{(A21.14)}$$

The wave equation thus becomes

$$c^2.\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)=\frac{\partial^2\phi}{\partial t^2} \qquad \text{(A21.15)}$$

or the symmetrical form (r being independent of t)

$$c^2\frac{\partial^2(r\phi)}{\partial r^2}=\frac{\partial^2(r\phi)}{\partial t^2} \qquad \text{(A21.16)}$$

The reader should verify this result by an alternative proof, viz. equating the rate of growth of the mass of fluid in a spherical shell to the difference of fluid flux across the inner and outer surfaces of the shell.

The solution of this wave equation is

$$\phi=\frac{1}{r}F_1(r-ct)+\frac{1}{r}F_2(r+ct) \qquad \text{(A21.17)}$$

the first term on the right-hand side representing a divergent wave, for if t increases by τ, say, then r must increase by $c\tau$ for its value to remain unchanged, and the second a convergent wave focusing on the centre of the disturbance. It should be noted, however, that this wave motion does not extend inwards to $r=0$, where it would assume an infinite value (cf. circular waves on a membrane) and this restriction implies that the dimensions of the source itself must be considered. Because of the factor $\frac{1}{r}$ the amplitude of spherical disturbances, unlike plane waves, falls off with distance from the source even with zero attenuation, they do not change shape, however, as do cylindrical waves (the three-dimensional analogue of circular waves). In the case of a cylindrical generator it may be shown that an issuing pulse of sound has a sharp beginning but no ending, thus leaving a "wake" behind it, which mathematical analysis indicates is the general feature of waves in an *even* number of dimensions.

Considering only the diverging wave and assuming the disturbance is S.H.M., *i.e.*

$$\phi=\frac{1}{r}F_1(r-ct)=\frac{A}{r}\cos k(r-ct) \qquad \text{(A21.18)}$$

where $k=\frac{2\pi}{\lambda}$, then the particle-velocity

$$u=-\frac{\partial\phi}{\partial r}=\frac{Ak}{r}\sin k(r-ct)+\frac{A}{r^2}\cos k(r-ct) \qquad \text{(A21.19)}$$

The second term of the expression will decrease more rapidly than the first as the wave spreads out, so that the particle-velocity u will become $\propto\frac{1}{r}$, and the kinetic energy of the wave motion per unit volume will follow the inverse square law, viz. $\propto\frac{1}{r^2}$.

It follows that the expressions for the excess pressure and condensation are respectively,

$$p=\rho\frac{\partial\phi}{\partial t}=\frac{\rho Akc}{r}\sin k(r-ct) \quad . \quad . \quad . \quad (A21.20)$$

and
$$s=\frac{1}{c^2}\,\frac{\partial\phi}{\partial t}=\frac{Ak}{rc}\sin k(r-ct) \quad . \quad . \quad . \quad (A21.21)$$

In the case of the *converging* wave

$$u=-\frac{\partial\phi}{\partial(-r)}=-\frac{Ak}{r}\sin k(r+ct)-\frac{A}{r^2}\cos k(r+ct)=A'\sin[k(r+ct)+\theta]$$

where
$$\tan\theta=\frac{1}{rk},\ \text{and}\ s=-\frac{Ak}{rc}\sin k(r+ct) \quad . \quad (A21.22)$$

The reader should verify that for this wave the particle-velocity undergoes a total phase change of π with respect to the condensation in passing through the focus.

APPENDIX 22

The Theory of the Acoustic Horn

An acoustic horn is not an amplifying device for it does not contain a separate source of power, but it provides a means of more effectively matching the relatively high impedance of a small metal diaphragm to the low impedance of the outer air into which the sound is propagated. In effect the horn is an acoustic transformer and its performance will depend upon its shape, the best form being found to have an exponentially diverging cross-section.

In the simple theory of the horn developed below it is tacitly assumed that the fundamental acoustic equations for small displacements and small particle-velocities are applicable, and that the diameter of the horn at any point is small compared with the wave-length of the sound to be propagated. This last restriction implies that the phase is approximately uniform over any section perpendicular to the axis x of the horn (Fig. A22.1). Furthermore, it is assumed that there is no transverse particle displacement or, in other words, that the walls of the horn remain rigid.

By considering the equilibrium of the mass of the gas confined between two planes at x and $x+\delta x$ (Fig. A22.1) it follows that the continuity equation (p. 401) is valid, viz. $s=-\frac{1}{A}\frac{\partial}{\partial x}(A\eta)$ where A is the area of cross-section of the horn at x, η is the particle displacement

and s is the appropriate condensation. Hence the excess pressure

$$p=Ks=-\frac{c^2\rho}{A}\,\frac{\partial}{\partial x}(A\eta) \quad . \quad . \quad . \quad . \quad (A22.1)$$

where K is the bulk modulus and ρ the density of the medium and c is the velocity of propagation of sound.

From (A22.1) by differentiation it follows that

$$\frac{\partial^2 p}{\partial t^2}=-\frac{c^2\rho}{A}\frac{\partial}{\partial x}\left(A\frac{\partial^2\eta}{\partial t^2}\right) \quad . \quad . \quad . \quad . \quad (A22.2)$$

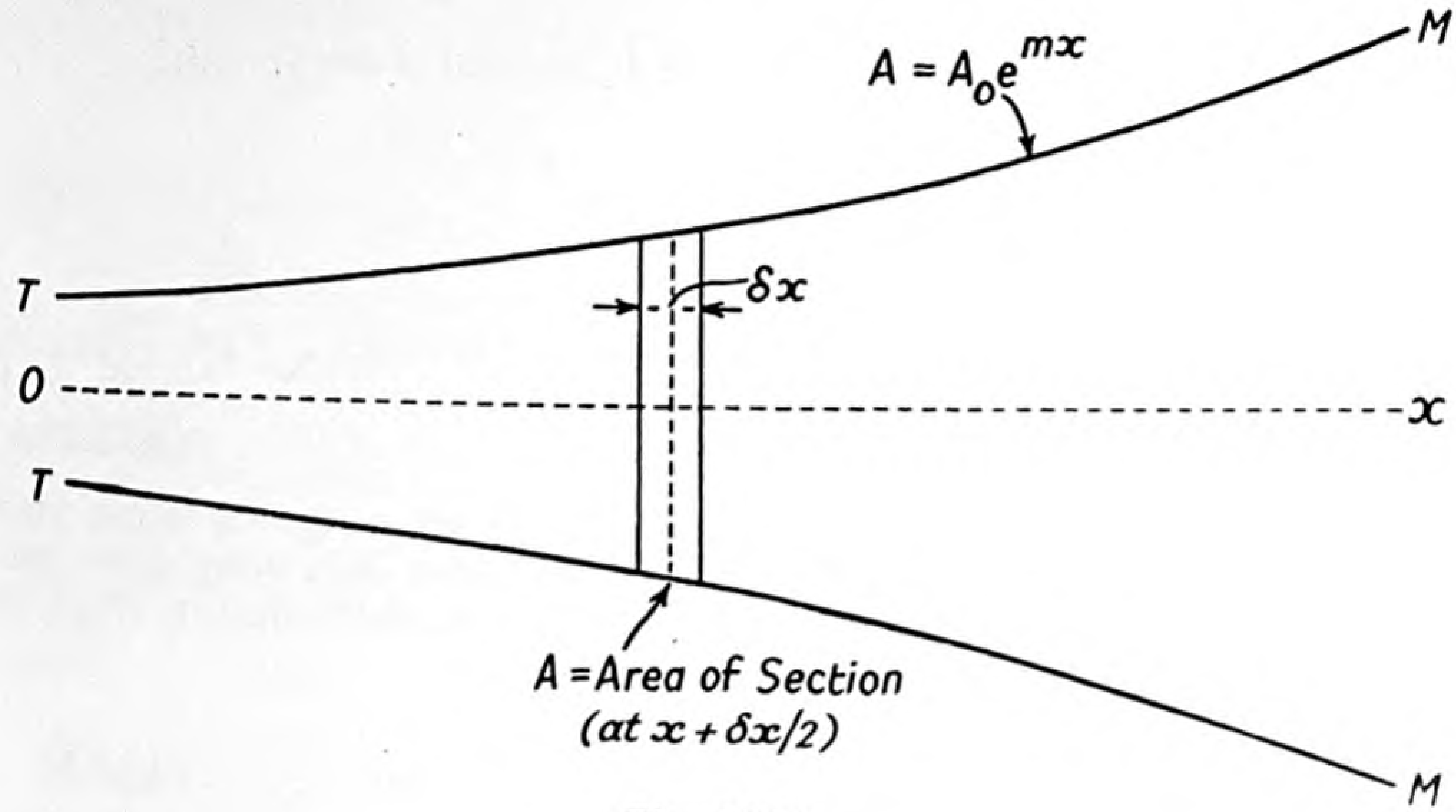


Fig. A22.1.

But the equation of motion of the small prism of gas considered will be given by combining equation (A19.9) with the relation $u=\frac{\partial\eta}{\partial t}$, i.e.

$$\rho\frac{\partial^2\eta}{\partial t^2}=-\frac{\partial p}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (A22.3)$$

Combining equations (A22.2) and (A22.3)

$$\frac{\partial^2 p}{\partial t^2}=\frac{c^2\rho}{A}\frac{\partial}{\partial x}\left(\frac{A}{\rho}\frac{\partial p}{\partial x}\right)=c^2\left[\frac{\partial^2 p}{\partial x^2}+\frac{1}{A}\frac{\partial p}{\partial x}\frac{\partial A}{\partial x}\right]$$

or

$$\frac{\partial^2 p}{\partial t^2}=c^2\frac{\partial^2 p}{\partial x^2}+c^2\frac{\partial p}{\partial x}\frac{\partial}{\partial x}(\log A) \quad . \quad . \quad . \quad . \quad . \quad . \quad (A22.4)$$

which is the *fundamental horn equation* expressed in terms of the excess pressure.

In the case of the exponential horn the area of section varies according to the law $A=A_0e^{mx}$, where A_0 and m are constants.

Equation (A22.4) now becomes

$$\frac{\partial^2 p}{\partial x^2}+m\frac{\partial p}{\partial x}=\frac{1}{c^2}\cdot\frac{\partial^2 p}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (A22.5)$$

If the pressure varies harmonically with a frequency $\frac{\omega}{2\pi}=\frac{kc}{2\pi}$, i.e.

$p=p_x e^{j\omega t}$ for any given value of x, then on substitution equation (A22.5) becomes

$$\frac{\partial^2 p_x}{\partial x^2}+m\frac{\partial p_x}{\partial x}+k^2 p_x=0 \quad . \quad . \quad . \quad . \quad \text{(A22.6)}$$

The solution of this equation is

$$p_x=e^{-\frac{mx}{2}}\left\{A_1 \cos\sqrt{k^2-\frac{m^2}{4}}\,.x+A_2 \sin\sqrt{k^2-\frac{m^2}{4}}\,.x\right\}$$

By a suitable change in the origin of x this may be put into the form

$$p_x=Ae^{-\frac{mx}{2}}\cos\sqrt{k^2-\frac{m^2}{4}}\,.x \quad [A_1, A_2 \text{ and } A \text{ are all real}]$$

$$p=Ae^{-\frac{mx}{2}}\cos\left(\sqrt{k^2-\frac{m^2}{4}}\,.x\right)\cos\omega t$$

$$=Be^{-\frac{mx}{2}}\left[\cos\left(\omega t-\sqrt{k^2-\frac{m^2}{4}}\,.x\right)+\cos\left(\omega t+\sqrt{k^2-\frac{m^2}{4}}\,.x\right)\right]$$

$$\text{where } A=2B \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A22.7)}$$

The first term of this expression represents an outgoing wave and the second term an incoming wave. In the case of a long horn the reflection from the end may be neglected so that the solution (A22.7) may be reduced to

$$p=Be^{-\frac{mx}{2}}\cos\left(\omega t-\sqrt{k^2-\frac{m^2}{4}}\,.x\right) \quad . \quad . \quad \text{(A22.8)}$$

The attenuation of the wave indicated by the exponential factor is due to a lateral release of pressure brought about by the widening section, since the theory has tacitly assumed no viscous dissipation of energy.

The phase-velocity defined by $\dfrac{\omega}{\sqrt{k^2-\dfrac{m^2}{4}}}$ is seen to vary with the frequency $\left(\dfrac{\omega}{2\pi}\right)$ and thus the exponential horn shows sound *dispersion.* Furthermore, the *velocity approaches* an *infinite value* for $k^2=\dfrac{m^2}{4}$, *i.e.* for a frequency $f_0=\dfrac{\omega_0}{2\pi}=\dfrac{kc}{2\pi}=\dfrac{mc}{4\pi}$. This critical frequency is known as the *cut-off frequency* (cf. analogous electromagnetic wave-guide) and implies that the horn will almost cease to radiate *below* this frequency, although the amount of energy transmitted rises very rapidly above this frequency. In other words, the exponential horn behaves as a high pass acoustical filter, the particle-velocity and pressure becoming 90° out of phase below the critical frequency, as do the current and voltage in the corresponding electrical filter. The specific acoustic impedance of the horn at the throat (considering only the outgoing waves) will be given by $z_0=R-jX=|Z|_e{}^{-j\phi}$, where

$z_0=\left(\frac{p}{u}\right)_{x=0}$. Hence it follows from (A22.8) that

$$p=\text{real part } Be^{-\frac{mx}{2}+j(\omega t-\delta kx)} \text{ where } B \text{ is a constant,}$$

$$\delta=\sqrt{1-\frac{m^2}{4k^2}}=\sqrt{1-\frac{f_0^2}{f^2}}=\sqrt{1-\left(\frac{\lambda}{\lambda_0}\right)^2}$$

and f_0 and λ_0 refer to the frequency and wave-length respectively at the cut-off frequency.

Now since $\frac{\partial u}{\partial t}=\frac{-1}{\rho}\frac{\partial p}{\partial x}$ it is easily deduced that for $f>f_0$

$$u=-\frac{p}{j\omega\rho}\left[-\frac{m}{2}-\frac{j\omega\delta}{c}\right]=\frac{p}{\rho c}\left[\delta-j\left(\frac{f_0}{f}\right)\right].$$

Consequently

$$Z=\left(\frac{p}{u}\right)=\frac{\rho c}{\left[\delta-j\left(\frac{f_0}{f}\right)\right]}=\rho c\sqrt{1-\left(\frac{f_0}{f}\right)^2}+j.\rho c\left(\frac{f_0}{f}\right),$$

and hence

$$R=\rho c\sqrt{\left[1-\left(\frac{f_0}{f}\right)^2\right]}, \quad \text{(A22.9)}$$

$$X=-\rho c\left(\frac{f_0}{f}\right) \quad \text{(A22.10)}$$

and

$$\phi=\sin^{-1}\left(\frac{f_0}{f}\right). \quad \text{(A22.11)}$$

The power (W) radiated from the throat, if u_0 denotes the appropriate velocity amplitude, will be expressed by $W=\frac{1}{2}u_0^2.A_0\rho c\sqrt{1-\left(\frac{f_0}{f}\right)^2}$. An inspection of the above results reveals that *above* the *cut-off* frequency the impedance, $Z=\sqrt{R^2+X^2}$, has a *constant magnitude* ρc, below this critical frequency, however, the horn acts as a pure reactance to the sound source and the outgoing energy becomes reflected back to the source. This absence of energy transference is further emphasised by noting that for $f<f_0$ the characteristic wave impedance becomes purely imaginary, and in the ideal horn there is attenuation but no phase shift. These conditions receive important applications in wave-guide attenuators.

APPENDIX 23

Radiation from an Acoustic Source Vibrating Harmonically

(a) Single source. The case dealt with mathematically is the *simple* source consisting of a sphere vibrating *harmonically*, the radius r_0 of the sphere being small compared with the wave-length λ of the emitted sound. When the wave-length is *much* greater than the linear dimensions of the radiating body then, provided all the elements of the latter are vibrating in phase, the radiation will be almost independent of the shape of the radiator. Hence, for example, the open end of any

wood-wind instrument may be regarded as a simple source of strength equal to $B\overline{U}$, where B is the cross-sectional area and $\overline{U}$ is the average particle-velocity at the open end. Now the rate of working per unit area at a distance r from a *simple* source, *i.e.* the *power* transmitted through unit area, will be given by (p. 414)

$$\frac{dW}{dt}=up=A\left[\frac{k}{r}\sin k(r-ct)+\frac{1}{r^2}\cos k(r-ct)\right]\left[\frac{\rho Akc}{r}\sin k(r-ct)\right]$$

$$=\frac{\rho c.A^2k^2}{2r^2}\left[1-\cos\ 2k(r-ct)+\frac{1}{kr}\sin 2k(r-ct)\right] \quad . \quad \text{(A23.1)}$$

The average power $$\frac{\overline{dW}}{dt}=\frac{\rho cA^2k^2}{2r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A23.2)}$$

Now the strength S of the source is defined by its *maximum* rate of emission of fluid, *i.e.* by $S=4\pi r_0^2u_0$, where u_0 is the *amplitude* of the particle-velocity at the surface ($r=r_0$) of the simple source.

It follows from equation (A21.19) (p. 413) that

$$4\pi r_0^2u=4\pi r_0^2.A\left[\frac{k}{r_0}\sin k(r-ct)+\frac{1}{r_0^2}\cos k(r-ct)\right]$$

$$=4\pi A[kr_0\sin k(r-ct)+\cos k(r-ct)] \quad . \quad . \quad \text{(A23.3)}$$

Hence, as r is made very small

$$S=4\pi r_0^2u_0=4\pi A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A23.4)}$$

The average power of the source per unit area at radius r, combining equations (A23.2) and (A23.4), is therefore

$$\frac{\overline{dW}}{dt}=\frac{\rho cA^2k^2}{2r^2}=\frac{\rho cS^2k^2}{32\pi^2r^2} \quad . \quad . \quad . \quad . \quad \text{(A23.5)}$$

and the average *total* power of the source

$$=4\pi r^2.\frac{\overline{dW}}{dt}=\frac{\rho cS^2k^2}{8\pi} \quad . \quad . \quad . \quad . \quad . \quad \text{(A23.6)}$$

(b) Doublet (or double source). As already stated (p. 410), an acoustic doublet consists of two small simple sources each of the same numerical strength (S') but opposite in phase and separated by a vanishingly small distance δl. Hence, defining the effective strength by $S=S'\delta l$, the velocity potential at a point distant r from the *doublet* (p. 412) will be given by

$$\phi=A'\delta l\frac{\partial}{\partial l}\left[\frac{\cos k(r-ct)}{r}\right]=-A\frac{\partial}{\partial r}\left[\frac{\cos k(r-ct)}{r}\right]\cos\theta \quad \text{(A23.7)}$$

since $$\frac{\partial r}{\partial l}=-\cos\theta,\ A=\frac{S}{4\pi}\text{ and }A'=\frac{S'}{4\pi}.$$

It follows that

$$\phi=Ak\cos\theta\left[\frac{\cos k(r-ct)}{kr^2}+\frac{\sin k(r-ct)}{r}\right] \quad . \quad \text{(A23.8)}$$

and hence the particle-velocity (in a radial direction) is given by

$$u_r=-\frac{\partial\phi}{\partial r}=-A\cos\theta\left[\left(\frac{2}{r^3}-\frac{k^2}{r}\right)\cos k(r-ct)+\frac{2k}{r^2}\sin k(r-ct)\right] \quad \text{(A23.9)}$$

and the corresponding excess pressure p_r at r is

$$p_r=\rho\frac{\partial\phi}{\partial t}=\frac{\rho ckA\cos\theta}{r}\left[\frac{\sin k(r-ct)}{r}-k\cos k(r-ct)\right] \quad \text{(A23.10)}$$

The energy radiated per unit area per second in an outward direction defined by θ will therefore be

$$u_rp_r=\frac{\rho ckA^2\cos^2\theta}{r}\left[k\cos k(r-ct)-\frac{\sin k(r-ct)}{r}\right]$$
$$\times\left[\left(\frac{2}{r^3}-\frac{k^2}{r}\right)\cos k(r-ct)+\frac{2k}{r^2}\sin k(r-ct)\right]$$

Hence average power per unit area

$$=\frac{\rho ck^4A^2\cos^2\theta}{2r^2}=\frac{\rho ck^4S^2\cos^2\theta}{32\pi^2r^2} \quad \text{(A23.11)}$$

It follows from this equation (as pointed out on p. 394) that the doublet becomes more efficient as a radiator as the wave-length $\left(\propto\frac{1}{k}\right)$ decreases. It should be noted that the *transverse* component of the particle-velocity in the field of a doublet will be given by

$$u_t=-\frac{1}{r}\frac{\partial\phi}{\partial\theta}.$$

APPENDIX 24

Larmor's Derivation of Radiation Pressure Formula

The pressure exerted by light waves incident upon a surface was predicted by Maxwell in connection with his electromagnetic theory of light, and it was subsequently verified by Lebedew, Nichols and Hull, and others. The phenomenon is a universal property of wave motion and the following proof due to Larmor has general application. He supposes a reflecting surface to be moved normally to meet the incident waves and as a result the reflected train of waves (cf. Doppler effect) will be of shorter wave-length (or greater frequency) than the incident train and may be shown to possess a greater *activity* (*i.e.* energy outgoing per second). This energy increase is presumed, therefore, to be due to the moving mirror doing work against the pressure of the waves.

Let $\eta=a\sin k(x+vt)$ represent the displacement due to the incident wave train travelling with velocity v in the *negative* direction along the x-axis, a, k and t having their usual significance. If the reflector is moving from the origin with velocity u, then the position at time t will be given by $x=ut$ and the reflected train may be represented by $\eta_r=a_r\sin k_r(x-vt)$ where the suffix r refers to the values of the various quantities for the reflected train. The resultant displacement at the

surface of the reflector must be zero at all times, *e.g.* at $x=ut$, then $\eta+\eta_r=0$.

Hence it follows that

$$a\sin k(x+vt)+a_r\sin k_r(x-vt)=0$$

but $x=ut$;

$$\therefore\ a\sin kt(u+v)+a_r\sin k_rt(u-v)=0 \quad . \quad . \quad \text{(A24.1)}$$

This relation must hold for all values of t and so

$$a=+a_r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A24.2)}$$

$$k(v+u)=k_r(v-u) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A24.3)}$$

Now $k=\frac{2\pi}{\lambda}$, $k_r=\frac{2\pi}{\lambda_r}$ where λ_r is the wave-length of the reflected train, hence from (A24.3) it follows that

$$\frac{v-u}{v+u}=\frac{\lambda_r}{\lambda}=\frac{n}{n_r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A24.4)}$$

where n, n_r refer to the frequency of the incident and reflected waves respectively.

At *constant amplitude* the energy E per unit volume of a wave train is proportional to the (frequency)^2, i.e. $E=\beta n^2$ and $E_r=\beta n_r^2$ where E_r refers to the reflected train and β is a constant. Hence from (A24.4)

$$\frac{E}{E_r}=\frac{n^2}{n_r^2}=\left(\frac{v-u}{v+u}\right)^2 \text{ or } E=E_r\left(\frac{v-u}{v+u}\right)^2 \quad . \quad . \quad \text{(A24.5)}$$

and

$$E+E_r=2E\frac{(v^2+u^2)}{(v-u)^2} \quad . \quad . \quad . \quad . \quad \text{(A24.6)}$$

Let the energies conveyed *per second* per unit cross-sectional area, *i.e.* the *activities*, of the incident and reflected trains be denoted by e and e_r respectively, then $e=(v+u)E$ and $e_r=(v-u)E_r$. It follows, therefore, from (A24.5) that

$$\frac{e_r}{e}=\left(\frac{v+u}{v-u}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A24.7)}$$

But the work done per second against the radiation pressure p_s is

$$p_su=e_r-e=e\left(\frac{2u}{v-u}\right)$$

or

$$p_s=\frac{2e}{(v-u)}=2E\left(\frac{v+u}{v-u}\right) \quad . \quad . \quad . \quad . \quad \text{(A24.8)}$$

Alternatively the pressure may be expressed in terms of the total energy density in front of the reflector, viz. $E+E_r$, and from (A24.6) and (A24.8) it follows that

$$p_s=\frac{v^2-u^2}{v^2+u^2}(E+E_r) \quad . \quad . \quad . \quad . \quad . \quad \text{(A24.9)}$$

For a reflector at rest, *i.e.* as $\frac{u}{v}\rightarrow 0$, then (A24.8) becomes

$$p_s=2E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A24.10)}$$

Hence in this case the radiation pressure is equal to the *total* energy density of the radiation in front of the reflector. In the case of radiation enclosed within a chamber the pressure exerted on the walls, being

equally distributed in all directions, will be equal to *one-third* of the energy density.

If p_{max} and u_{max} refer respectively to the maximum excess pressure and particle-velocity in a sound wave, then $p_{max}=\rho c u_{max}$, where ρ is the density of the medium and c is the velocity of sound. Also $E=\frac{1}{2}\frac{p_{max}^2}{\rho c^2}=\frac{1}{2}\rho u_{max}^2$; therefore $p_s=2E=\rho u_{max}^2=\frac{p_{max}}{c}.u_{max}$, or

$$\frac{p_s}{p_{max}}=\frac{u_{max}}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A24.11)}$$

For a perfectly *absorbing* surface the reflected wave is absent and A24.10 becomes $P_s=E$, where E corresponds to E_d (p. 43). In a more rigorous analysis than that of Larmor, Rayleigh* deduced the expression $P_s=(\gamma+1)E$ for the mean pressure acting on a disc set normal to the axis of a tube and situated at a node. Whereas Rayleigh had considered the sound wave confined within a tube, Brillouin† and Langevin later assumed the sender and receiver to form a standing wave system in an infinite medium and deduced $P_s=2E$. The two formulae for P_s will thus only give numerical agreement under isothermal conditions, *i.e.* when Boyle's Law ($\gamma=1$) holds. Radiation pressure measurements are usually performed in the "free" gas conditions of Brillouin when they are independent of the equation of state of the gas.

APPENDIX 25

Transverse Sound Waves Propagated in Rigid Tubes

In the measurement of acoustic impedance by the standing-wave method Hartig and Swanson found difficulty, after a certain impressed frequency, in obtaining uniformly spaced nodes and antinodes. This effect, which had previously been reported by Davis and Evans in England, was attributed to the existence of "transverse" sound waves analogous to the electromagnetic waves of a dielectric wave-guide. In the electrical case electromagnetic energy cannot be propagated down the guide in the zero mode (0, 0), excepting that the pipe contains an insulated *conductor* within its interior. This limitation does not exist in the acoustic problem and L. Brillouin was the first to suggest the employment of a balanced sound generator, *i.e.* an "acoustic dipole," to eliminate the 0, 0 mode. It should be noted, however, that the theoretical study of the normal modes of vibration of the air within a cylindrical tube had been partly considered earlier by Duhamel (in 1849) and dealt with more fully by Rayleigh about 25 years later.

The experimental verification of the theory has only been possible within the last decade, and Hartig and Swanson in their work first considered the internal normal transverse oscillations which are not dependent on z, which is measured along the tube axis. At the walls

* *Phil. Mag.*, **3**, pp. 338-346, 1902.

† *Rev. d'Acoustique*, **5**, p. 99, 1936.

of the tube, viz. $r=R$, the particle-velocity $\frac{\partial p}{\partial r}=0$ and the solution of the general wave equation for cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}+\frac{\partial^2 p}{\partial z^2}=\frac{1}{c^2}\frac{\partial p^2}{\partial t^2} \quad . \quad . \quad (A25.1)$$

leads to a distribution of *excess* pressure analogous to the voltage distribution of the so-called *H type* electromagnetic wave in a cylindrical guide.

It is hoped that these "transverse" waves will provide an accurate means of investigating the acoustic impedance of materials for angles of incidence other than normal.

APPENDIX 26

Reciprocity Theorem

The reciprocity theorem is yet another law commonly met in electrical network theory which has important counterparts in acoustical and mechanical systems. The theorem is applicable to an electrical system which is devoid of internal sources of energy, its circuit elements must also show complete reversibility, and strict linearity must exist between the currents and E.M.F.s. If these conditions are satisfied by the network, then *it may be shown that an E.M.F. placed in branch A produces the same current in branch B as it would in branch A, if it were placed in branch B.* The theorem may be more generally expressed as

$$\sum_{m=1}^{m=n} E_m' \, I_m'' = \sum_{m=1}^{m=n} E_m'' \, I_m'$$

where the two sets of E.M.F.s (E) are all harmonic, of the same frequency, and act in n points of an invariable network to produce the respective current distributions, i.e. E_m' and E_m'' give rise to I_m' and I_m'' respectively.

In a mechanical system comprising moving parts having mass, compliance and mechanical resistance, reciprocity exists if the system is invariable and conforms to the same restrictions as for the electrical case, stated symbolically

$$F_1'x_1''+F_2'x_2''+ \; . \; . \; . \; . \; . \; . = F_1''x_1'+F_2''x_2'+ . \; . \; . \; .$$

where forces F_n' and F_n'' give rise to displacements x_n' and x_n'' respectively. It is often more convenient when using electrical circuit analogies of the mechanical system to replace the displacements by velocities.

Electrical-mechanical system. An extension of the reciprocity theorem is its application to interconnected systems such as the mechanical and electrical types discussed above. The validity of this extended use is dependent upon the linearity and reversibility of the *interconnections*, *e.g.* the presence of ferro-magnetic materials would

require the use of small amplitudes only (see Ballantine, *Proc. I.R.E.*, 1929).

If $E_1'I_1'$ represent the E.M.F. and current, and F' and v' the force and velocity of one set of variables in the electrical and mechanical meshes respectively, then if E'', I'', F'', and v'' refer to a second set

$$\sum_{m=1}^{m=n}(E_m'\, I_m'' + F_m'\, v_m'') = \sum_{m=1}^{m=n}(E_m''\, I_m' + F_m''\, v_m') \qquad \text{(A26.1)}$$

The above equation is correct if E and I are expressed in E.M. (or E.S.) C.G.S. units and F and v are in dynes and in cm. per sec. respectively. The simplest example is when all the forces except two are zero and the equation of reciprocity becomes $E'I''=F''v'$; which implies that if a unit E.M.F. E' impressed in the electrical system produces a certain velocity v' in the mechanical system, then a unit mechanical force F'' acting in the mechanical system will create a current I'' in the electrical network which is numerically the same as the velocity v'. An example of such an electrical-mechanical system would be a coil moving in the field of a permanent magnet, as in the construction of a loud-speaker.

Acoustical reciprocity theorem. This theorem may be simply stated thus: if irrotational harmonic vibrations of small amplitude are propagated in a medium of uniform density and an excess pressure p' produces a particle-velocity v', and similarly p'' produces v'', then the *surface* integral $\iint(v''p' - v'p'')_n ds = 0$, where the integration is taken over the boundaries of the volume.

The irrotational waves are assumed to be of a simple compressional type and the theorem is applicable to gas, liquid, or solid media, if the appropriate elastic moduli are employed.

In the special case of a "free" acoustical field in which only two excess pressures p' and p'' are involved it follows that $p'v''=p''v'$, where v' and v'' are the particle-velocities concerned. The theorem is also applicable to an acoustical system of lumped constants analogous to the electrical network and is subject to similar restrictions; in this case the volume current (X) replaces the particle-velocity of the "free" field.

In acoustical-mechanical systems the interconnections are rather akin to those between an electrical network and its radiating aerial, for the acoustical forces are essentially mechanical and the velocities are physical quantities; both mechanical and acoustical systems also employ the same system of units for forces and velocities. It should be noted, however, that it is the *force per unit area* of the mechanical system which is equivalent to the acoustical pressure. An important application of the reciprocity theorem to acoustical-mechanical systems is its use in deducing the transmitting properties of a sound transducer from its receiving properties and vice-versa. This result will be made evident by considering the system of two diaphragms D_1 and D_2, shown in Fig. A26.1. By the reciprocity theorem, if p', the force *per unit area* impressed on the diaphragm D_1 at A_1, produces a velocity v' at A_2, then this will be equal to the velocity v'' which would exist at A_1 due to an equal force p'', impressed on unit area at A_2.

Interconnected electrical, mechanical and acoustical systems. Since reciprocal relations have been shown to exist in electrical-mechanical and in mechanical-acoustical systems, it follows that they should hold in a composite system involving all three systems interconnected in the above order. A simple example of such a composite system is shown in Fig. A26.2 where a generator of E.M.F. E' energises the loud-speaker $L.S.$ to produce a volume current X' at the point A in

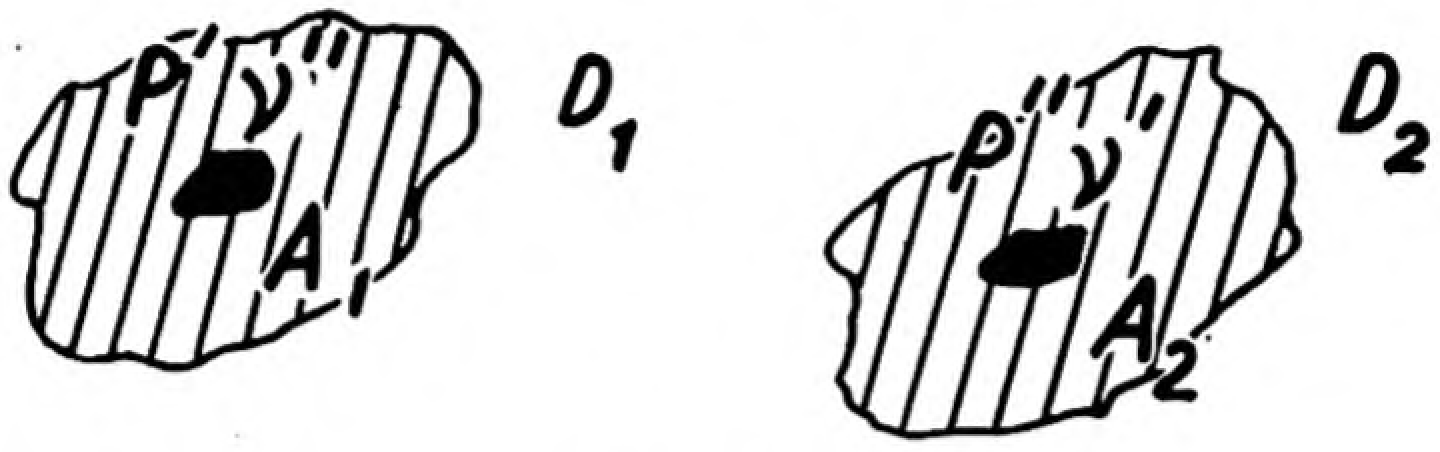


Fig. A26.1.

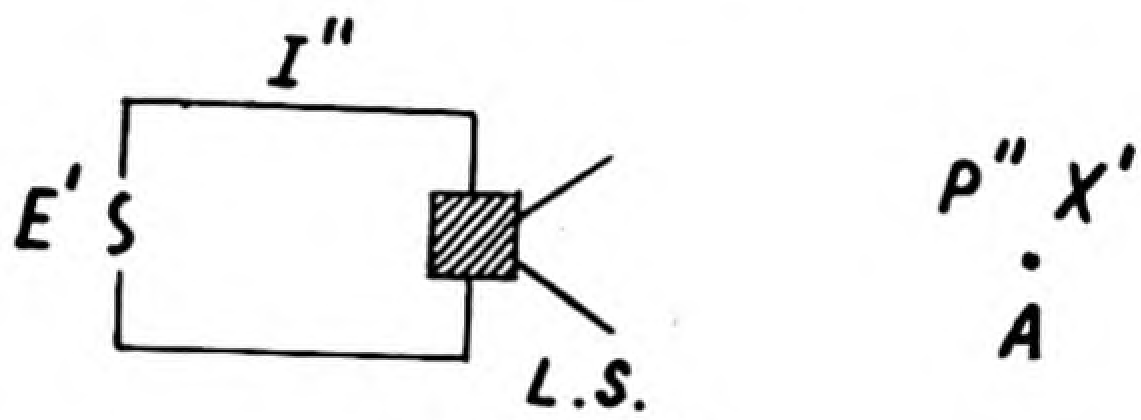


Fig. A26.2.

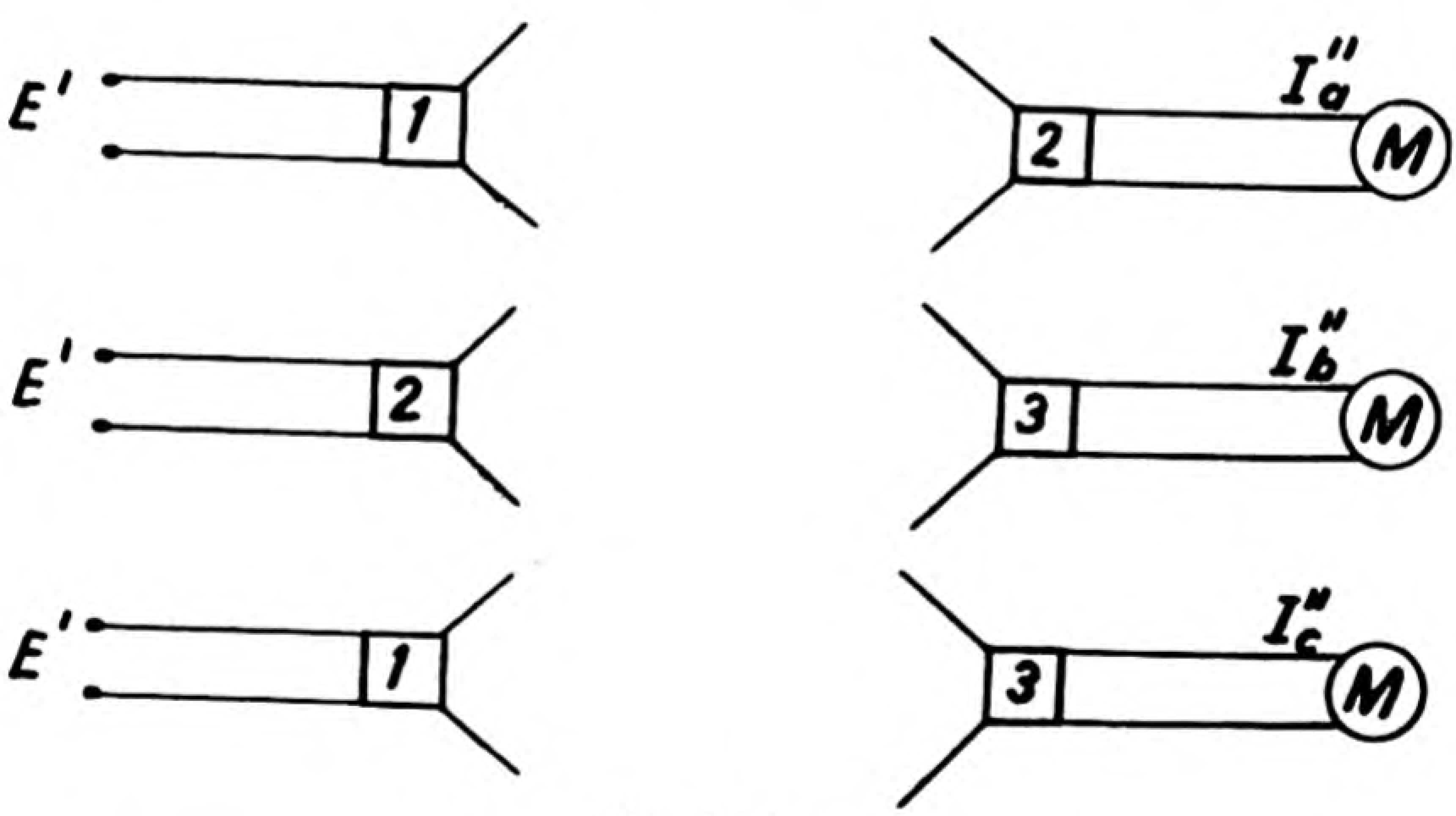


Fig. A26.3.

the sound field. Then assuming the loud-speaker (or other transducer) to be reversible, it follows that a pressure P'', numerically equal to E', exerted at A, will produce a current I'' in the electrical circuit equal to the previously produced volume-current X' at A. If a second transducer circuit is set up in the sound field then the reciprocal relations between the two circuits become the same as if the system were entirely electrical, viz. $E'I''=E''I'$.

Frequency calibration of a transducer by application of extended reciprocity relations. In the method to be described (due to Ballantine) only one of the transducers (No. 2 in Fig. A26.3) needs to

be reversible, for Nos. 1 and 3 are solely employed as a source and receiver respectively. The source-transducer is excited by a generator E' of variable frequency and the current I' is measured by, say, a meter M in the receiver-transducer circuit, the two transducers being placed so that (*a*) there is inappreciable interaction between them, and (*b*) the sound wave front is essentially planar. Let $\phi_1(\omega)$, $\phi_2(\omega)$ and $\phi_3(\omega)$ denote the frequency characteristics of the three transducers respectively, so that dealing with case (*a*), Fig. A26.3, the overall frequency characteristics of transmission will be $\phi_{12}(\omega)=\phi_1(\omega)\phi_2(\omega)$. Similarly for cases (*b*) and (c) the corresponding expressions will be

$$\phi_{23}(\omega)=\phi_2(\omega)\phi_3(\omega) \text{ and } \phi_{13}(\omega)=\phi_1(\omega)\phi_3(\omega)$$

respectively. These equations give as solutions

$$\phi_1(\omega)=\left(\frac{\phi_{12}(\omega).\phi_{13}(\omega)}{\phi_{23}(\omega)}\right)^{\frac{1}{2}}, \quad \phi_2(\omega)=\left(\frac{\phi_{12}(\omega)\phi_{23}(\omega)}{\phi_{13}(\omega)}\right)^{\frac{1}{2}}$$

and

$$\phi_3(\omega)=\left(\frac{\phi_{13}(\omega)\phi_{23}(\omega)}{\phi_{12}(\omega)}\right)^{\frac{1}{2}}.$$

Only the relative and not the absolute values of ϕ_{12}, ϕ_{13}, and ϕ_{23} are required if, as is usually the case, only the relative variations of the frequency response of a transducer is desired. A noteworthy extension of reciprocity methods of sound measurement is described by W. R. Maclean in a recent paper (*J. Acoust. Soc. America*, Vol. **12**, 1940) in which he shows how it is possible to make absolute acoustic measurements without the use of a primary standard.

APPENDIX 27

Average Value of the Square of a Sinusoidal Alternating Quantity. Formulae for Plane Waves

Let an alternating quantity be represented by the equation $y=a \sin \theta$ where y is the instantaneous value of the quantity for any angular position θ (radians) from the zero position on the circle of reference, and a is the maximum possible value of y, *i.e.* its amplitude of motion.

It follows that at any instant $y^2=a^2 \sin^2 \theta$ and the average value over a *complete* number of cycles of this quantity is

$$\overline{y^2}=\frac{1}{2\pi}\int_0^{2\pi} a^2 \sin^2\theta d\theta$$

$$=\frac{a^2}{4\pi}\int_0^{2\pi}(1-\cos 2\theta)d\theta$$

$$=\frac{a^2}{4\pi}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$=\frac{a^2}{4\pi}\left[2\pi-0\right]=\frac{a^2}{2}.$$

The square root of $\overline{y^2}$ is known as the Root Mean Square (R.M.S.) value of the alternating quantity and is thus given by

$$\sqrt{\overline{y^2}}=\sqrt{\frac{a^2}{2}}=\frac{a}{\sqrt{2}}$$

Formulae for a plane sound wave of small amplitude

$$\frac{\partial^2\eta}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2}$$

$$\frac{\partial^2 p}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

$$\frac{\partial^2 s}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}$$

$$c=\sqrt{\frac{\gamma P}{\rho}},\qquad \gamma P=\rho c^2,\qquad s=-\frac{\partial\eta}{\partial x},$$

$$\Delta p=\gamma Ps=\rho c^2 s=-\rho c^2\frac{\partial\eta}{\partial x}$$

$$\Delta T=\left(1-\frac{1}{\gamma}\right)\frac{T.\Delta p}{P}=-(\gamma-1).\frac{\partial\eta}{\partial x}.T.$$

η, p and s refer to the particle displacement, to the excess pressure and to the condensation of the medium respectively, while P and ρ are the static values of the pressure and density. Δp is a finite change in the excess pressure and γ is the ratio of the principal specific heats of the gas, while ΔT represents the change in temperature of the medium due to the passage of a plane sound wave.

APPENDIX 28

Deep-water Waves

Velocity of deep-water surface waves. Fig. 5.15 (p. 69) indicates that deep-water surface waves are propagated by a localised circulation of the water particles near the surface; it is assumed here that surface tension forces are negligible. Then the period (T) of the *wave* motion will also represent the time taken by a *particle* to complete one circulatory path. Furthermore, it follows that the instantaneous velocity of the *particle* at C will be in the direction of the wave-motion and equal to $v=\frac{2\pi r}{T}$, where r is the radius of the circle traversed by the particle. Similarly the particle-velocity at J will be $-v=-\frac{2\pi r}{T}$, *i.e.* exactly opposite in sense to the wave-velocity (C). Now to maintain the wave-form of Fig. 5.15 stationary with time a *backward* velocity c is imparted to the whole mass of water, and the energy of a mass m of water situated at C and J respectively is considered. The decrease of potential energy between these crest and trough positions will be equivalent to the corresponding gain of kinetic energy, assuming no attenuation of the wave due to viscosity.

Hence $$\frac{m}{2}\left[\left(-c-\frac{2\pi r}{T}\right)^2-\left(-c+\frac{2\pi r}{T}\right)^2\right]=mg\times 2r,$$

whence $c=\frac{gT}{2\pi}=\frac{g}{2\pi}\cdot\frac{\lambda}{c}$, where λ is the wave-length of the wave.

It follows that $$c=\sqrt{\frac{g\lambda}{2\pi}} \quad \ldots\ldots \quad \text{(A28.1)}$$

In the case of deep-water waves of *finite* amplitude Rayleigh deduced the more exact expression, $c=\sqrt{\frac{g\lambda}{2\pi}}\left(1+\frac{a^2}{\left(\frac{\lambda}{2\pi}\right)^2}\right)^{\frac{1}{2}}$, where a is defined in Fig. 5.20.

Effect of surface tension. The effect of surface tension is to create a difference of pressure between the two sides of the liquid-air interface, which is given by $\frac{T}{R}$, where T is the surface tension (dynes per cm.) and R is the radius of curvature of the free liquid surface at the point considered. Assuming the profile $OPAB$ (Fig. A28.1) of the wave to

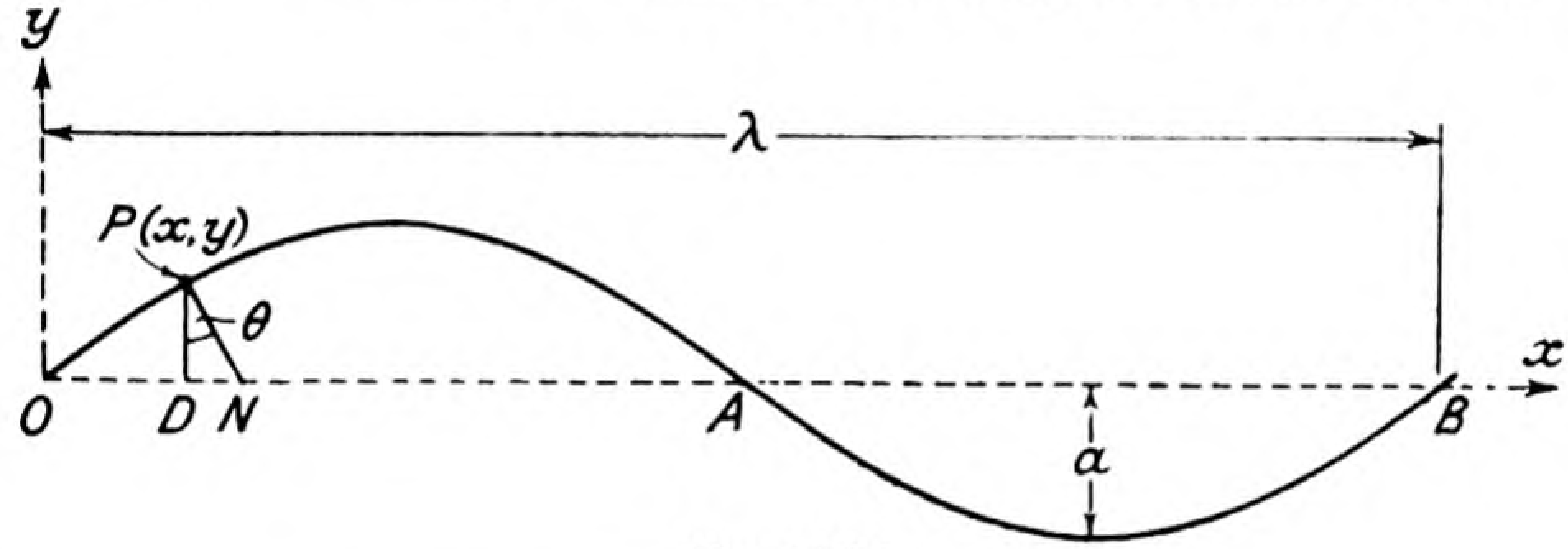


Fig. A28.1.

be sinusoidal in form and taking the origin of coordinates at 0, it follows that the vertical displacement of P above the undisturbed level is $y=a\sin 2\pi\frac{x}{\lambda}$, where x is the abscissa of P, λ is the wave-length (cf. $2\pi R$ of Fig. 5.20) and a is the amplitude of the waves.

Since the curve slope $\frac{dy}{dx}$ is small compared with unity under the conditions considered, it follows that the radius of curvature (R) at P is given by $\frac{1}{R}=\frac{d^2y}{dx^2}=-\frac{4\pi^2}{\lambda^2}.y$. Hence the excess pressure along the *inward* drawn normal to the curve at P is $+\frac{4\pi^2T}{\lambda^2}.y$.

Consider now an element of wave surface at P of *unit width* perpendicular to the plane of the paper and a length ds along the wave profile, then the *force* along $PN=\frac{4\pi^2T}{\lambda^2}y.ds$, or along $PD=\frac{4\pi^2T}{\lambda^2}y.ds\cos\theta$, *i.e.* $\frac{4\pi^2T}{\lambda^2}ydx$, since $\cos\theta=\frac{dx}{ds}$. The corresponding *pressure* exerted at D will be $4\pi^2Ty/\lambda^2$.

The total excess *pressure* at D above atmospheric is therefore $\rho g y+\frac{4\pi^2T}{\lambda^2}y=\rho y\left(g+\frac{4\pi^2T}{\lambda^2\rho}\right)$.

Now in the case of deep-water gravity waves considered above, the velocity of propagation was shown to be $c=\sqrt{\frac{\lambda}{2\pi}g}$, therefore for capillary waves when ρg becomes $\rho\left(g+\frac{4\pi^2T}{\lambda^2\rho}\right)$ it would seem fair to assume that the velocity is given by

$$c=\sqrt{\frac{\lambda}{2\pi}\left(g+\frac{4\pi^2T}{\lambda^2\rho}\right)}=\sqrt{\frac{g\lambda}{2\pi}+\frac{2\pi T}{\lambda\rho}} \quad . \quad . \quad \text{(A28.2)}$$

APPENDIX 29

Shallow-water Waves

Velocity of shallow-water waves. If the liquid depth h is much less than the wave-length λ, the localised circulations are destroyed by the proximity of the "bed" below, and the velocity of the fluid particles can be considered as sensibly uniform throughout the liquid depth. The problem may now be treated similarly to the flow of a fluid through a pipe of gradually varying section, by imposing a velocity in the reverse direction to the wave so that the *form* of the latter remains stationary with time.

If unit width *perpendicular* to the plane of the paper be considered, the liquid may be supposed to be divided into a number of tubes of flow which will have an equal cross-sectional area at every vertical section when the liquid surface is undisturbed. In the disturbed state it follows from the above assumptions that the cross-sectional areas of all these tubes of flow at a *given* vertical section will be equal. Considering a tube of flow in the surface of the water where the elevation above the static level is η and the velocity of flow C_η, the energy U per unit volume of the fluid of density ρ is $U=\rho g(h+\eta)+\frac{1}{2}\rho C_\eta{}^2$ $=\rho g(h+\eta)+\frac{\rho C^2}{2}\left(\frac{h}{h+\eta}\right)^2$, since the total volume of fluid crossing any vertical section must be constant and therefore $Ch=C_\eta(h+\eta)$.

Since U is independent of η it follows that $\frac{\partial U}{\partial \eta}=0$, which gives the condition, assuming η small compared with h, that $C=\sqrt{gh}$, which defines the velocity of propagation of what is sometimes called the Lagrangian wave.

APPENDIX 30

Definition of Standing-wave Ratio

Fig. A30.1 shows the formation of a standing-wave system (either acoustic or electrical) by the superposition of a direct wave (going from left to right) with the reflected wave travelling in the reverse

direction along a transmission pipe or line. Assuming the attenuation of the waves in the transmitting medium to be negligible, the successive maxima (electrical potential V or excess sound pressure P as the case may be) will be separated by half wave-lengths $\left(\frac{\lambda}{2}\right)$ and the minima will appear symmetrically between these maxima. The values of the sound pressure (or electrical potential) at any point in the transmission system will vary periodically with the frequency of the source, so that in practice it is the root-mean-square values which will be measured.

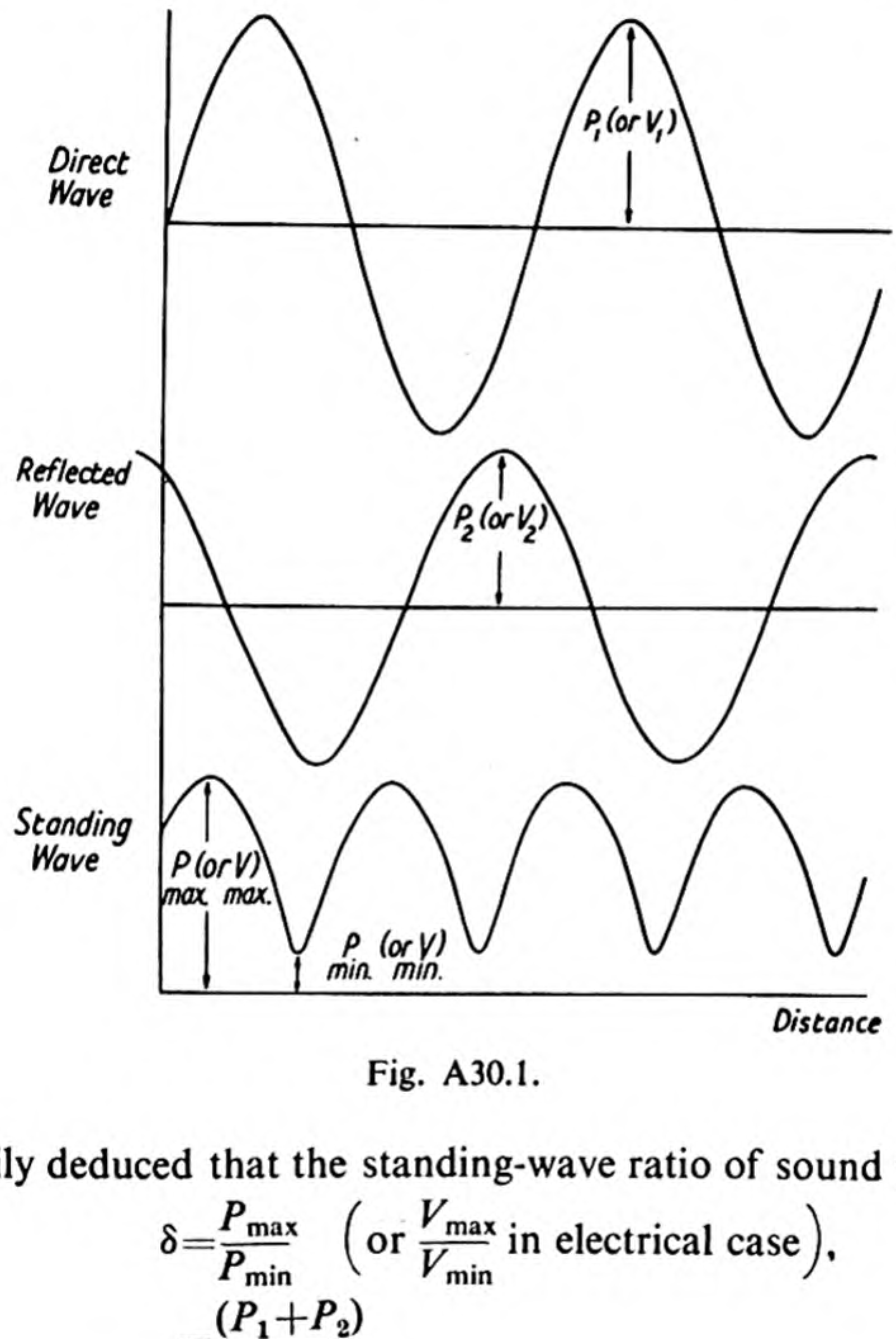


Fig. A30.1.

It is easily deduced that the standing-wave ratio of sound pressure is

$$\delta=\frac{P_{\text{max}}}{P_{\text{min}}}\quad\left(\text{or }\frac{V_{\text{max}}}{V_{\text{min}}}\text{ in electrical case}\right),$$

$$=\frac{(P_1+P_2)}{(P_1-P_2)}.$$

Hence $$\frac{P_2}{P_1}=\left(\frac{\delta-1}{\delta+1}\right).$$

It follows that the ratio of the *reflected* to the *incident energy* will be given by $\frac{E_2}{E_1}=\frac{P_2^{\,2}}{P_1^{\,2}}=\left(\frac{\delta-1}{\delta+1}\right)^2$. If z denote the impedance of the sample acting as reflector, and R_0 is the wave impedance of the gaseous medium (for air $R_0=42$ c.g.s.), it may be shown that $\frac{P_2}{P_1}=\frac{z-R_0}{z+R_0}$.

TABLE OF CONSTANTS

Velocity of sound, etc., in materials at 15° C. (approximately)

Material	*Young's Modulus* (dynes per sq. cm.)	*Density* (grm. per c.cm.)	*Velocity of Sound* (metres per sec.)	*Specific Acoustic Resistance* (grm. per sec. per sq. cm.)
Aluminium	$7{\cdot}3 \times 10^{11}$	2·7	5200	140×10^4
Beryllium	$12{\cdot}7 \times 10^{11}$	1·8	8400	150×10^4
Cadmium	$5{\cdot}3 \times 10^{11}$	8·6	2500	215×10^4
Copper	$11{\cdot}0 \times 10^{11}$	8·9	3500	310×10^4
Iron, cast..	$9{\cdot}0 \times 10^{11}$	7·8	3400	270×10^4
,, wrought	$20{\cdot}0 \times 10^{11}$	7·9	5100	400×10^4
Lead	$1{\cdot}7 \times 10^{11}$	11·4	1200	130×10^4
Magnesium	$4{\cdot}0 \times 10^{11}$	1·7	4800	82×10^4
Nickel	$21{\cdot}0 \times 10^{11}$	8·8	4900	430×10^4
Platinum	$17{\cdot}0 \times 10^{11}$	21·4	2800	600×10^4
Rhodium..	$30{\cdot}0 \times 10^{11}$	12·4	4900	610×10^4
Brass	$9{\cdot}5 \times 10^{11}$	8·4	3400	290×10^4
Beryllium-copper ..	$12{\cdot}5 \times 10^{11}$	8·2	3900	320×10^4
Monel	$18{\cdot}0 \times 10^{11}$	8·8	4500	400×10^4
Steel	$19{\cdot}5 \times 10^{11}$	7·7	5050	390×10^4
Glass, hard	$8{\cdot}7 \times 10^{11}$	2·4	6000	144×10^4
,, soft	$6{\cdot}0 \times 10^{11}$	2·4	5000	120×10^4
Marble	$3{\cdot}8 \times 10^{11}$	2·6	3800	99×10^4
Quartz, fused	$5{\cdot}2 \times 10^{11}$	2·7	4400	118×10^4
,, parl. to O.A. ..	$10{\cdot}3 \times 10^{11}$	2·7	6200	168×10^4
,, perp. to O.A. ..	$8{\cdot}0 \times 10^{11}$	2·7	5400	146×10^4
Ice	$0{\cdot}94 \times 10^{11}$	0·92	3200	29×10^4
Ash (with grain).. ..	$1{\cdot}3 \times 10^{11}$	0·64	4500	29×10^4
Cellulose acetate, moulded	$2{\cdot}1 \times 10^{10}$	1·3	1300	17×10^4
Methyl methacrylate, moulded	$2{\cdot}8 \times 10^{10}$	1·2	1500	18×10^4
Paper, parchment ..	$4{\cdot}8 \times 10^{10}$	1·0	2200	22×10^4

Velocity of sound, etc., in fluids at 15° C. (approximately)

Fluid	*Bulk Modulus* (dynes per sq. cm.)	*Density* (grm. per c.cm.)	*Velocity of Sound* (metres per sec.)	*Specific Acoustic Resistance* (grm. per sec. per sq. cm.)
Alcohol (methyl)	$9{\cdot}5 \times 10^{10}$	0·81	1240	$10{\cdot}0 \times 10^4$
Benzene ..	$11{\cdot}2 \times 10^{10}$	0·90	1170	$10{\cdot}5 \times 10^4$
Chloroform ..	$12{\cdot}0 \times 10^{10}$	1·5	983	$14{\cdot}7 \times 10^4$
Water, pure ..	$20{\cdot}0 \times 10^{10}$	1·0	1440	$14{\cdot}4 \times 10^4$
,, sea ..	$23{\cdot}3 \times 10^{10}$	1·03	1500	$15{\cdot}5 \times 10^4$
Turpentine ..	$13{\cdot}3 \times 10^{10}$	0·87	1330	$11{\cdot}6 \times 10^4$
Mercury ..	$263{\cdot}0 \times 10^{10}$	13·6	1400	$190{\cdot}0 \times 10^4$
Air	$0{\cdot}00014 \times 10^{10}$	0·00122	341	41·7
Hydrogen ..	$0{\cdot}00015 \times 10^{10}$	0·00009	1270	11·4
Oxygen ..	$0{\cdot}00014 \times 10^{10}$	0·00143	317	45·5
Carbon dioxide	$0{\cdot}00013 \times 10^{10}$	0·00198	258	51·2
Nitrogen ..	$0{\cdot}00014 \times 10^{10}$	0·00125	336	42·0

Acoustic absorption coefficients (at 500 c.p.s.)

Brick wall, unpainted	0·03
„ „ painted	0·02
Carpet, unlined	0·15 to 0·20
„ felt lined	0·20 to 0·35
Fabric curtains, hung straight, light	0·11
„ „ „ „ heavy	0·50
Plaster on tile, brick, or concrete	0·01 to 0·03
„ „ lath	0·03 to 0·04
Linoleum, cork tile, or rubber	0·03 to 0·06
Glass	0·03
Marble or glazed tile	0·01
Metal or wood chairs, each	0·2 approx.
Theatre seats, heavily upholstered, each	2.8 „
Desks, each	0·5 to 2·0
Audience, per adult seated (depending on nature of seating)	3.0 to 4.3
Acousti-celotex (a fibrous commercial product), depending on thickness	0·35 to 0·80

An open window (O.W.) is taken as the standard of reference, the coefficient for such a surface being unity.

Absorption coefficient and frequency

	125 c.p.s.	500 c.p.s.	4,000 c.p.s.
Audience, per adult seated	1·0 to 2·0	3·0 to 4·3	4·0 to 6·5
Seat, leather upholstered, each	1·2	1·6	2·1
Brick wall, unpainted	0·024	0·03	0·05
Fabric curtains, heavy	0·1	0·5	0·9
Felt (one inch thick)	0·1	0·52	0·44

Definitions

A **cycle** is a complete repetition of the recurrent values of a periodic phenomenon.

The time required for one cycle of a periodic quantity is the **period,** and is expressed in seconds.

Frequency is the rate of repetition of the cycles of a periodic phenomenon, and is the reciprocal of the period. The unit is the cycle per second and is occasionally termed the hertz.

Pulsatance is equal to the product of 2π and the frequency.

Bel is a unit used in the comparison of the magnitudes of powers (acoustical or electrical). The number of bels expressing the relative magnitude of two powers is the logarithm to the base 10 of the ratio of the powers. If the **natural** logarithm of the power ratio is used the unit is known as the **neper,** which is employed often in electrical transmission practice.

Decibel is one-tenth of the bel.

Phon is a unit of equivalent loudness, defined as follows:—

The standard tone shall be a plane sinusoidal sound wave train coming from a position directly in front of the observer and having a frequency of 1000 cycles per second.

The listening shall be done with both ears, the standard tone and the sound under measurement being heard alternately and the standard tone being adjusted until it is judged by a normal observer to be as loud as the sound under measurement. The intensity level of the standard tone shall be measured in the free progressive wave and the reference level shall be taken to be that corresponding to an R.M.S. sound pressure of 0·0002 dyne per sq. cm.

When, under the above conditions, the intensity level of the standard tone is n decibels above the stated reference intensity, the sound under measurement is said to have an equivalent loudness of n phons.

The **bar** is a unit of pressure and is equal to one million dynes per sq. cm.

The **microbar** is the C.G.S. unit of pressure and equals 1 dyne per sq. cm.

Acoustic energy density at a point is the sound energy per unit volume at that point. The unit is the erg per cm.^{3}.

Sound intensity (I), is defined in terms of the *effective* (*i.e.* R.M.S. for sine waves) sound pressure p, velocity c, and density of the medium ρ, by the relation

$$I = \frac{p^2}{\rho c}.$$

This is equal to the sound energy flux per unit area for a plane or spherical free progressive wave having the same values for p and c in the same medium of density ρ. The unit is the erg per cm.^{2} per sec.

Sound energy flux through an area is the mean flow of sound energy per unit time through that area, and is expressed in ergs per sec. The sound energy flux through an area A perpendicular to the direction of propagation is equal to $\frac{p^2.A}{\rho c}$ ergs per sec., p being the R.M.S. sound pressure.

Loudness is that *subjective* quality of a sound which determines the auditory sensation produced by the sound.

Impedance

There are three different designations of acoustic impedance, which are respectively employed under different conditions.

Specific acoustic impedance or unit area impedance (Z) is employed in transmission line problems and is defined as the complex ratio of the sound pressure to the sound particle-velocity at the point in the medium in which sound waves are being propagated.

Analogous or **acoustical impedance** (Z_A), see p. 311, is used at low frequencies when dealing with "lumped" circuits and is defined by the complex ratio of sound pressure to the volume flow, or current.

Radiation impedance (Z_r) is the portion of the mechanical impedance due to the sound field, and occurs in the calculation of the coupling between a vibrating surface and the sound waves.

The electrical impedance of all electrical sources of sound is a function of the acoustical load, and if the vibrating element, *e.g.* diaphragm of a loud-speaker, is held fixed, the corresponding electrical impedance as measured at the input terminals is termed the *blocked impedance* (Z_B). The impedance Z_V measured when the element is free to vibrate will differ from Z_B, and $(Z_V - Z_B)$ is known as the *motional impedance* (Z_M), and this is related to the acoustical impedance at the vibrating surface.

EXAMPLES

1. A particle is simultaneously subjected to three simple harmonic motions, all of the same period and acting in the same direction. If the amplitudes of the motions are 0·25, 0·20, and 0·15 mm. respectively and the phase difference between the first and second is 45°, and between the second and the third is 30°, determine the resultant amplitude of the particle.

[0·517 mm.]

2. A particle is acted on simultaneously by twelve simple harmonic motions, all of the same period and amplitude. If a common phase difference exists between each successive motion, determine its value for the resultant displacement of the particle to be zero.

[30°.]

3. A uniform rod of length l swings freely in a vertical plane about a horizontal axis at one end and executes *small* oscillations about its position of rest. Show that these are simple harmonic of period $2\pi\sqrt{\left(\frac{2l}{3g}\right)}$ sec. What is the period of oscillation when a particle of mass one-half that of the rod is attached to its lower end?

$\left[2\pi\sqrt{\frac{7l}{6g}}.\right]$

4. A helical spring of mass 45 gm. supports a mass of 150 gm. from its lower end. An addition of 15 gm. would cause it to stretch a further 3 cm. The load is now raised 3 cm. and released. Calculate its period of vibration.

If after the first descent the load just touches a mass of 15 gm. which adheres, what time will elapse before the highest point is reached, and where is this point? Ignore damping losses.

[1·2 sec.; 6 cm. above lowest point; 0·645 sec.]

5. A pulsating spherical surface of radius 1·0 cm. performing a simple harmonic motion given by $\eta = \cdot 01 \sin (0{\cdot}2\pi)t$ is the source of spherical waves within an infinite homogeneous medium. If the speed of propagation is 9 cm. per sec., determine the displacement of a particle 90 cm. from the origin of the disturbance, 90 sec. after the source has commenced to vibrate.

Also calculate the phase difference between any two particles situated 10 cm. apart on any normal drawn from the vibrating surface.

[Energy falls off inversely as square of distance for spherical waves, hence amplitude varies inversely as distance.

Therefore $\qquad \eta = \frac{\cdot 01}{91} \sin \left(\frac{36 \times 90}{9}\right) = 0.$

Phase difference $\qquad = \left(\frac{10}{\lambda}\right) 2\pi \text{ radians} = \frac{2\pi}{9} \text{ radians.}$]

6. In a long line of particles which are in contact with one another the "head" particle is subjected to a S.H.M. given by $\eta = a \sin 8\pi t$, and gives rise to the propagation of a wave motion along the line having a velocity of 500 cm. per sec.

Calculate the displacement of a particle distant 30 cm. from the "head" particle 3 sec. after the latter has commenced to move from its equilibrium position.

[Time for wave to travel 30 cm. = 0·06 sec.

Hence $\quad \eta = a \sin 8\pi(3 - \cdot 06) = -0{\cdot}998a.$]

7. Define simple harmonic motion.

A metal cylinder 16 cm. long of density 8·9 floats upright in mercury. Find the period of small vertical oscillations, ignoring frictional and other effects. [1·104 sec.]

8. Find the period of oscillation of a liquid in a vertical U-tube in terms of its cross-sectional area and the length of the liquid column. What is the effect on the period of inclining the plane of the tube through 60°? Neglect any frictional effects. [Period is increased to $\sqrt{2}$ original period.]

9. A helical spring hangs vertically and is extended by 3 cm. when a load of 50 gm. is hung on the lower end. If the spring is now extended by 0·5 cm. and released, determine the period of the vibrations and the total vibrational energy of the system. [0·35 sec.; $8{\cdot}18\times10^3$ ergs.]

10. The frequency of the second overtone of an organ pipe is the same as that of the third overtone of an open pipe. Compare the lengths of the pipes, allowing an end-correction at the mouth of the pipe of 3·3*R*, and of 0·6*R* at the open end, the diameters being 10 cm. and the frequency of the overtone 680 c.p.s. Calculate (*a*) the fundamental frequencies of each pipe, and (*b*) the length of each pipe. [136 c.p.s.; 170 c.p.s.; 45 cm.; 80·5 cm.]

11. A particle is subjected simultaneously to two S.H.Ms. of the same period T but executed in perpendicular directions. If these S.H.Ms. are given by

(1) $x=a\sin\dfrac{2\pi t}{T},\qquad y=b\sin\dfrac{2\pi t}{T},$

(2) $x=a\sin\dfrac{2\pi t}{T},\qquad y=b\sin\left(\dfrac{2\pi t}{T}-\dfrac{\pi}{3}\right),$

(3) $x=a\sin\dfrac{2\pi t}{T},\qquad y=b\sin\left(\dfrac{2\pi t}{T}-\dfrac{\pi}{2}\right),$

determine graphically the resultant motion in the three cases.

12. When a train of plane waves of length 280 cm. traverses a medium, individual particles execute a periodic motion given by the equation $\eta=3\sin\left(\dfrac{2\pi t}{8}+\theta\right)$.

(i) Determine the velocity of propagation of the waves.

(ii) Calculate the phase difference for two positions of the *same* particle which are occupied at time intervals 0·8 sec. apart.

(iii) Determine the phase difference at any *given instant* of two particles 200 cm. apart, and

(iv) Find the displacement of a certain particle 1·5 sec. after its displacement is 2·4 cm.

[(i) 35 cm. per sec.; (ii) 36°; (iii) 257·1°; (iv) 2·58 or −0·75 cm. Note re (iv) $\eta=3\sin\left(\dfrac{2\pi t}{8}+\theta\right)\equiv3\sin2\pi\left(\dfrac{t}{8}+\dfrac{x}{\lambda}\right)$, hence assume particle at $x=0$ whence $\eta=3\sin 45t$, converting to angular degrees.]

13. Show that in a train of progressive harmonic sound waves half of the energy is kinetic and half potential. Derive an expression for the energy density and for the intensity of such a train of waves, and hence describe how the minimum amplitude of vibration for audibility in air has been determined.

14. Distinguish between combination tones and beat tones, and describe how they may be produced.

15. What is meant by the principle of superposition? Apply this principle to explain the formation of standing waves.

A train of sound waves of amplitude 0·001 cm. is propagated along a *wide* tube, and is reflected without loss of amplitude from an open end. If the wave-length is 36 cm., what is the amplitude of vibration at a point 33 cm. inside the pipe? [0·00173 cm.]

16. Show that a transverse disturbance in a stretched string travels with a velocity given by the expression $\left(\dfrac{\text{tension}}{\text{mass per unit length}}\right)$.

How does this determine the natural frequencies of the string?

Show that the time taken for a disturbance to pass along a string of length l cm. and of mass m gm. per cm. is constant when the tension is ml^2 gm. wt., and calculate this time. [0·032 sec.]

17. A metal wire of mass 9·8 gm. is stretched with a tension of 10 Kgm. wt. between two rigid supports one metre apart. The wire passes at its mid-point between the poles of a permanent horseshoe-magnet and it vibrates in resonance when carrying an alternating current of frequency f c.p.s. Determine the frequency of this supply, and discuss the effect of using an iron wire and replacing the permanent magnet by an electromagnet energised by the A.C. supply.

[50 c.p.s.; the wire will vibrate twice for every cycle of the current.]

18. Derive expressions for the velocities of propagation of transverse and longitudinal waves along a flexible wire under tension. Compare the fundamental frequencies of such a wire for each type of wave, given that the wire is elongated by 1 in 1000 due to the tensile force.

$$\left[f_T=\frac{1}{2e}\sqrt{\frac{T}{m}};\ f_L=\frac{1}{2e}\sqrt{\frac{E}{\rho}};\ \frac{f_L}{f_T}=10\sqrt{10}.\right]$$

19. A siren, situated 170 metres from a vertical cliff, gives out a tone which rises from zero frequency uniformly to 340 c.p.s. in 5 sec. and then drops uniformly to zero in the same time. A beating effect due to reflection from the cliff is heard. Explain this, and indicate how the beats vary in frequency for different positions of observation.

[An observer situated near the siren will hear beats 1 sec. after siren commences, their frequency being 68 per sec. This persists for 4 sec., then drops to zero during the next half-second, and rises steadily in the next half-second to a uniform 68 beats per sec.]

20. What is meant by "non-linearity"? Discuss two particular examples occurring in electrical or acoustical systems.

21. Describe the dust-tube method of Kundt for the comparison of the velocities of sound in different gases. How can the results be applied to determine the ratio of the principal specific heats of a gas?

22. Explain the production of beats, and derive an expression for their frequency.

23. Show, for a gas obeying the perfect gas equation, that if $\gamma=1{\cdot}4$, the velocity of sound in it is equal to 0·68 times the R.M.S. molecular velocity at the same temperature.

$$\left[P=\frac{\rho}{3}(\text{R.M.S. Mol. Vely.})^2.\right]$$

24. The equation of motion of a damped oscillating system is given by

$$\frac{d^2\theta}{dt^2}+\alpha\frac{d\theta}{dt}+\beta\theta=0.$$

Given that its period is 5·0 sec. and the log. dec. $=1{\cdot}0$, plot the curve representing the motion.

Determine the character of the motion in the particular cases where (i) $\beta=\alpha^2/4$, (ii) $\alpha=10{\cdot}0$, and β has the same value as in the first example.

$$\left[\text{log dec.}=\frac{\alpha}{4}\text{(period).}\right]$$

25. A source of sound of frequency 525 c.p.s. is situated on a cliff at a point 280 ft. above the surface of a calm sea. At a distance of 1 mile alternate maxima and minima of sound are observed by a man climbing a vertical mast, the distance between adjacent maxima and minima being 10 ft. Explain this, and use the data to calculate the velocity of sound in air.

[1113 ft. per sec.]

26. What would be the effect on the reception of a wireless programme broadcast in a flat country simultaneously from two equally powerful transmitters on the same (medium) wave-length? Assume the transmitters to be many miles apart, and the effect of the upper atmosphere to be negligible. [See Fig. 6·8.]

27. Distinguish between progressive and standing waves.

A closed organ pipe is speaking its fundamental tone. Describe the motion of the air in the pipe and indicate how the pressure varies at different points within the pipe during one cycle. What adjustment is necessary to make the pipe speak the same fundamental when "opened"? Explain the difference in the quality of the notes.

28. Write a short précis of the lines quoted from Phineas Fletcher at the beginning of the book, and interpret their physical meaning.

29. Describe the cathode-ray oscillograph, and explain how it may be used to compare the frequencies of two sources of sound.

30. What do you understand by the Doppler effect?

Show how this effect operates in the case of radiation incident normally upon a moving reflector, and hence derive an expression for the pressure exerted by the radiation when the receiving surface is perfectly reflecting.

31. Assuming the formula for the velocity or propagation of ripples over a liquid surface, viz. $c^2=\frac{g\lambda}{2\pi}+\frac{2\pi T}{\rho\lambda}$ in the usual notation, deduce an expression for the group velocity of the ripples. Use your expression to find the limiting value of the velocity: (*a*) in the case of the shortest ripples, and (*b*) for very long waves.

$$\left[U=c\left\{1-\frac{1}{2}\left[\frac{\frac{g\lambda}{2\pi}-\frac{2\pi T}{\rho\lambda}}{\frac{g\lambda}{2\pi}+\frac{2\pi T}{\rho\lambda}}\right]\right\};\ (a)\ U=\tfrac{3}{2}c;\ \ (b)\ U=\frac{c}{2}.\right]$$

32. Give a critical account of the methods available for measuring the velocity of sound in air.

33. Show that the amplitude of the temperature change due to the passage of a plane sound wave in a gas is given by

$$\Delta T=\left(1-\frac{1}{\gamma}\right)T-\frac{\Delta P}{P},$$

where γ is the ratio of the principal specific heats, ΔP the excess pressure, and T and P refer to the equilibrium values of temperature and pressure respectively.

(**Note.** $\frac{\Delta T}{T}=\frac{\Delta P}{P}+\frac{\Delta V}{V}$.)

34. Explain the need for tempering a musical scale; how is it calculated and realised?

35. Prove that the energy of a string of mass M vibrating in any number of loops is equivalent to $\frac{M(2v)^2}{2}$, in which v is the maximum velocity of an antinode.

36. Show how, for a particle executing damped harmonic vibrations, the ratio of successive velocities of transit through the zero depends on the logarithmic decrement (k) and the frequency (n).

$\left[-\frac{k}{n}.\right]$

37. Briefly describe the principle of the cathode-ray oscillograph and its use in acoustic measurements. Give circuit diagrams, etc., to indicate the working of: (a) a linear, and (b) a circular time-base, and state their particular applications.

38. Calculate the amplitude of temperature fluctuations in air caused by the passage of a plane sound wave of intensity 50 ergs per sq. cm. per sec.

[$\gamma=1{\cdot}40$; $\rho c=41{\cdot}7$ c.g.s.; $\Delta P=\sqrt{4170}$; $\Delta T=0{\cdot}0053^\circ$ C.]

39. Define the *decibel* and the *phon*, explaining the difference between them. What significance has the decibel in physiological acoustics?

40. Show how the velocity of sound in dry air may be deduced from the observed value (C_m) in moist air at 17° C., given that the partial pressures of argon and water vapour at the time of experiment are respectively 0·95 per cent. and 1·85 per cent. of the total pressure. (Ratios of principal specific heats for water vapour, argon, dry air (also oxygen and nitrogen) are 1·260, 1·667, and 1·400 respectively.)

[Ans. $\frac{C_o}{C_m}=\sqrt{\frac{\gamma_o}{\gamma_m}\cdot\frac{T_o}{T_m}}=\sqrt{\frac{1{\cdot}4015\times 273}{1{\cdot}4000\times 290}}=0{\cdot}97$.

Note.—$(\gamma-1)=\frac{R}{C_v}$ since $C_p-C_v=R$; $C_v=C_{v1}m_1+C_{v2}m_2+C_{v3}m_3$

where m_1, m_2, and m_3 are respective masses of argon, water vapour and nitrogen-oxygen in unit mass of moist air.

Assuming gas equation $Pv=RT$, the final form for γ_m is

$$\gamma_m-1=\frac{1}{\frac{1}{\gamma_1-1}\cdot\frac{P_1}{P}+\frac{1}{\gamma_2-1}\cdot\frac{P_2}{P}+\frac{1}{\gamma_3-1}\cdot\frac{P_3}{P}}$$]

41. A string AB is plucked as shown in Fig. E.1 where l is the total length of the string. Determine the relative intensities of the harmonics.

$$\left[A_m=\frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}dx\right.$$

$$=\frac{2}{l}\int_0^{a_1}\frac{x\delta}{a}\sin\frac{m\pi x}{l}dx+\frac{2}{l}\int_{a_1}^{a_2}\sin\frac{m\pi x}{l}dx$$

$$+\frac{2}{l}\int_{a_2}^{l}\left(\frac{l-x}{l-a_2}\right)\delta.\sin\frac{m\pi x}{l}dx$$

$$=\frac{2\delta}{m^2\pi^2}\left[\frac{l}{a_1}\sin\frac{m\pi a_1}{l}-\frac{l}{(l-a_2)}\sin m\pi+\frac{l}{(l-a_2)}\sin\frac{m\pi a_2}{l}\right]$$

$$=\frac{2\delta}{m^2\pi^2}B_m.$$

$$\left.\therefore\ \text{Intensity}\propto m^2A_m{}^2\propto\frac{1}{m^2}[B_m]^2.\right]$$

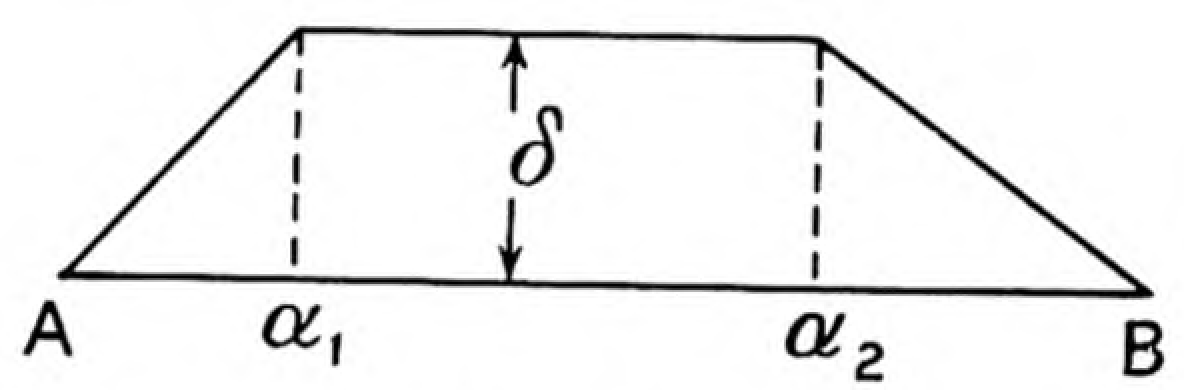


Fig. E.1.

42. Write an account of the phenomenon of resonance, quoting examples of its occurrence in different branches of physics.

43. Distinguish between "amplitude" resonance and "velocity" resonance for a mechanical vibrating system. Calculate the frequencies corresponding to these conditions for a system which has a mass of 200 gm., a stiffness factor of $3{\cdot}0\times10^7$ dynes per cm., and a damping coefficient of 400 dynes sec. per cm.

Determine also the logarithmic decrement and the mechanical "Q" of the system.

["Velocity" frequency=61·6 c.p.s.; "amplitude" frequency smaller but not distinguishably so. log. dec.=0·0081; Q=387.]

44. What is meant by the term *impedance* as applied to a mechanical system? Discuss fully the analogy between such a vibrating system and an electrical circuit.

45. Give an account of any modern work on the mechanism of production and on the constitution of vowel sounds.

46. A plane wave of sound is incident obliquely on a plane surface of separation between two fluids. Discuss the conditions governing the resulting reflection and refraction. If both fluids have the same elasticity, determine the angle at which no reflection occurs.

47. Derive expressions for the energy density and the energy current for plane sound waves in a gas. Such waves of frequency 512 c.p.s. in oxygen have an amplitude of 0·0012 mm. Assuming the density to be 0·001429 gm.

per cc. and the velocity of sound to be 317 m.p.s., determine (i) the energy-density of the radiation, (ii) the energy-current per unit area.

[(i) 0·000107 erg per cm.^{3}, (ii) 3·38 erg per cm.^{2} per sec.]

48. Discuss the conditions governing the reflection and refraction of a plane wave of sound incident at the surface of separation of two media and obtain expressions for the coefficients of reflection and transmission.

49. A composite rod is made by joining the ends of two homogeneous materials of uniform cross-sections S_1 and S_2 respectively. Longitudinal waves travelling in one rod strike the common interface. Show that for no reflection to occur at the junction, $\dfrac{S_1}{S_2}$ must equal $\left(\dfrac{\rho_2 E_2}{\rho_1 E_1}\right)^{\frac{1}{2}}$, where ρ and E signify respectively the density and Young's modulus of elasticity of the material of a rod respectively.

What change of phase occurs, if any, in the reflected wave for the cases where (i) $E_2=E_1$, and (ii) $\rho_2=\rho_1$?

50. Obtain expressions for the reflection and transmission coefficients of sound waves on reaching the interface between two gases.

If a sound wave originates in hydrogen and passes, by normal incidence, into oxygen, what fraction of the initial energy emerges?

Density of hydrogen, 0·0000898 gm. per cc.

Density of oxygen, 0·001429 gm. per cc.

Ratio of principal specific heats of each gas = 1·4.

$$\left[\frac{\text{Emergent Amplitude}}{\text{Incident Amplitude}}=\frac{2}{\frac{v_1}{v_2}+1}=\frac{2}{5};\ \text{Energy}=\rho v\times(\text{Amplitude})^2;\right.$$

$$\left.\text{Ans.}=\tfrac{16}{25}.\right]$$

51. What is meant by "tempering" a musical scale? Explain how it is accomplished, and why it is necessary with some instruments and not with others.

52. Discuss the physical principles underlying two of the following: violin, pianoforte, trombone, pipe-organ.

53. In some instruments the overtones and fundamental do not necessarily form exact intervals. Explain this with reference to one particular instrument.

54. Discuss the physical factors which affect the intelligibility of speech.

55. Assuming that the intensity, I (ergs per cm.^{2}), of a spherical sound wave at a distance r (cm.) from the source is given by $I=\dfrac{Pe^{-\alpha r}}{4\pi r^2}$, where P is the power output (ergs per sec.) of the source, determine the distance travelled in water by such a wave of frequency 1 Mc.p.s. before it is attenuated to $\dfrac{1}{e}$ of its original intensity, $\alpha=\dfrac{2\mu k^2}{\rho c}$, where k, ρ, c have their usual significance and μ (0·0114 c.g.s. for water) is the viscosity of the medium. [100 metres.]

56. Distinguish between forced and free vibrations, and discuss briefly their importance in acoustical and electrical phenomena.

57. Explain what is meant by relaxation oscillations, and give examples of their occurrence in various branches of physics.

58. Assuming that the amplitude (A') of the radiation *scattered* from small objects varies *directly* as the volume (V) of these obstacles and as the

amplitude (A) of the incident waves, and *inversely* as the distance r from the obstacle of the point under consideration; find by the method of dimensions the law of dependence of the scattered amplitude upon wave-length (λ).

$$\left[A'=k\frac{AV\lambda^{-2}}{r};\ \text{hence scattered intensity} \propto (A')^2 \propto \frac{1}{\lambda^4}.\right]$$

59. Obtain, by the method of dimensions, the expression for the frequency of transverse vibration of a stretched string in terms of its length, mass per unit length, and tension.

A rope of mass 490 gm. is 6 metres in length and is stretched with a tension equal to the weight of 1·2 kg. What frequency of transverse vibration will make it vibrate in loops 2 metres in length? [1·5 c.p.s.]

60. Use the method of dimensions to show the dependence of the velocity of sound in a gas upon the density, pressure, and dynamic viscosity.

$$\left[C=A\sqrt{\frac{P}{\rho}}.\right]$$

61. Assuming that the velocity of deep water waves is a possible function of wave-length, density, and gravitational attraction, use the method of dimensions to determine this dependence. [$v \propto \sqrt{g\lambda}$.]

62. Use the method of dimensions to show how the time of vibration (T) of a tuning-fork depends upon its linear dimensions and the Young's modulus (Y) and density (ρ) of the material of the fork. [$T=kLY^{-\frac{1}{2}}\rho^{\frac{1}{2}}$.]

63. Show that the solution of the equation representing the free oscillations of a lightly damped mechanical system can be expressed as $x=$real part. $Ae^{-\left(\frac{1}{2Q}\right)\omega t}e^{3\omega t}$, where x is the displacement at any time t, A is a constant, and Q and ω have their usual significance. If $\frac{\omega_0}{2\pi}$ is the natural frequency of the system in the absence of damping, verify also that for small damping $\omega=\omega_0\sqrt{1-\left(\frac{1}{2Q}\right)^2}$.

64. A source of sound is situated at one end of a rigid tube which is closed at the other end by a partly reflecting surface. Discuss the wave motion set up when the source emits a continuous note of constant amplitude.

65. Discuss the difference in principle between the measurement of absolute intensities in sound waves and such measurements in other branches of physics.

66. Discuss the basic principles governing the efficiency of sound receivers.

67. Describe fully any experimental work on the determination of the acoustical impedances of orifices.

68. Establish the differential equation applicable to small lateral vibrations of a uniform elastic bar.

69. Discuss briefly the vibrations of a tuning-fork in relation to the transverse oscillations of a uniform elastic bar.

70. Derive the differential equation for spherically symmetrical motion in a fluid medium of density ρ and bulk modulus k, expressing your result in terms of the velocity potential ϕ and distance r from the centre of symmetry.

71. Derive an expression for the propagation of plane longitudinal waves in a solid elastic medium of infinite extent.

Discuss briefly the phenomena of wave propagation involved in geophysical prospecting.

72. Describe briefly the application of "optical" methods to acoustic measurements.

73. Derive an expression for the resonant frequency of a narrow-neck resonator and describe how you would investigate its response in practice.

74. Describe and discuss critically the instruments and methods employed in the measurement of sound intensity.

75. The shadow picture of a moving bullet reveals that the shock-wave at the nose makes an angle of 26° with the direction of the bullet. Determine the speed of the bullet, assuming a standard atmospheric pressure and a temperature of 12° C. [773 m.p.s.]

76. A square membrane of 3·0 cm. side is subject to a uniform tension of 10 dynes per cm. If the density of the material is 0·12 gm. per sq. cm., what will be the frequencies of the fundamental and the lowest overtone of the membrane? [64 c.p.s., 108 c.p.s.]

77. A cubical room of side 5 m. has an average absorption coefficient of 0·2 for the ceiling and floor, and of 0·04 for the walls. What are the reverberation times for those waves (*a*) which strike floor and ceiling, (*b*) for those which do not, (*c*) for the complete room? Give all the resonant frequencies possible between 0 and 100 c.p.s. and construct a spectrum (p. 389).

[(*a*) and (*b*) are special cases of (A15.1).

(*a*) M.F.P. = 5 m.; N (p. 297) = $c/500 = 68$; T (p. 297) = 0·91 sec.; (*b*) $T = 5{\cdot}0$ sec.; (*c*) M.F.P. (A15.2) = $4 \times 5^3/6 \times 5^2 = 10/3$ m.; $N = 102$, $\bar{a} = {\cdot}093$, $T = 1{\cdot}4$ sec.]

78. Describe a tube method for the determination of the absorption coefficient, at normal incidence, of a substance in the form of a disc. Discuss how these results compare with those derived from reverberation measurements.

79. A thin wire of mass 4·0 gm. is stretched horizontally between two points 100 cm. apart with a tension of $6{\cdot}4 \times 10^5$ dynes.

Determine (*a*) the frequency of the fundamental mode, (*b*) the number of vibrational modes in the wave-length range 4·0 to 5·0 cm., and (*c*) the energy of the mode whose frequency is 400 c.p.s. when the displacement of the string at its centre is 2×10^{-3} cm.

[(*a*) 20 c.p.s.; (*b*) 10; (*c*) 25·3 ergs.

Energy $= 4\pi^2 f^2 M u^2$, where u = mean square displacement for all elements of wire.]

80. Determine the velocities of longitudinal and transverse elastic waves in aluminium, given that Young's modulus (E) and Poisson's ratio (σ) for aluminium are respectively $7{\cdot}2 \times 10^{11}$ dynes per sq. cm. and 0·33, and its density is 2·7 gm. per cm.^{3}.

Calculate the total number of modes of vibration in a centimetre cube within the region 10^{13} to $10{\cdot}1 \times 10^{12}$ c.p.s.

$$\left[C_L = \sqrt{\frac{E}{\rho} \frac{(1-\sigma)}{(1+\sigma)(1-2\sigma)}}; \quad C_T = \sqrt{\frac{E}{\rho} \cdot \frac{2}{(1+\sigma)}}; \right.$$

$$\left. dn = 4\pi \left(\frac{1}{C_L{}^3} + \frac{2}{C_T{}^3} \right) \nu^2 d\nu = 2{\cdot}4 \times 10^{12}. \right]$$

81. What do you understand by Fourier analysis? Discuss briefly its importance in physical problems, and give a detailed analysis of some particular problem in acoustics.

82. Assuming that an electron volt is the energy acquired by an electron in dropping through a p.d. of one volt, calculate the wave-length of a particle having an energy of one electron volt in the case of (*a*) an electron and (*b*) a proton. The potential energy is to be assumed zero.

Mass of electron $=9{\cdot}0\times10^{-28}$ gm.
Mass of proton $=1{\cdot}66\times10^{-24}$ gm.
1 electron volt $=1{\cdot}602\times10^{-12}$ erg.
Planck's constant $=6{\cdot}628\times10^{-27}$ erg sec.

[(*a*) $1{\cdot}25\times10^{-7}$ cm.; (*b*) $2{\cdot}90\times10^{-9}$ cm.]

83. Explain what is meant by the acoustical properties of a room. If the reverberation time for an auditorium is 1·8 sec. when 200 people are present and drops to 1·2 sec. when 600 people are present, estimate the number present when the reverberation time is 1·6 sec. [300.]

84. Electromagnetic radiation in the form of plane harmonic waves fills a cubical box of side d and satisfies the condition that the displacement is always zero at the bounding faces. Show that the number of possible modes of oscillation within the box for frequencies between ν and $\nu+\Delta\nu$ is given by $\frac{4\pi d^3}{c^3}\cdot\nu^2\Delta\nu$, where c is the velocity of propagation of the radiation.

85. Discuss briefly the fundamental principles governing the acoustics of rooms.

Derive an expression for the reverberation period of a room and indicate how a knowledge of this period can be applied to the determination of acoustic absorption coefficients.

86. A plane harmonic sound wave, defined by $\eta=Be^{j(\omega t-kx)}$, where η is the displacement, B is a constant, and ω, t, k, and x have their usual significance, is travelling in the positive x-direction through a long cylindrical tube whose cross-section at a certain point changes abruptly from A_1 to A_2. Determine the fraction of the incident energy which is reflected at the change of section $\left(k=\frac{\omega}{c}\text{ where } c \text{ is velocity of sound}\right)$.

[Apply boundary conditions of continuity of pressure and volume current. Ans. $\left(\frac{A_2-A_1}{A_2+A_1}\right)^2$.]

87. Derive an expression for the acoustic impedance $\left(i.e.\ \frac{\text{excess pressure}}{\text{volume current}}\right)$ for the incident wave in the previous problem.

Note: $\Delta p=Ks=-\rho c^2\partial\eta/\partial x$
$\dot{X}=A_1\partial\eta/\partial t$

88. Describe the principles of magnetostrictive and piezoelectric generators respectively, and discuss their relative advantages.

89. Write a brief essay on the applications of ultrasonic vibrations.

90. Derive an expression for the velocity of propagation in a gas of sound waves of large amplitude. Discuss the significance of your result in explaining the effects of explosive waves in air.

91. The multivibrator (Fig. E.2) is a two-stage resistance coupled amplifier in which the input of the first valve is provided by the output of the second valve, but since each valve introduces a phase shift of π radians the circuit will be oscillatory. The frequency of the oscillations is chiefly controlled by the values of the grid capacitances and resistances, but it is also a function

of the plate voltages and valve constants. The expression for the frequency of a given valve working at a constant voltage is therefore of the form $\dfrac{k}{C_1R_{G1}+C_2R_{G2}}$, where k is a constant and C and R are in farads and ohms respectively. In the more common symmetrical type of multivibrator $C_1=C_2$, $R_{G1}=R_{G2}$ and $R_{L1}=R_{L2}$. The wave-forms of the grid voltage and plate current are roughly saw-tooth and square respectively, this irregularity in shape signifying the presence of a large number of harmonics. By injecting a voltage of constant frequency into the multivibrator circuit a rich source of harmonics (up to several hundred) is obtained, the point of injection (and also whether the multivibrator is symmetrical or otherwise) controlling the nature of the harmonics, *e.g.* odd or even, present.

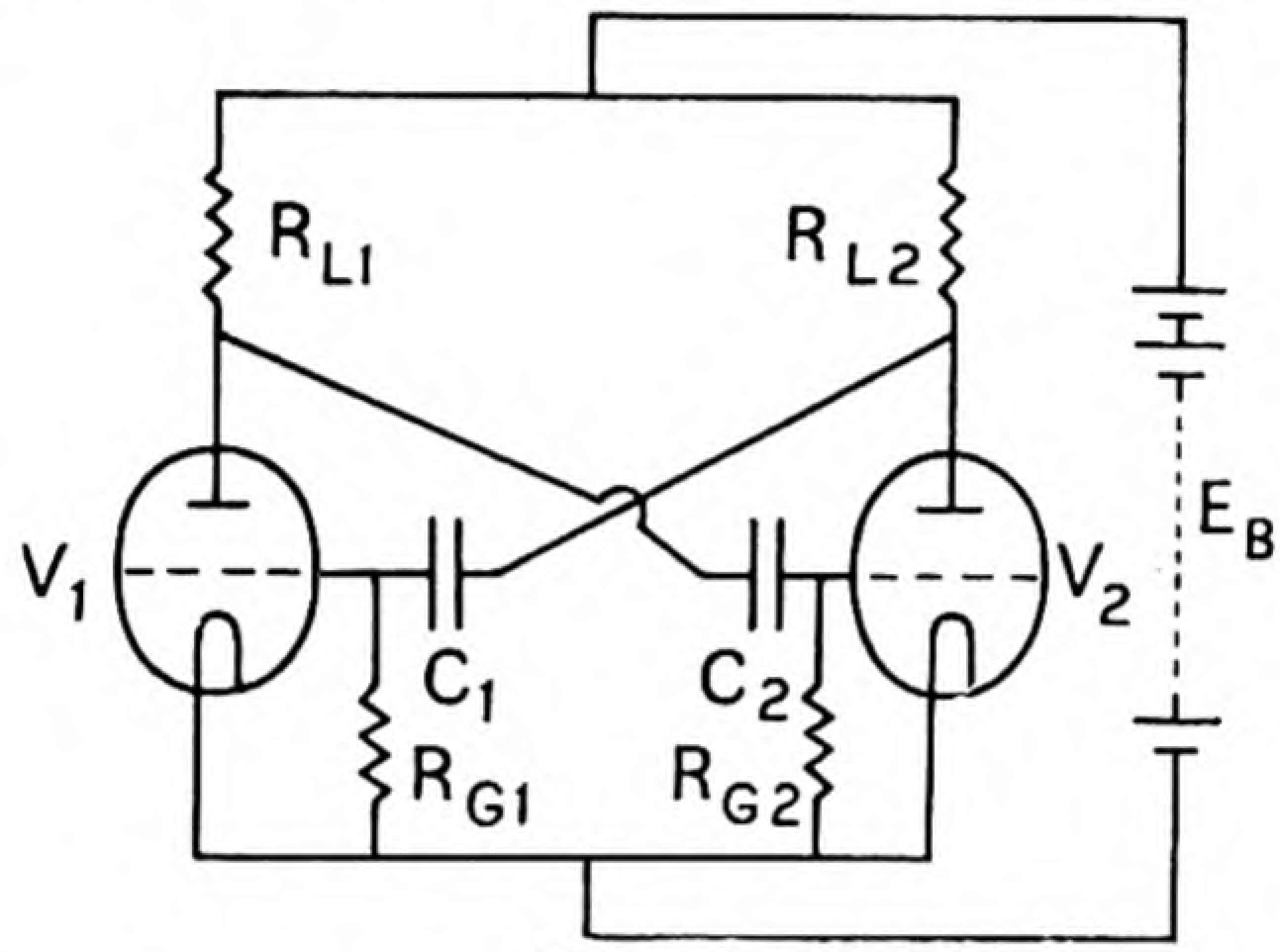


Fig. E.2.

In a particular multivibrator of the symmetrical form for which $k=2{\cdot}5$, $R_G=80{,}000$ ohm; determine the value of C for the fundamental frequency to be (*a*) 0·02 c.p.s., (*b*) 200 c.p.s., and (*c*) 2000 c.p.s.

92. Show by means of Fourier analysis that the harmonic composition of a square wave of amplitude E (*i.e.* a discontinuous function having only two values $+E$ and $-E$) is given by

$$y=f(x)=\frac{4}{\pi}E\left(\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x-\frac{1}{7}\cos 7x+\ \ldots\right)$$

$$i.e.\quad y=\frac{4}{\pi}\sum_{n=1,3,5,\text{etc.}}^{\infty}(-1)^{\frac{(n-1)}{2}}\cos\frac{nx}{n}.$$

93. A periodic force $F_0 \sin \omega t$ acts on a mass M which is subjected to an elastic restraint S per unit displacement and a damping force R per unit velocity. If the damping of the system is increased, what is the effect on the power dissipation in the system?

[For values of $R>\left(\dfrac{S-M\omega^2}{\omega}\right)$ the power dissipation is decreased.

Note.—Power Dissipation $=R\dot{x}^2$ and Average Power Dissipation $=R\left(\dfrac{v}{\sqrt{2}}\right)^2$ where $v=(\dot{x})_{\max}$ is the velocity amplitude.]

94. Use the relation on page 429 to plot a graph showing the relation between the percentage of reflected to incident energy for values of the standing wave ratio from unity to eight.

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